Abstract
Let $(R, \mathfrak{m})$ be a quasi-unmixed local ring and $I$ an equimultiple ideal of $R$ of analytic spread $s$. In this paper, we introduce the equimultiple coefficient ideals. Fix $k \in \{1, \ldots, s\}$. The largest ideal $L$ containing $I$ such that $e_i(I) = e_i(L)$ for each $i \in \{1, \ldots, k\}$ and each minimal prime of $I$ is called the $k$-th equimultiple coefficient ideal denoted by $I_k$. It is a generalization of the coefficient ideals introduced by Shah for the case of $\mathfrak{m}$-primary ideals. We also see applications of these ideals. For instance, we show that the associated graded ring $G_I(R)$ satisfies the $S_1$ condition if and only if $I^n = (I^n)_1$ for all $n$.

1. Introduction
Let $(R, \mathfrak{m})$ be a quasi-unmixed local ring of dimension $d$ and $I$ an $\mathfrak{m}$-primary ideal of $R$. Shah [10] showed the existence of unique largest ideals $I_k$ ($1 \leq k \leq d$) lying between $I$ and $\overline{I}$ such that the $k + 1$ Hilbert coefficients of $I$ and $I_k$ coincide, that is, $e_i(I) = e_i(I_k)$ for $0 \leq i \leq k$. These ideals are called coefficient ideals. They have been studied in some articles such as [2], [5], [6] and [10]. In [10], it was found that if $I$ contains a regular element, then the Ratliff-Rush closure $I^*$ and the $d$-th coefficient ideal $I_d$ coincide; moreover the author studied the associated primes of the associated graded ring $G_I(R)$. In [2], Ciupercă studies the relationship between the $S_2$-ification of the extended Rees algebra $R = R[It, t^{-1}]$ and the cited ideals. In [6], when $R$ is a domain, it is shown that the associated Ratliff-Rush ideal $I^*$ of $I$ is the contraction to $R$ of the extension of $I$ to its blowup $B(I) = \{R[I/a]_P \mid a \in I - 0, P \in \text{spec}(R[I/a])\}$, i.e., $I^* = \bigcap \{IS \cap R \mid S \in B(I)\}$. If further $R$ is analytically unramified, it is shown in [5], that the coefficient ideals $I_k$ are also contracted from a blowup $B(I)(k)$ which is obtained from $B(I)$ by a process similar to “$S_2$-ification”.

The paper is organized as follows: in section 2, we generalize the notion of coefficient ideals (introduced by Shah); we work with an equimultiple ideal $I$, that is, $\text{ht}(I) = s(I)$, where $s = s(I)$ is the analytic spread of $I$. We make
use of Böger’s theorem (on Hilbert-Samuel multiplicity) to show the existence of unique largest ideals $I_k$ (we use the same notation as used by Shah) lying between $I$ and $T$, and satisfying $e_i(I_p) = e_i((I_k)_p)$, for $0 \leq i \leq s$ and every minimal prime $p$ of $I$. We call them equimultiple coefficient ideals. Given an ideal $J$, we denote the unmixed part of $J$ by $J^u$. We show that if $I$ contains a regular element then $I_s = (I^*)^u$, which shows $I_s$ is an unmixed ideal. In fact we verify that all the equimultiple coefficient ideals are unmixed (Theorem 2.12).

In section 3, we give a criterion to control the height of the associated primes of $G_1(R)$ (Theorem 3.2). As a consequence of this, we show that if $G_1(R)$ satisfies $S_1$, so does $G_{1^m}(R)$ for every $m$ (Corollary 3.6). In [8], Noh and Vasconcelos showed that if $R$ is a Cohen-Macaulay ring, the Rees algebra $R[I_1]$ satisfies $S_2$ and $I$ is an equimultiple ideal, then all the powers $I^n$ are unmixed ideals. In this work, we verify the same result when $R$ is only a quasi-unmixed ring satisfying the $S_2$ condition. Finally, we give a way to provide associated graded rings $G_1(R)$ satisfying the $S_1$ condition (Corollary 3.20).

2. Equimultiple coefficient ideals

In this section, we show the existence of the equimultiple coefficient ideals and also we introduce a refined version for their existence. We show that all of them are unmixed ideals, and find their primary decompositions components. It is also seen how coefficient ideals control the height of the associated primes of $G_1(R)$. For example, the associated graded ring $G_1(R)$ satisfies the $S_1$ condition if and only if $(I^n)_1 = I^n$ for all $n$. As consequence, if $G_1(R)$ satisfies the $S_1$ condition then $G_{1^m}(R)$ satisfies the $S_1$ condition for all $m$. Finally, we give a way to construct associated graded rings satisfying the $S_1$ condition.

Let $I$ be an ideal in a ring $R$. An element $r \in R$ is said to be integral over $I$ if there exist an integer $n$ and elements $a_i \in I^i$, $i = 1, \ldots, n$, such that

$$r^n + a_1r^{n-1} + a_2r^{n-2} + \cdots + a_{n-1}r + a_n = 0.$$ 

The set of all elements that are integral over $I$ is called the integral closure of $I$, and it is denoted by $\overline{I}$.

Below, we recall the well known theorem of Böger on Hilbert coefficients. Let $\text{Min}(R)$ denote the set of minimal prime ideals of the ring $R$. Thus, $\text{Min}(R/I)$ is the set of minimal prime ideals of $I$.

**Theorem 2.1 (Böger [11, Corollary 11.3.2]).** Let $(R, m)$ be a quasi-unmixed local ring, and let $I \subseteq J$ be two ideals such that $I$ is equimultiple. Then $J \subseteq \overline{I}$ if and only if $e_0(I_p) = e_0(J_p)$ for every $p \in \text{Min}(R/I)$.

