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A BOAS-TYPE THEOREM FOR α-MONOTONE FUNCTIONS

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Abstract

We define the class of α-monotone functions using fractional integrals. For such functions we prove a Boas-type result on the summability of the Fourier coefficients.

1. Introduction

In this paper we study summability properties of a sequence $\{a_k\}_{k=0}^{\infty}$ and integrability properties of the corresponding trigonometric series

$$f(x) = \sum_{k=0}^{\infty} a_k \cos \pi k x, \quad x \in [0, 1].$$

1.1. The Hardy-Littlewood and Boas theorems

One of the first results in this area is the well-known theorem of Hardy and Littlewood ([12], [24, XII,§6], [3, §6]).

THEOREM A. *Let $1 < p < \infty$ and $a = \{a_k\}_{k=0}^{\infty}$ be a non-increasing non-negative sequence such that $a_k \to 0$ as $k \to \infty$. Then*

$$\|f\|_{L_p(0,1)} \sim \left(\sum_{k=0}^{\infty} (k+1)^{p-2} a_k^p \right)^{1/p}.$$

Throughout this paper, we denote by C a positive constant that may be different on different occasions. In addition, $T \sim S$ means that $\frac{1}{C} S \leq T \leq CS$.

There are several generalizations of Theorem A. In particular, Sagher [18] replaced the monotonicity by the quasi-monotonicity condition. In [23] and

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[6] Dyachenko and Tikhonov further generalized this result by considering the general monotonicity condition. Later more general result was stated in [8]. Dyachenko in [9] proved similar result for piecewise monotonic functions of several variables.

The counterpart result to Theorem A is the following theorem by Hardy and Littlewood ([24, XII,§6], [3, §6])

Theorem B. *Let* $1 < p < \infty$, $f(x)$ *be a non-negative non-increasing integrable function on* $[0, 1]$, *then*

$$\|a\|_{\ell_p} \sim \left(\int_0^1 x^{p-2}(f(x))^p dx\right)^{1/p}.$$

Similar questions were studied in the setting of the Lorentz spaces. We will use the following notation.

Let μ be Lebesgue measure on $[0, 1]$ and let f be a μ-measurable function on $[0, 1]$, then by f^* we denote the non-increasing rearrangement of f, i.e.,

$$f^*(t) = \inf\left\{\sigma : \mu\{x \in [0,1] : |f(x)| > \sigma\} \le t\right\}.$$

Definition 1.1. Let $0 < p \le \infty$ and $0 < q \le \infty$. Then the *Lorentz space* $L_{p,q}$ is the set of μ-measurable functions for which, the functional

$$\|f\|_{L_{p,q}} := \begin{cases} \left(\int_0^1 \left(t^{1/p} f^*(t)\right)^q \frac{dt}{t}\right)^{1/q}, & \text{for } 0 < p < \infty \text{ and } 0 < q < \infty, \\ \sup\limits_t t^{1/p} f^*(t), & \text{for } 0 < p \le \infty \text{ and } q = \infty, \end{cases}$$

is finite.

We will denote by $\ell_{p,q}$ similarly defined spaces of sequences.

Analogues of Theorem A for the Lorentz spaces were proved by Sagher [18], Dyachenko and Nursultanov [11] (see also [15]), and Booton [4]. In [11] the authors considered trigonometric series with α-monotone coefficients. These series were introduced by Dyachenko in [7] (see also [10]).

The corresponding analogue of Theorem B for the Lorentz spaces was stated in Boas's book [3, p. 36] and was proved in [19].

Theorem C. *Let* $1 < p < \infty$ *and* $f(x)$ *be as in Theorem B. Then*

$$\|a\|_{\ell_{pq}} \sim \|f\|_{L_{p',q}}.$$

Similar results in the two-dimensional case were proved in [22]. The corresponding result for the Fourier transform was proved by Sagher in [20]. Later on, further generalizations of Sagher's results were obtained in [13]–[14].

The main goal of this paper is to generalize Theorem C replacing the monotonicity condition by the weaker condition of α-monotonicity.

1.2. α-monotonicity

We will need the following definition of Riemann-Liouville's fractional integrals and derivatives.