The next two remarks may be found in [10]. They are used when we localize an equimultiple ideal $I$ at a minimal prime $P$. 


Remark 2.2. Let \((R, \mathfrak{m})\) be a Noetherian local ring and \(\dim R \geq 1\). Suppose \(I \subseteq J\) are \(\mathfrak{m}\)-primary ideals and fix \(k\) such that \(1 \leq k \leq d\). Then for all large \(n\), \(e_i(I) = e_i(J)\) with \(0 \leq i \leq k\) if and only if \(\ell(J^n/I^n) \leq P(n)\), where \(P(n)\) is some polynomial in \(n\) of degree at most \(d - (k + 1)\).

Proof. It suffices to observe that, for large \(n\),

\[
\ell(J^n/I^n) = \ell(R/I^n) - \ell(R/J^n) = \sum_{i=0}^{d} (-1)^i [e_i(I) - e_i(J)] \binom{n + d - i - 1}{d - i}.
\]

Remark 2.3. Let \((R, \mathfrak{m})\) be a Noetherian local ring with \(\dim R \geq 1\). Suppose \(I \subseteq I' \subseteq J\) are \(\mathfrak{m}\)-primary ideals and fix \(k\) such that \(1 \leq k \leq d\). Then \(e_i(I) = e_i(J)\) with \(0 \leq i \leq k\) if and only if \(e_i(I') = e_i(J)\) with \(0 \leq i \leq k\).

Proof. We just use that \(\ell(I'^n/I^n) \leq \ell(J^n/I^n)\) and apply Remark 2.2.

By Böger’s Theorem, it is easy to see that \(\overline{I}\) is the unique largest ideal \(L\) which satisfies \(L \supseteq I\) and \(e_0(I_v) = e_0(L_v)\) for every \(v \in \text{Min}(R/I)\). In the next result we generalize the notion of coefficient ideals, firstly introduced by Shah in [10], for a more general case where \(I\) is an equimultiple ideal.

Theorem 2.4 (Existence of the equimultiple coefficients ideals). Let \((R, \mathfrak{m})\) be a quasi-unmixed local ring. Assume \(R/\mathfrak{m}\) is infinite and \(\dim R = d \geq 1\). Let \(I\) be an equimultiple ideal with \(s = s(I)\). Then there exist unique largest ideals \(I_k\), for \(1 \leq k \leq s\), containing \(I\) such that

1. \(e_i(I_{v}) = e_i((I_k)_{v})\), for \(0 \leq i \leq k\) and every \(v \in \text{Min}(R/I)\), and
2. \(I \subseteq I_1 \subseteq \cdots \subseteq I_s \subseteq \overline{I}\).

Proof. Let \(s = s(I)\) denote the analytic spread of \(I\). By the known Ratliff-Rush theorem, we have \(s = \dim R_v\), for any \(v \in \text{Min}(R/I)\). For each \(k = 1, \ldots, s\), consider the set

\[V_k = \{L \mid L\ is an ideal of R\ such that L \supseteq I\ and e_i(I_v) = e_i(L_v), for every 0 \leq i \leq k and v \in \text{Min}(R/I)\}.\]

Firstly note that if \(L \in V_k\ then e_i(I_v) = e_i(L_v)\), for every \(v \in \text{Min}(R/I)\), and in particular, by Böger’s Theorem, \(L \subseteq \overline{I}\).

Since \(I \in V_k\ and R\ is Noetherian there exists a maximal element \(J \in V_k\). We prove \(J\ is unique. Let \(L \in V_k\ and x \in L. Since I \subseteq (I, x) \subseteq L\, we have by Remark 2.3 that e_i(I_v) = e_i((I, x)_v) = e_i(J_v), for 0 \leq i \leq k\ and v \in \text{Min}(R/I). Then I is a reduction of (I, x), so that (I, x)^t+1 = (I, x)^tI, for some t. So x^t+1 \in (I, x)^tI \subseteq (J, x)^tJ. Hence, (J, x)^t+1 = (J, x)^tJ and
then \((J, x)^n = (J, x)^t J^{n-t}\), for \(n \geq t\). Fix \(\mathfrak{p} \in \text{Min}(R/I)\). We have, for all \(n \geq t\),

\[
\ell(((J, x)^n)_\mathfrak{p}/J^n_\mathfrak{p}) = \ell\left(\left((J, x)^t J^{n-t}\right)_\mathfrak{p}/J^n_\mathfrak{p}\right) = \ell\left((J^n_\mathfrak{p}, (J^{n-1}x)_\mathfrak{p}, \ldots, (J^{n-t}x^t)_\mathfrak{p})/J^n_\mathfrak{p}\right)
\]

\[
\leq \sum_{i=1}^{t} \ell\left((J^{n-i}x^i)_\mathfrak{p} + J^n_\mathfrak{p}/J^n_\mathfrak{p}\right) \leq \sum_{i=1}^{t} \ell\left((J^{n-i}x^i)_\mathfrak{p} + J^n_\mathfrak{p}/I^n_\mathfrak{p}\right)
\]

\[
\leq \sum_{i=1}^{t} \left[\ell((J^{n-i}x^i)_\mathfrak{p} + I^n_\mathfrak{p}/I^n_\mathfrak{p}) + \ell(J^n_\mathfrak{p}/I^n_\mathfrak{p})\right]
\]

\[
\leq \sum_{i=1}^{t} \left[\ell(I^{n-i}x^i)_\mathfrak{p} + I^n_\mathfrak{p}/I^n_\mathfrak{p}\right] + \ell\left((J^{n-i}x^i)_\mathfrak{p} + I^n_\mathfrak{p}/(I^{n-i}x^i)_\mathfrak{p} + I^n_\mathfrak{p}\right) + \ell(J^n_\mathfrak{p}/I^n_\mathfrak{p})
\]

\[
\leq \sum_{i=1}^{t} \left[\ell(J^n_\mathfrak{p}/I^{n-i}_\mathfrak{p}) + \ell((I, x)_\mathfrak{p}/I^n_\mathfrak{p}) + \ell(J^n_\mathfrak{p}/I^n_\mathfrak{p})\right].
\]