DEFINITION 1.2. Let $f(x) \in L_1(0,1)$, $0 < \alpha < 1$. Then the integral

$$I^\alpha f(x) = (I^\alpha_{1-} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^1 \frac{f(t)}{(t-x)^{1-\alpha}}\,dt \quad \text{for } x < 1.$$

is called fractional integral of order α. By $\mathscr{D}^\alpha f(x)$ we denote the fractional derivative of order α $(0 < \alpha < 1)$, i.e.,

$$\mathscr{D}^\alpha f(x) = (\mathscr{D}^\alpha_{1-} f)(x) = -\frac{d}{dx}(I^{1-\alpha} f)(x).$$

REMARK 1.3. If $\alpha = 1$, then the fractional derivative $\mathscr{D}^1$ is understood as follows: $\mathscr{D}^1 f(x) := -f'(x)$. Also, the fractional integral I^0 is understood as: $I^0 f(x) := f(x)$.

DEFINITION 1.4. Let $0 < \alpha \le 1$. We say that a non-negative function f is α-monotone (or belongs to the class M_α), if $I^{1-\alpha} f(x)$ is a non-increasing, absolutely continuous function on $[0, 1]$.

The condition of absolute continuity is related to the following Lemma [21, p. 44] (see also [5, Lemma 2]).

LEMMA 1.5. *Suppose that* $0 < \alpha \le 1$, $f(x) \in L_1(0,1)$, *and let the fractional integral* $I^{1-\alpha} f(x)$ *be an absolutely continuous function on* $[0, 1]$. *Then the following equality holds*

$$I^\alpha \mathscr{D}^\alpha f(x) = f(x) - \frac{f_{1-\alpha}(1)}{\Gamma(\alpha)}(1-x)^{\alpha-1},$$

where

$$f_{1-\alpha}(x) = I^{1-\alpha} f(x).$$

Note that if $\alpha = 1$, then from Lemma 1.5 we obtain the fundamental theorem of calculus for Lebesgue integration.

1.3. The main result

As usual, we denote $p' = p/(p-1)$. The main result of this paper is the following.

THEOREM 1.6. *Let* $\alpha \in (0,1]$, $1/\alpha < p < \infty$, $1 \le q \le \infty$. *Let also* f *be an integrable function,* $f \in M_\alpha$, *and* $a_k = \int_0^1 f(x)\cos \pi k x\,dx$, $k \in \mathbb{N}_0$. *Then*

$$\|a\|_{\ell_{p,q}} \sim \left(\int_0^1 \left(t^{1/p'}\overline{f}(t)\right)^q \frac{dt}{t}\right)^{1/q},$$

where

$$\overline{f}(t) = \sup_{y\ge t} \frac{1}{y}\left|\int_0^y f(s)\,ds\right|.$$

REMARK 1.7. From assertion 1 in [17] it follows that there exists $C > 0$ such that

$$\left(\int_0^1 \left(t^{1/p}\overline{f}(t)\right)^q \frac{dt}{t}\right)^{1/q} \le C\|f\|_{L_{p'q}}.$$

REMARK 1.8. If $p < 1/\alpha$, then Theorem 1.6 does not hold in general. A counterexample is given in Section 5.

REMARK 1.9. If $\alpha = 1$, then the results of Theorem 1.6 coincide with the results of Theorem C for non-increasing, absolutely continuous functions.

Indeed, if $f(x)$ is a non-increasing function, then $f^*(t) = f(t)$ a.e. on $[0,1]$. Hence,

$$\overline{f}(t) = \sup_{y\ge t}\frac{1}{y}\left|\int_0^y f(s)\,ds\right| \ge \frac{1}{t}\int_0^t f(s)\,ds \ge f(t) = f^*(t).$$

REMARK 1.10. The counterpart of Theorem 1.6 for trigonometric series with α-monotone coefficients was proved in [11].

The paper is organized as follows. In Section 2 we introduce the net-type spaces $N_{p,q;\alpha}$ and study their properties. Section 3 provides some properties of the Fourier coefficients of functions from the net-type spaces. These properties are used in Section 4 to prove Theorem 1.6. Finally, in Section 5 we construct a counterexample to Theorem 1.6 in the case $p < 1/\alpha$.