Since \(e_i(I_\mathfrak{p}) = e_i((I, x)_\mathfrak{p})\) and \(e_i(J_\mathfrak{p}) = e_i((J, x)_\mathfrak{p})\) holds for \(0 \leq i \leq k\) and every \(\mathfrak{p} \in \text{Min}(R/I)\), one can conclude by Remark 2.2 that \(e_i(J_\mathfrak{p}) = e_i((J, x)_\mathfrak{p})\) holds for \(0 \leq i \leq k\) and every \(\mathfrak{p} \in \text{Min}(R/I)\). But \(J\) is maximal in \(V_k\), so \(L \subseteq J\) and therefore \(J\) is the unique maximal in \(V_k\). This ideal is denoted by \(I_k\).

**Definition 2.5.** The ideals \(I_k\) above obtained will be called *equimultiple coefficient ideals*.

Let \(I^* = \bigcup_{n \geq 1} (I^{n+1} : I^n)\) be the Ratliff-Rush ideal. It is known that \(I^*\) is the unique largest ideal \(L\) which satisfies \(L \supseteq I\) and \(L^n = I^n\) for large \(n\). Moreover, by localizing at each \(\mathfrak{p} \in \text{Min}(R/I)\), we have \(e_i(I_\mathfrak{p}) = e_i((I^*)_\mathfrak{p})\), for \(0 \leq i \leq s\) and every \(\mathfrak{p}\) minimal prime of \(I\). By the above theorem one has \(I^* \subseteq I_s\).

An ideal \(I\) is said to be a *Ratliff-Rush ideal* if \(I^* = I\).

**Corollary 2.6.** Assume the hypothesis of Theorem 2.4. Then \(I \subseteq J \subseteq I_k \subseteq \overline{I}\) if and only if \(I \subseteq J\) and \(e_i(I_\mathfrak{p}) = e_i(J_\mathfrak{p})\), for \(1 \leq i \leq k\) and every \(\mathfrak{p} \in \text{Min}(R/I)\).

**Corollary 2.7.** Assume the hypothesis of Theorem 2.4. All the coefficient ideals \(I_k\) are Ratliff-Rush ideals.

**Proof.** We know that \((I_k)^n_\mathfrak{p} = ((I_k)^*)_\mathfrak{p}^n\), for \(n \gg 0\) and any prime \(\mathfrak{p}\). In particular, \(e_i((I_k)_\mathfrak{p}) = e_i(((I_k)^*)_\mathfrak{p})\), for \(0 \leq i \leq s\) and any \(\mathfrak{p} \in \text{Min}(R/I)\). So by maximality of \(I_k\), we have \((I_k)^* \subseteq I_k\).
Notation 2.8. Given an ideal $J \subseteq R$, let $J^u$ denote the unmixed part of $J$.

The next result shows that the coefficient ideal $I_*$ is an unmixed ideal if $I$ contains a regular element. Later, we will see that in fact $I_*$ is unmixed anyway (see Theorem 2.12).

**Proposition 2.9.** Assume the setup of Theorem 2.4 with $I$ containing a regular element. Then $I_* = (I^u)^u$. In particular, $I_*$ is an unmixed ideal.

**Proof.** For simplicity of notation, let $J = (I^u)^u$ denote the unmixed part of the Ratliff-Rush closure $I^*$. We have $\text{Min}(R/I) = \text{Min}(R/I^*).$

Because of the first condition in Theorem 2.4, we have $(I_*)_p \subseteq (I_p)^* = I_p^*$, for every $p \in \text{Min}(R/I).$ But $(I_*)_p \subseteq I_p^*$ for any $p \in \text{Min}(R/J) = \text{Ass}(R/J).$ Therefore, $I_* \subseteq J$.

**Proposition 2.10.** Assume the setup of Theorem 2.4 and let $J \supseteq I$ be an equimultiple ideal. Then

1. if $J \subseteq I_k$ then $I_k = J_k$;
2. if there exists one positive integer $m$ such that $J^m \subseteq (I^m)_k$, then $J^n \subseteq (I^n)_k$ for all positive integers $n$;
3. $((I^m)_k^n)_k = (I^{mn})_k$, for all positive integers $m, n$.

**Proof.** Fix $k$. For item (1), it suffices to use $\ell(R/I_p^n) - \ell(R/I_k^n) \leq \ell(R/I_p^m) - \ell(R/I_k^m)$, for each prime $p \in \text{Min}(R/I)$ and the fact that the last term is, for large $n$, a polynomial of degree at most $s - (k + 1)$.

Now we show (2). We have $e_i(I_p^m) = e_i(J_p^m)$, for $0 \leq i \leq k$ and every $p \in \text{Min}(R/I^m)$, by Corollary 2.6. By using coefficients ideals for primary case, we have, for each minimal prime $p$, that $I_p^m \subseteq J_p^m \subseteq (I_p^m)_k$. By [5, Proposition 3.2], $I_p^n \subseteq J_p^n \subseteq (I_p^n)_k$, for all $n$, so that for each minimal prime $p$, we have $e_i(I_p^n) = e_i(J_p^n)$, for $0 \leq i \leq k$ and all $n$. Corollary 2.6 gives then $J^n \subseteq (I^n)_k$ for all $n$.

Item (3) is a combination of the two previous items.

**Remark 2.11.** If $I$ contains a regular element, by Proposition 2.10 and Corollary 2.7 we have $(I^*)_k = I_k = (I_k)^*,$ since $(I^*)_k = I_k.$ In particular, the unmixed part of a Ratliff-Rush ideal is also a Ratliff-Rush ideal.

**Theorem 2.12.** Assume the setup of Theorem 2.4. The coefficient ideals of $I^u$ are $(I^u)_k = (I_k)^u$ and all the coefficient ideals of $I$ are unmixed ideals.

**Proof.** Firstly we construct a specific chain from $I^u$ formed by the unmixed part of the coefficient ideals of $I$ and after show that its terms are the coefficient
ideals of $I^u$. We have $((I_1)_{t})_{p} = (I_1)_{t} \leq \overline{T}_{p}$ for any $p \in \text{Ass}(R/\overline{T})$. So $(I_1)_{t} \leq \overline{T}$. Moreover, $((I_2)_{t})_{p} = (I_2)_{p} \leq ((I_1)_{t})_{p}$ for any $p \in \text{Ass}(R/(I_1)_{t})$ so that $(I_2)_{t} \leq (I_1)_{t}$. Inductively the desired chain is constructed.

Now let $I_{t} \subseteq J_{t} \subseteq \cdots \subseteq J_1 \subseteq \overline{T}$ be the coefficient ideals of $I_{t}$. Fix $k$. So $J_k$ is the unique largest ideal for which $e_i((I_{t})_{p}) = e_i((J_k)_{p})$, for $0 \leq i \leq k$ and any $p \in \text{Min}(R/I_{t})$. Moreover, $e_i((I_{t})_{p}) = e_i(I_{p}) = e_i((I_k)_{p}) = e_i(((I_k)_{t})_{p})$ for $0 \leq i \leq k$ and any $p \in \text{Min}(R/I_{t})$. Hence, $J_k \leq I_k \leq (I_k)_{t}$ and therefore $(I_k)_{t} = J_k$, for each $k$.

By Proposition 2.10, $(I_{t})_{k} = I_k$. Therefore, $I_k$ is an unmixed ideal for each $k$.

The next result expresses the primary decomposition components of the coefficient ideals $I_k$, besides giving another way to show they are unmixed ideals.

**Proposition 2.13.** Assume the setup of Theorem 2.4. Let $p_1, \ldots, p_r$ be the minimal primes of $I$. Then $I_k$ has the following primary decomposition

$$I_k = ((I_{p_1})_k \cap R) \cap \cdots \cap ((I_{p_r})_k \cap R).$$

Furthermore, $(I_{p})_{t} = (I_{p})_{t}$ for every prime ideal $p$.

**Proof.** Fix $k \in \{1, \ldots, s\}$. To simplify notation, let $J_i$ and $H_i$ denote $(I_{p_i})_k$ and $(I_{p_i})_k \cap R$, respectively, where $1 \leq i \leq r$. Since $J_i$ is $p_i R_{p_i}$-primary, $H_i$ is $p_i$-primary. Set $H = H_1 \cap \cdots \cap H_r$. Then $H_{p_i} = (H_i)_{p_i} = J_i$ for each $i$, as $(H_j)_{R_{p_i}} = R_{p_i}$ for every $j \neq i$. By definition of equimultiple coefficient ideals one may then conclude $I_k = H$.

Hence, the second part follows by observing that

$$(I_{p})_{t} = ((I_{p_1})_k \cap R_{p}) \cap \cdots \cap ((I_{p_r})_k \cap R_{p})$$

and

$$(I_{k})_{t} = ((I_{p_1})_k \cap R)_{t} \cap \cdots \cap ((I_{p_r})_k \cap R)_{t},$$

where $p_1, \ldots, p_r \leq p$ and $p_{t+1}, \ldots, p_r \not\leq p$.

Below, we recall the definition of some blowup algebras which are important for the present paper.

**Definition 2.14.** Let $R$ be a ring, $I$ an ideal and $t$ an indeterminate over $R$. The Rees algebra of $I$ is the subring of $R[t]$ defined as

$$R[It] := \oplus_{n \geq 0} I^n t^n.$$

The extended Rees algebra of $I$ is the subring of $R[t, t^{-1}]$ defined as

$$R[It, t^{-1}] := \oplus_{n \in \mathbb{Z}} I^n t^n,$$
where, by convention, for any non-positive integer \( n \), \( I^n = R \). The associated graded ring of \( I \) is

\[
G_I(R) := \bigoplus_{n \geq 0} \frac{I^n}{I_{n+1}} = R[It]/IR[It] = R[It, t^{-1}]/t^{-1}R[It, t^{-1}].
\]

For simplicity, from now on, we also denote the extended Rees algebra of \( I \) by \( \mathcal{R} \).

**Theorem 2.15.** Let \((R, \mathfrak{m})\) be a local ring and \( I \) an equimultiple ideal. If \(((I^n)^*)^u = I^n\) for all \( n \), then \( \text{ht}(P) < s \) for every \( P \in \text{Ass}(G_I(R)) \).

**Proof.** It is easy to see that the hypothesis implies \((I^n)^* = I^n\) and \((I^n)^u = I^n\) for all \( n \). Let \( P \in \text{Ass}_R(\mathcal{R}/t^{-1}\mathcal{R}) \) and \( \mathfrak{p} = P \cap R \). Initially we assume \( R \) is a domain. By the Dimension Inequality, one has

\[
\text{ht}(P) - \text{ht}(\mathfrak{p}) \leq 1 - \text{tr.deg}_k R/P.
\]

We claim that \( \text{tr.deg}_{R/\mathfrak{p}} \mathcal{R}/P \neq 0 \). Suppose the contrary. Note first that we can assume \( \mathfrak{p} \) is maximal, since \(((I^n)^*)^u = I^n\) and \( \text{ht}(P) = \text{ht}(P_\mathfrak{p}) \). Then \( \mathcal{R}/P \) is a finitely generated algebra over the field \( k := R/\mathfrak{p} \). Hence, \( \dim \mathcal{R}/P = \text{tr.deg}_k \mathcal{R}/P = 0 \). It implies \( \mathcal{R}/P \) is a field, as \( \mathcal{R}/P \) is a domain. Therefore, \( R/\mathfrak{p} \) and \( \mathcal{R}/P \) are isomorphic; whence \( G_+ \subseteq P/t^{-1}\mathcal{R} \), which is a contradiction since \( G_+ \) is a regular ideal.