2. The net-type spaces

DEFINITION 2.1. Let $\alpha \in (0,1]$, $0 < p, q \le +\infty$ and μ be the Lebesgue measure on $[0,1]$. We will say that a μ-measurable function $f \in N_{p,q;\alpha}$, if

$$\|f\|_{N_{p,q;\alpha}} := \begin{cases} \left(\int_0^1 \left(t^{1/p}\widetilde{f}(t,\alpha)\right)^q \frac{dt}{t}\right)^{1/q}, & \text{for } 0<p<\infty \text{ and } 0<q<\infty, \\ \sup_t t^{1/p}\widetilde{f}(t,\alpha), & \text{for } 0<p\le\infty,\ q=\infty, \end{cases}$$

is finite, where

$$\widetilde{f}(t,\alpha) = \begin{cases} \sup_{y\ge t} \frac{1}{y^\alpha}\left|\int_0^y \frac{f(s)}{(y-s)^{1-\alpha}}\,ds\right|, & \text{for } t\in[0,1], \\ 0, & \text{for } t\notin[0,1]. \end{cases}$$

Note that if $\alpha = 1$, then these spaces coincide with the net spaces $N_{p,q}(M)$ in [17] (see also [16]), where $M = \{[y,1] : [y,1] \subseteq [0,1]\}$. The properties of the net spaces were studied in detail in [15]. Let us present several important properties of the $N_{p,q,\alpha}$ spaces.

LEMMA 2.2. *We have*

(i) $N_{p,q_1;\alpha} \hookrightarrow N_{p,q_2;\alpha}$, *for* $q_1 \le q_2$;

(ii) $N_{p_2,q_2;\alpha} \hookrightarrow N_{p_1,q_1;\alpha}$, *for* $p_1 < p_2$, $0 < q_1, q_2 \le +\infty$.

PROOF. (i) Let us consider two cases.

Case 1: $q_2 = +\infty$. Suppose $f \in N_{p,q_1;\alpha}$. Then using monotonicity of the function $\widetilde{f}(t,\alpha)$, we get

$$\begin{aligned} \|f\|_{N_{p,\infty;\alpha}} &= \sup_{t\in[0,1]} t^{1/p}\widetilde{f}(t,\alpha) = \sup_{t\in[0,1]} \left(\frac{p}{q_1}\int_0^t s^{q_1/p-1}ds\right)^{1/q_1}\widetilde{f}(t,\alpha) \\ &= C\sup_{t\in[0,1]}\left(\int_0^t \left(\widetilde{f}(t,\alpha)\right)^{q_1} s^{q_1/p-1}ds\right)^{1/q_1} \\ &\le C\sup_{t\in[0,1]}\left(\int_0^t \left(\widetilde{f}(s,\alpha)\right)^{q_1} s^{q_1/p-1}ds\right)^{1/q_1} \\ &\le C\left(\int_0^1 \left(\widetilde{f}(s,\alpha)s^{1/p}\right)^{q_1}\frac{ds}{s}\right)^{1/q_1} = C\|f\|_{N_{p,q_1;\alpha}}. \end{aligned}$$

Case 2: $q_2 < +\infty$. Let $f \in N_{p,q_1;\alpha}$. Then

$$\|f\|^{q_2}_{N_{p,q_2;\alpha}} = \int_0^1 \big(s^{1/p}\widetilde{f}(s,\alpha)\big)^{q_2}\frac{ds}{s}$$
$$= \int_0^1 \big(s^{1/p}\widetilde{f}(s,\alpha)\big)^{q_2-q_1}\big(s^{1/p}\widetilde{f}(s,\alpha)\big)^{q_1}\frac{ds}{s}.$$

Hence, from the first case we obtain

$$\|f\|^{q_2}_{N_{p,q_2;\alpha}} \le \int_0^1 \big(\sup_{s\in[0,1]} s^{1/p}\widetilde{f}(s,\alpha)\big)^{q_2-q_1}\big(s^{1/p}\widetilde{f}(s,\alpha)\big)^{q_1}\frac{ds}{s}$$
$$= \|f\|^{q_2-q_1}_{N_{p,\infty;\alpha}}\int_0^1 \big(s^{1/p}\widetilde{f}(s,\alpha)\big)^{q_1}\frac{ds}{s} = \|f\|^{q_2-q_1}_{N_{p,\infty;\alpha}}\|f\|^{q_1}_{N_{p,q_1;\alpha}}$$
$$\le C\|f\|^{q_2-q_1}_{N_{p,q_1;\alpha}}\|f\|^{q_1}_{N_{p,q_1;\alpha}} = C\|f\|^{q_2}_{N_{p,q_1;\alpha}}.$$