In conclusion, we can write \( P = (t^{-1}\mathcal{R} : at^r) \), for some homogeneous element \( at^r \in \mathcal{R} \setminus t^{-1}\mathcal{R} \). Hence, \( Pat^r \in I^{r+1}t^r \), for some integer \( r \geq 0 \), so that \( \mathfrak{p} = (I^{r+1} : a) \). This means \( \mathfrak{p} \in \text{Ass}(R/I^{r+1}) \). By hypothesis, \( \mathfrak{p} \) is a minimal prime of \( I^{r+1} \), so \( \text{ht}(\mathfrak{p}) = s \). As \( \text{tr.deg}_{R/\mathfrak{p}} \mathcal{R}/P \neq 0 \), through the above inequality, we obtain \( \text{ht}(P) \leq s \) and therefore \( \text{ht}(P/t^{-1}\mathcal{R}) < s \).

The case for which \( R \) is not a domain is similar. It follows by taking a minimal prime \( Q \) contained in \( P \) such that \( \text{ht}(P) = \text{ht}(P/Q) \) and after going modulo a minimal prime.

**Remark 2.16.** The equation \(((I^n)^*)^u = I^n\), for all \( n \), is equivalent to have \((I^n)^* = I^n\), for all \( n \), and \((I^n)^u = I^n\), for all \( n \). Moreover if \( I \) is a regular ideal and only \((I^n)^u = I^n\), for \( n \gg 0 \), the condition \((I^n)^* = I^n\), for all \( n \), implies \((I^n)^u = I^n\), for all \( n \), since \( \text{Ass}(\mathcal{R}/(I^n)^*) \subseteq \text{Ass}(\mathcal{R}/(I^{n+1})^*) \), for all \( n \geq 1 \) (see [9, p. 14]).
3. Equimultiple coefficient ideals, associated graded ring and Serre’s condition ($S_1$)

In this section, we see necessary and sufficient conditions for the associated graded ring $G_1(R)$ to satisfy the $S_1$ condition. It has a relation to the concept of equimultiple coefficient ideals. Moreover, in Theorem 3.2 and Theorem 3.17, we generalize Theorem 4 from [10], which concern the height of associated prime ideals of $G_1(R)$. In particular, we obtain that $G_1(R)$ satisfies the $S_1$ condition if and only if $(I^n)_1 = I^n$, for all $n$.

**Lemma 3.1.** Let $(R, \frako{m})$ be a quasi-unmixed local ring and let $I$ be a proper ideal of $R$. Then we have the following:

1. If $\text{ht}(P) < k$ for every $P \in \text{Ass}(G_1(R))$, then $I^n_p = (I^n_p)_k$, for all $n$ and every $\frako{p} \in \text{Min}(R/I)$;

2. Suppose all the powers $I^n$ are unmixed ideals, then the converse of (1) is valid.

**Proof.** To show (1), let $\frako{p} \in \text{Min}(R/I)$. Then for every $P_\frako{p} \in \text{Ass}_{\frako{m}}(\frako{R}_{\frako{p}}/t^{-1}\frako{R}_{\frako{p}})$ we have $\text{ht}(P_\frako{p}) = \text{ht}(P) < k$. We use then the result [10, Theorem 4] to complete the assertion. For the other assertion, let $P \in \text{Ass}_{\frako{m}}(\frako{R}/t^{-1}\frako{R})$. We have $\frako{p} = P \cap R \in \text{Ass}(R/I^n)$ which is minimal on $I^n$ by assumption. Therefore, $\text{ht}(P) = \text{ht}(P_\frako{p}) < k$ and the result follows.

**Theorem 3.2.** Let $(R, \frako{m})$ be a quasi-unmixed local ring and let $I$ be an equimultiple ideal.

1. If $(I^n)_k = I^n$, for all $n$, then $\text{ht}(P) < k$, for every $P \in \text{Ass}(G_1(R))$.

2. If $\text{ht}(P) < k$, for every $P \in \text{Ass}(G_1(R))$, then $(I^n)_k = (I^n)_u$, for all $n$.

In particular, $G_1(R)$ satisfies the $S_1$ condition if and only if $(I^n)_1 = I^n$, for all $n$.

**Proof.** If $(I^n)_k = I^n$, we have in particular that $(I^n)_u = I^n$, for all $n$. Further by localizing we obtain $(I^n_\frako{p})_k = ((I^n)_\frako{p})_u = I^n_\frako{p}$, for every minimal prime $\frako{p}$ of $I$, by Proposition 2.13. Now one just applies Lemma 3.1 to conclude item (1).

To show (2), firstly let $\frako{p}$ be a minimal prime of $I$ and consider the localization $S^{-1}R[It, t^{-1}]$, where $S = R \setminus \frako{p}$. For each associated prime $S^{-1}P \in \text{Ass}_{S^{-1}R[It, t^{-1}]}(S^{-1}R[It, t^{-1}]/t^{-1}S^{-1}R[It, t^{-1}])$, we have $\text{ht}(S^{-1}P) = \text{ht}(P) < k$. Thus, by [10, Theorem 4], we obtain, for each $\frako{p} \in \text{Min}(R/I)$, that $I^n_\frako{p} = (I^n_\frako{p})_u$, for all $n$. In particular, $I^n_\frako{p} = ((I^n)_\frako{p})_u$ by Proposition 2.13, so that $(I^n)_k = (I^n)_u$, for all positive integers $n$. 

Now we consider the case $k = 1$. If $\frak{p} \in \text{Ass}(R/I^n)$, then there exists a $P \in \text{Ass}(R/I^{-1})$ such that $P = \frak{p} \cap R$. Firstly assume $R$ is a domain. By using the Dimension Formula, we obtain $ht(\frak{p}) = ht(P) - 1 + t$, where

$$t := \text{tr.deg}_{R/\frak{p}} R = \text{tr.deg}_{R/\frak{p}} R \leq \text{tr.deg}_{\frak{p}/\frak{p}} R = s.$$ 

Therefore, each associated prime $\frak{p} \in \text{Ass}(R/I^n)$ is actually a minimal prime of $I^n$, for every $n$, as required. The general case may be derived by taking a minimal prime $Q$ of $R$ such that $Q \subseteq P$ and $ht(P) = ht(P/Q)$. The converse is immediate from (1).

**Corollary 3.3.** Let $(R, \frak{m})$ be a quasi-unmixed, analytically unramified domain satisfying the $S_2$ condition and let $I$ be an equimultiple ideal such that $ht(I) \geq 2$. If $\bigoplus_{n \geq 0} I^n$ is the $S_2$-ification of the Rees algebra $R[It]$ and $ht(P) < k$, for every $P \in \text{Ass}(G_1(R))$, then $(I^n)_k \subseteq I_n$.

**Proof.** This follows directly from [2, Proposition 2.10] and Theorem 3.2.

The Theorem 3.2 derives the following result, firstly introduced by Noh and Vasconcelos [8, Theorem 2.5] for the less general case which $R$ is a Cohen-Macaulay ring.

**Corollary 3.4.** Let $R$ be a quasi-unmixed ring satisfying $S_2$ and $I$ an equimultiple ideal containing a regular element. If $R[It]$ satisfies $S_2$, then all the powers $I^n$ are unmixed ideals.

**Proof.** We may assume $R$ is local. It then suffices to use [1, Theorem 1.5] and later apply Theorem 3.2.

**Remark 3.5.** Grothe, Hermann and Orbanz [4, Theorem 4.7] showed that if $I$ is an equimultiple ideal of a Cohen-Macaulay local ring $(R, \frak{m})$, then the Cohen-Macaulayness of $G_1(R)$ implies the Cohen-Macaulayness of $G_{1^m}(R)$, for all $m \geq 1$. Also Shah [10, Corollary 5(C)] showed the same result when $R$ is just quasi-unmixed but $I$ an $\frak{m}$-primary ideal.

Below we see that a similar result for the $S_1$ condition can be obtained immediately through coefficient ideals.

**Corollary 3.6.** Let $(R, \frak{m})$ be quasi-unmixed local ring and $I$ an equimultiple ideal. If $G_1(R)$ satisfies $S_1$, then so does $G_{1^m}(R)$, for all $m \geq 1$.

Due to the above result and [1, Theorem 1.5], we obtain the following.

**Corollary 3.7.** Let $(R, \frak{m})$ be a quasi-unmixed local ring satisfying $S_2$ and $I$ an equimultiple ideal containing a regular element. If $R[It]$ satisfies $S_2$, then so does $R[I^{m^2}t]$, for all $m \geq 1$. 


Theorem 3.8. Let \((R, \mathfrak{m})\) be a quasi-unmixed local ring and \(I\) an equi-
multiple ideal of analytic spread \(s\). If depth \(G_I(R) \geq k\), where \(1 \leq k \leq s\), then \((I^n)_j = (I^n)^u\), for all \(n\) and \(s + 1 - k \leq j \leq s\).

Proof. By using the fact that \(G \otimes_R R_p\) is flat over \(G\), one can conclude that depth \(G_I(R) \geq k\) implies depth \(G_{I_p}(R_p) \geq k\), for each \(p\) prime. So by [10, Theorem 5], we have \(I^n_p = (I^n)_j\), for all \(n\), and each minimal prime \(p\) of \(I\). Hence, \(I^n_p = ((I^n)_j)_p\) and therefore \((I^n)_j = (I^n)^u\), for all positive integers \(n\), as all coefficients ideals are unmixed ideals.

Remark 3.9. As can be seen in the above proof, if \(I\) is an arbitrary equim-
multiple ideal and depth \(G_I(R) + \geq k\), one has \(I^n_p = (I^n)_j\), for all \(n\), \(s + 1 - k \leq j \leq s\), and each minimal prime \(p\) of \(I\).

Proposition 3.10. Let \((R, \mathfrak{m})\) be a quasi-unmixed local ring satisfying the \(S_{s+1}\) condition and \(I\) an ideal with \(\text{grade} \ I = \text{ht}(I)\) which is equimultiple. Suppose \(s(I_p) = \mu(I_p)\), for every \(p \in \text{Min}(R/I)\). Then \(G_I(R)\) satisfies \(S_1\).