(ii) By property (i), it is enough to prove the following embedding

$$N_{p_2,\infty;\alpha} \hookrightarrow N_{p_1,q_1;\alpha}.$$

Let $f \in N_{p_2,\infty;\alpha}$, i.e., $\sup_t t^{1/p_2}\widetilde{f}(t,\alpha) < \infty$. Then

$$\|f\|^{q_1}_{N_{p_1,q_1;\alpha}} = \int_0^1 \big(s^{1/p_1}\widetilde{f}(s,\alpha)\big)^{q_1}\frac{ds}{s} = \int_0^1 \big(s^{1/p_2}\widetilde{f}(s,\alpha)\big)^{q_1}s^{(1/p_1-1/p_2)q_1}\frac{ds}{s}$$
$$\le \int_0^1 \big(\sup_{s\in[0,1]} s^{1/p_2}\widetilde{f}(s,\alpha)\big)^{q_1}s^{(1/p_1-1/p_2)q_1}\frac{ds}{s}$$
$$= \|f\|^{q_1}_{N_{p_2,\infty;\alpha}}\int_0^1 s^{(1/p_1-1/p_2)q_1}\frac{ds}{s} = C\|f\|^{q_1}_{N_{p_2\infty;\alpha}}.$$

LEMMA 2.3. *Let* $\alpha \in (0,1]$ *and* $0 < p, q, h < +\infty$. *Then*

$$\|f\|_{N_{p,q;\alpha}} \sim \left(\sum_{k=0}^\infty \big(2^{-k/p}\widetilde{f}(2^{-k},\alpha)\big)^q\right)^{1/q}$$
$$\sim \left(\sum_{k=0}^\infty\left(\int_{2^{-(k+1)}}^{2^{-k}} t^{h/p-1}\big(\widetilde{f}(t,\alpha)\big)^h dt\right)^{q/h}\right)^{1/q}.$$

PROOF. We have

$$\|f\|_{N_{p,q;\alpha}} = \left(\int_0^1 \big(t^{1/p}\widetilde{f}(t,\alpha)\big)^q\frac{dt}{t}\right)^{1/q} = \left(\sum_{k=0}^\infty\int_{2^{-(k+1)}}^{2^{-k}}\big(t^{1/p}\widetilde{f}(t,\alpha)\big)^q\frac{dt}{t}\right)^{1/q}.$$

The first equivalence follows from

$$\big(2^{-k/p}\widetilde{f}(2^{-k},\alpha)\big)^q 2^{-q/p}\ln 2 \le \int_{2^{-k-1}}^{2^{-k}} \big(t^{1/p}\widetilde{f}(t,\alpha)\big)^q \frac{dt}{t}$$
$$\le \big(2^{-(k+1)/p}\widetilde{f}(2^{-(k+1)},\alpha)\big)^q 2^{q/p}\ln 2$$

and the second one from

$$(\ln 2)^{1/h} 2^{-(k+1)/p}\widetilde{f}(2^{-k},\alpha) \le \left(\int_{2^{-(k+1)}}^{2^{-k}} t^{h/p-1}\big(\widetilde{f}(t,\alpha)\big)^h dt\right)^{1/h}$$
$$\le (\ln 2)^{1/h} 2^{-k/p}\widetilde{f}(2^{-(k+1)},\alpha).$$

Now we will study the interpolation between the $N_{p,q;\alpha}$ spaces. Let (A_0, A_1) be a compatible pair of quasi-normed spaces and

$$K(t,a) = K(t,a;A_0,A_1) = \inf_{a=a_0+a_1}(\|a_0\|_{A_0} + t\|a_1\|_{A_1}), \quad a \in A_0 + A_1$$

be the Peetre K-functional ([2]).