Proof. By hypothesis, there exists a minimal reduction \(J\) of \(I\) generated by a regular sequence of length \(s\) and \(J_p = I_p\), for each \(p \in \text{Min}(R/I)\), since \(I_p\) has no proper reduction. Once \(R\) satisfies the \(S_{s+1}\) condition, we have \(J\) is unmixed. Hence, \(I = J\) is generated by a regular sequence of length \(s\). In particular the generating set of \(I\) form a quasi-regular sequence, thus grade \(G_I(R) + \geq s\). Moreover there is an isomorphism of graded rings

\[ A = (R/I)[X_1, \ldots, X_s] \cong G_I(R), \]

where \(A\) is a polynomial ring with coefficients in \(R/I\). We can then conclude that \(\text{Ass}_R(I^i/I^{i+1}) = \text{Ass}_R(R/I)\), for each \(i\). By using the exact sequence

\[ 0 \rightarrow I^i/I^{i+1} \rightarrow R/I^{i+1} \rightarrow R/I^i \rightarrow 0, \]

it follows by induction that \(\text{Ass}_R(R/I^n) = \text{Ass}_R(R/I)\), for \(n \geq 1\). Since \(I\) is unmixed, we conclude that \(I^n = (I^n)^u\), for all \(n\). The result follows then by Theorem 3.8 and Theorem 3.2.

An ideal \(I\) is a locally complete intersection if \(\text{ht}(I_p) = \mu(I_p)\) for each \(p \in \text{Ass}(R/I)\).

Corollary 3.11. Let \((R, \mathfrak{m})\) be a quasi-unmixed local ring satisfying the \(S_{s+1}\) condition and \(I\) an ideal with grade \(I = \text{ht}(I)\) which is equimultiple. Suppose \(I\) is a locally complete intersection. Then \(G_I(R)\) satisfies \(S_1\).

Remark 3.12. In the set-up of Proposition 3.10, we obtain \((I^n)_s = \cdots = (I^n)_1 = I^n\) for all positive integers \(n\).
**Definition 3.13.** Let $R$ be a local ring and $I$ a proper ideal of $R$. The *reduction number* $r(I)$ of $I$ is defined to be

\[
r(I) = \min\{n \mid \text{there exists a minimal reduction } J \text{ of } I \text{ such that } I^{n+1} = JJ^n\}.
\]

**Proposition 3.14.** Let $(R, \mathfrak{m})$ be a quasi-unmixed local ring satisfying the $S_{s+1}$ condition and $I$ an ideal with $\text{grade } I = \text{ht}(I)$ which is equimultiple. Suppose some power $I^i$ with $r(I^i) \leq 1$ is an unmixed ideal. If $\text{grade } G_1(R) \geq k$, where $1 \leq k \leq s$, then $(I^n)_j = I^n$, for all $n$ and $s+1-k \leq j \leq s$.

**Proof.** Because of Theorem 3.8, it suffices to show $(I^n)_u = I^n$, for all $n$. Further, by [6, (1.2)], we have $(I^n)^* = I^n$, for all $n$. Hence, $\text{Ass}(R/I^n) \subseteq \text{Ass}(R/I^{n+1})$, for all $n$, by [9, Lemma 6.6]. By hypothesis, we then get $\text{Ass}(R/I^n) = \text{Min}(R/I^n)$, for all $n \leq i$. By assumption on $r(I^i)$, there exists a minimal reduction $J$ of $I^i$ such that $JJ^i = (I^i)^2$. Note that $J$ may be generated by $s(I^i) = s$ elements. Thus, the hypothesis grade $I = s$ gives that grade $J = s$, so that $J$ may be generated by a regular sequence of length $s$. Since $R$ satisfies $S_{s+1}$, the ideal $J$ is unmixed.

Now consider the exact sequence

\[
0 \longrightarrow J/JJ^i \longrightarrow R/JJ^i \longrightarrow R/J \longrightarrow 0,
\]

where $J/JJ^i \simeq (R/I^i)^s$. Since $J$ is an unmixed ideal one may then conclude that $(I^i)^2$ is unmixed. By considering the following exact sequence

\[
0 \longrightarrow J^2/J^2I^i \longrightarrow R/(I^i)^3 \longrightarrow R/J^2 \longrightarrow 0,
\]

we get $(I^i)^3$ is unmixed. We have then obtained that $I^n$ is unmixed for infinitely many $n$. Therefore, all the powers $I^n$ are unmixed ideals.

**Lemma 3.15.** Let $(R, \mathfrak{m})$ be a quasi-unmixed local ring of infinite residue field and $I$ an equimultiple ideal. For all $N \geq 1$ and all reduction $x = x_1, \ldots, x_t$ of $I^N$, we have

\[
(I^{N+1}: x_1, \ldots, x_k) \subseteq I_k, \quad \text{for } 1 \leq k \leq d.
\]

**Proof.** It is easy to see we may assume $x$ is a minimal reduction. Fix any $N \geq 1$ and let $x_1, \ldots, x_s$ be a minimal reduction of $I^N$. By [10, Theorem 2], we have $(I^{N+1}: x_1, \ldots, x_k) \subseteq (I_p)_k$, for each $p \in \text{Min}(R/I)$. Hence, for each $p \in \text{Min}(R/I)$, the equality $e_i((I^{N+1}: x_1, \ldots, x_k)_p) = e_i(I_p)$ is true for $0 \leq i \leq k$. The result then follows by maximality of $I_k$. 

Lemma 3.16. Let \((R, m)\) be quasi-unmixed ring and \(I\) an equimultiple ideal with analytic spread \(s\). Let \(x_1, \ldots, x_s \in I^N\), for some \(N \geq 1\), and let \(x'_1, \ldots, x'_s\) be their images in \(I^N/I^{N+1}\). Let \(a_1, \ldots, a_t \in R\) be a system of parameters modulo \(I\). Then \(a'_1, \ldots, a'_t, x'_1, \ldots, x'_s\) is a system of parameters of \(G_1(R)\) if and only if \(x_1, \ldots, x_s\) form a minimal reduction of \(I^N\).