The space $(A_0, A_1)_{\theta,q}$, $0 < \theta < 1$, consists of all elements $a \in A_0 + A_1$ for which the functional

$$\|a\|_{(A_0,A_1)_{\theta,q}} = \begin{cases} \left(\displaystyle\int_0^\infty \big(t^{-\theta}K(t,a)\big)^q \frac{dt}{t}\right)^{1/q}, & \text{for } 0 < q < \infty, \\ \displaystyle\sup_{0<t<\infty} t^{-\theta}K(t,a), & \text{for } q = \infty, \end{cases}$$

is finite.

THEOREM 2.4. *Let* $\alpha \in (0,1]$, $0 < p_0 < p_1 < \infty$, $0 < q_0, q_1, q \le \infty$, *and* $0 < \theta < 1$. *Then*

$$(N_{p_0,q_0;\alpha}, N_{p_1,q_1;\alpha})_{\theta,q} \hookrightarrow N_{p,q;\alpha}, \quad \textit{where } \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

PROOF. By the embeddings $N_{p_i,q_i;\alpha} \hookrightarrow N_{p_i,\infty;\alpha}$, $i = 0, 1$ (see part (ii) of Lemma 2.2) it is enough to prove

$$(N_{p_0,\infty;\alpha}, N_{p_1,\infty;\alpha})_{\theta,q} \hookrightarrow N_{p,q;\alpha}.$$

Let $t \in (0,\infty)$, $s \in [0,1]$, $f \in (N_{p_0,\infty;\alpha}, N_{p_1,\infty;\alpha})_{\theta,q}$ such that $f = f_0 + f_1$ is any decomposition with $f_i \in N_{p_i,\infty;\alpha}$, $(i = 0,1)$. Then

$$\begin{aligned}
\widetilde{f}(s,\alpha) &= \sup_{y\ge s} \frac{1}{y^\alpha}\left|\int_0^y \frac{f(t)}{(y-t)^{1-\alpha}}\,dt\right| = \sup_{y\ge s} \frac{1}{y^\alpha}\left|\int_0^y \frac{f_0(t)+f_1(t)}{(y-t)^{1-\alpha}}\,dt\right| \\
&\le \sup_{y\ge s} \frac{1}{y^\alpha}\left|\int_0^y \frac{f_0(t)}{(y-t)^{1-\alpha}}\,dt\right| + \sup_{y\ge s} \frac{1}{y^\alpha}\left|\int_0^y \frac{f_1(t)}{(y-t)^{1-\alpha}}\,dt\right| \\
&= \widetilde{f_0}(s,\alpha) + \widetilde{f_1}(s,\alpha).
\end{aligned}$$

Denote $v(t) = t^{p_1p_0/(p_1-p_0)}$, $t \in (0,\infty)$. We have

$$\begin{aligned}
\sup_{v(t)\ge s} s^{1/p_0}\widetilde{f}(s,\alpha) &\le \sup_{v(t)\ge s} s^{1/p_0}\widetilde{f_0}(s,\alpha) + \sup_{v(t)\ge s} s^{1/p_0}\widetilde{f_1}(s,\alpha) \\
&= \sup_{v(t)\ge s} s^{1/p_0}\widetilde{f_0}(s,\alpha) + \sup_{v(t)\ge s} s^{1/p_0-1/p_1}\widetilde{f_1}(s,\alpha)s^{1/p_1} \\
&\le \sup_{v(t)\ge s} s^{1/p_0}\widetilde{f_0}(s,\alpha) + \sup_{v(t)\ge s} (v(t))^{1/p_0-1/p_1}\widetilde{f_1}(s,\alpha)s^{1/p_1} \\
&= \sup_{v(t)\ge s} s^{1/p_0}\widetilde{f_0}(s,\alpha) + t\sup_{v(t)\ge s} s^{1/p_1}\widetilde{f_1}(s,\alpha) \\
&\le \sup_{s\in[0,1]} s^{1/p_0}\widetilde{f_0}(s,\alpha) + t\sup_{s\in[0,1]} s^{1/p_1}\widetilde{f_1}(s,\alpha).
\end{aligned}$$

Hence,

$$\sup_{v(t)\ge s} s^{1/p_0}\widetilde{f}(s,\alpha) \le K\big(t, f; N_{p_0,\infty;\alpha}, N_{p_1,\infty;\alpha}\big).$$

Thus, for $0 < q \le \infty$ we get

$$\begin{aligned}
\|f\|_{\theta,q}^q := \|f\|^q_{(N_{p_0,\infty;\alpha},N_{p_1,\infty;\alpha})_{\theta,q}} &= \int_0^\infty \big(t^{-\theta}K(t, f; N_{p_0,\infty;\alpha}, N_{p_1,\infty;\alpha})\big)^q\frac{dt}{t} \\
&\ge \int_0^\infty \Big(t^{-\theta}\sup_{v(t)\ge s} s^{1/p_0}\widetilde{f}(s,\alpha)\Big)^q\frac{dt}{t} \\
&= C\int_0^\infty \Big(y^{-\theta(p_1-p_0)/(p_1p_0)}\sup_{y\ge s\ge 0} s^{1/p_0}\widetilde{f}(s,\alpha)\Big)^q\frac{dy}{y} \\
&\ge C\int_0^1 \Big(y^{-\theta(p_1-p_0)/(p_1p_0)}\sup_{y\ge s\ge 0} s^{1/p_0}\widetilde{f}(s,\alpha)\Big)^q\frac{dy}{y} \\
&= C\sum_{r=0}^\infty \int_{2^{-(r+1)}}^{2^{-r}} \Big(y^{-\theta(p_1-p_0)/(p_1p_0)}\sup_{y\ge s\ge 0} s^{1/p_0}\widetilde{f}(s,\alpha)\Big)^q\frac{dy}{y}.
\end{aligned}$$

By the monotonicity of the integrand we get

$$\begin{aligned}\|f\|_{\theta,q}^q &\geq C\sum_{r=0}^{\infty}\Big(2^{r\theta(1/p_0-1/p_1)}\sup_{2^{-(r+1)}\geq s\geq 0}s^{1/p_0}\widetilde{f}(s,\alpha)\Big)^q\int_{2^{-(r+1)}}^{2^{-r}}\frac{dy}{y}\\ &= C\sum_{r=0}^{\infty}\Big(2^{r\theta(1/p_0-1/p_1)}\sup_{2^{-(r+1)}\geq s\geq 0}s^{1/p_0}\widetilde{f}(s,\alpha)\Big)^q\\ &= C\sum_{r=0}^{\infty}\Big(2^{-r\left(\frac{(1-\theta)}{p_0}+\frac{\theta}{p_1}\right)}2^{\frac{r}{p_0}}\sup_{2^{-(r+1)}\geq s\geq 0}s^{1/p_0}\widetilde{f}(s,\alpha)\Big)^q.\end{aligned}$$

Hence,

$$\begin{aligned}\|f\|_{\theta,q}^q &\geq C\sum_{r=0}^{\infty}\big(2^{-r/p}2^{r/p_0}2^{-(r+1)/p_0}\widetilde{f}(2^{-(r+1)},\alpha)\big)^q\\ &= C\sum_{r=0}^{\infty}\big(2^{-r/p}2^{-1/p_0}\widetilde{f}(2^{-(r+1)},\alpha)\big)^q\\ &\geq C\sum_{r=0}^{\infty}\big(2^{-r/p}2^{-1/p_0}\widetilde{f}(2^{-r},\alpha)\big)^q\\ &= C\sum_{r=0}^{\infty}\big(2^{-r/p}\widetilde{f}(2^{-r},\alpha)\big)^q\geq C\|f\|_{N_{p,q,\alpha}}^q.\end{aligned}$$

This completes the proof of Theorem 2.4.

3. The net-type spaces and Fourier coefficients

We start with two auxiliary results.