Proof. Denote \(G_1(R)\) by \(G\) and the ideal \((a'_1, \ldots, a'_t)G_1(R)\) by \(L\). If the sequence \(a'_1, \ldots, a'_t, x'_1, \ldots, x'_s\) is a system of parameters of \(G\), we have \(\sqrt{(G/L)_+} = (x'_1, \ldots, x'_s)G/L\). Then \(G^n_+ \subseteq (x_1, \ldots, x_s)G + L\), for some positive integer \(n\). Thus, it is easy to see that \(G^n_+ \subseteq (x_1, \ldots, x_s)G + LG^n_+\). By Nakayama’s Lemma, one can conclude \(G^n_+ \subseteq (x_1, \ldots, x_s)G\). Therefore, \(x_1, \ldots, x_s\) form a minimal reduction of \(I^N\). The converse follows analogously to [4, Corollary 2.7].

The next result is also a generalization of Theorem 4 in [10] as may be easily observed.

Theorem 3.17. Let \((R, m)\) be a quasi-unmixed local ring and \(I\) an equimultiple ideal. Let \(a_1, \ldots, a_r\) be a system of parameters modulo \(I\) and let \(L = (a'_1, \ldots, a'_r)\) denote the ideal generated by their images in \(G_1(R)\). Fix \(k \in \{1, \ldots, s\}\). If \(I^n = (I^n)_k\), for all \(n\), then \(ht(P + L) < k + d - s\), for every \(P \in \text{Ass}(G_1(R))\).

Proof. Let \(P\) be an associated prime of \(G_1(R)\). Since \(P\) is graded we have \(P = (0' : y')\), where \(y'\) is the image in \(G_1(R)\) of \(y \in I^{n-1} - I^n\), for some \(n \geq 1\). Suppose \(ht(P + L) \geq k + d - s\). Then we obtain \(\dim((G/L)/P(G/L)) \leq s - k\). By [10, Lemma 2(E)] we can get a homogeneous system \(\bar{x}_1, \ldots, \bar{x}_s\) of parameters (of equal degree) for \(G/L\) such that \(\bar{x}_1, \ldots, \bar{x}_k \in P(G/L)\). Let \(x'_1, \ldots, x'_k \in G_1(R)\) be homogeneous inverse images of \(\bar{x}_1, \ldots, \bar{x}_k\) respectively so that \(x'_1, \ldots, x'_k \in P\). Since \(a'_1, \ldots, a'_r, x'_1, \ldots, x'_s\) is a system of parameters of \(G_1(R)\) ([4, Proposition 2.6]) we can then use Lemma 3.16 to obtain that \(x_1, \ldots, x_s\) is a minimal reduction of \(I^m\), for some \(m\). It is easy to see we may assume \(m = nN\). Hence, \(y(x_1, \ldots, x_k) \subseteq I^{n-1 + nN + 1}\), and so \(y \in ((I^n)^{N+1} : x_1, \ldots, x_k)\). By Lemma 3.15, one has \(y \in (I^n)_k = I^n\), which is a contradiction.

Definition 3.18. Let \(R\) be a local ring and \(I\) a proper ideal of \(R\). If for each \(p \in \text{Min}(R/I)\), the localization \(I_p\) has reduction number \(r(I_p) \leq t\), it is said that \(I\) has generic reduction number \(t\).

Proposition 3.19. Let \((R, m)\) be a Cohen-Macaulay local ring of infinite residue field and \(I\) an equimultiple ideal of analytic spread \(s\). Assume \(R/I\) satisfies the \(S_1\) condition and \(I\) has generic reduction number \(1\). Then \(G_1(R)\) satisfies the \(S_1\) condition.
Proof. Let $J$ be a minimal reduction of $I$ and consider the exact sequence
\[ 0 \longrightarrow JJI \longrightarrow RJI \longrightarrow R/J \longrightarrow 0, \]
where $JJI \cong (R/I)^s$. Hence, since $R$ is Cohen-Macaulay and because $I$ is unmixed, one may conclude $JI$ is unmixed. In this way, $I^2R_p = JIR_p$, for all $p \in \text{Ass}(R/JI) = \text{Min}(R/JI)$, and then $I^2 = JI$. Consider now the following exact sequence
\[ 0 \longrightarrow J^2/J^2I \longrightarrow R/I^3 \longrightarrow R/J^2 \longrightarrow 0. \]
Since the associated prime ideals of $J^n$ are the same as those of $J$, and $J^2/J^2I$ is isomorphic to a power of $R/I$, one may conclude that $I^3$ is unmixed. Inductively we obtain that all powers $I^n$ are unmixed ideals. On the other hand, if $p \in \text{Ass}_R(R/I)$, we have $R_p/I_p$ is a Cohen-Macaulay ring and hence, by [7, Proposition 26.12], we obtain that $G_{I_p}(R_p)$ is Cohen-Macaulay, for all minimal prime $p$ of $I$, and therefore we may use [10, Theorem 4] to obtain $I^n_p = (I^n_p)_1$, for all $n$ and all $p \in \text{Min}(R/I)$. Now one may just apply Lemma 3.1 to conclude the proof.

In [3], Corso and Polini indicated a method to provide ideals $I$ of reduction number 1. In this way, through above proposition we may produce associated graded rings $G_I(R)$ satisfying the $S_1$ condition.

Corollary 3.20. Let $(R, m)$ be a Cohen-Macaulay ring, $p$ a prime ideal of height $g$ such that $R_p$ is not a regular local ring and $J = (x_1, \ldots, x_g) \subseteq p$ a regular sequence. Set $I = J : p$. Then $G_I(R)$ satisfies $S_1$.

Proof. Since $\text{Ass}(R/I) \subseteq \text{Ass}(R/J)$, the ideal $I$ is unmixed. The result then follows from [3, Theorem 2.3].

Acknowledgements. Pedro Lima thanks Sathya Sai Baba for guidance. Both authors are very grateful to Professor Daniel Katz for his encouragement, advice and reviews. Pedro Lima thanks the University of Kansas for all facilities. Finally, the authors thank the referee for all useful suggestions.

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