LEMMA 3.1 (Hardy-Littlewood's inequality). *[1, p. 44] Suppose that f and g are μ-measurable functions on* $[0,1]$. *Then*

$$\int_0^1|f(x)g(x)|\,dx\leq\int_0^1 f^*(t)g^*(t)\,dt. \tag{3.1}$$

LEMMA 3.2. *Let* $\alpha\in(0,1]$, $y\in[0,1]$ *and* $k\in\mathbb{N}_0$. *Then*

$$\left|\int_0^y\frac{e^{\pi iks}}{(y-s)^{1-\alpha}}\,ds\right|\leq C\min\Big(y^\alpha,\frac{1}{\widetilde{k}^\alpha}\Big),$$

where

$$\widetilde{k}=\begin{cases}k, & \text{for } k\neq 0,\\ 1, & \text{for } k=0.\end{cases}$$

PROOF. If $k = 0$ then Lemma 3.2 is obvious. Let us consider a $k > 0$. Then by elementary calculations, we get

$$\left|\int_0^y \frac{e^{\pi iks}}{(y-s)^{1-\alpha}}\,ds\right| \le \int_0^y \frac{1}{(y-s)^{1-\alpha}}ds = \frac{y^\alpha}{\alpha}.$$

On the other hand

$$\left|\int_0^y \frac{e^{\pi iks}}{(y-s)^{1-\alpha}}\,ds\right| = \left|\int_0^{ky} \frac{e^{\pi it}}{(y-\frac{t}{k})^{1-\alpha}}\,\frac{dt}{k}\right| = \frac{1}{k^\alpha}\left|\int_0^{ky} \frac{e^{\pi it}}{(ky-t)^{1-\alpha}}\,dt\right|$$
$$= \frac{1}{k^\alpha}\left|\int_0^{ky} \frac{e^{\pi i(ky-z)}}{z^{1-\alpha}}\,dz\right| = \frac{1}{k^\alpha}\left|\int_0^{ky} \frac{e^{-\pi iz}}{z^{1-\alpha}}\,dz\right|.$$

Let us consider the function $\varphi(t) = \int_0^t \frac{e^{-\pi iz}}{z^{1-\alpha}}\,dz$. By the Dirichlet test the integral $\int_0^\infty \frac{e^{-\pi iz}}{z^{1-\alpha}}dz$ converges to some finite A. Hence, there exists $N > 0$ such that $|\varphi(t) - A| < 1$ for all $t > N$, and thus $|\varphi(t)| < |A| + 1$ for the same t. Note that the function $\varphi(t)$ is bounded on $[0, N]$, i.e., there exists $M > 0$ such that $|\varphi(t)| \le M$ for all $t \in [0, N]$. Hence, $|\varphi(t)| \le \max(M, |A| + 1) = C$. Thus, we have

$$\frac{1}{k^\alpha}\left|\int_0^{ky} \frac{e^{-\pi iz}}{z^{1-\alpha}}\,dz\right| \le C\frac{1}{k^{1-\alpha}}.$$

COROLLARY 3.3. *Let $\alpha \in (0, 1]$, $y \in [0, 1]$, $k \in \mathbb{N}_0$. Then*

$$\left|\int_0^y \frac{\cos \pi ks}{(y-s)^{1-\alpha}}\,ds\right| \le C \min\left(y^\alpha, \frac{1}{\widetilde{k}^\alpha}\right).$$

THEOREM 3.4. *Let $\alpha \in (0, 1]$, $p \in (1, 1/(1-\alpha)]$ and $0 < q \le \infty$. Also, let f be an integrable function on $[0, 1]$ with series expansion $\sum_{k\in\mathbb{N}_0} a_k e^{\pi ikx}$. Then, the following inequality*

$$\|f\|_{N_{p',q;\alpha}} \le C\|a\|_{\ell_{p,q}} \tag{3.2}$$

holds.

REMARK 3.5. Note that from the decomposition $\cos \pi kx = \frac{1}{2}(e^{\pi ikx} + e^{-\pi ikx})$ it follows that Theorem 3.4 is true for the cosine-series.

REMARK 3.6. In [16], Nursultanov obtained an inequality similar to (3.2) for the net spaces, $N_{p,q}$.

PROOF. We have

$$\begin{aligned}
\|f\|_{N_{p',\infty;\alpha}} &= \sup_{t\in[0,1]} t^{1/p'} \widetilde{f}(t,\alpha) = \sup_{t\in[0,1]} t^{1/p'} \sup_{t\le y\le 1} \frac{1}{y^\alpha}\left|\int_0^y \frac{f(s)}{(y-s)^{1-\alpha}}\,ds\right| \\
&\le \sup_{t\in[0,1]} \sup_{t\le y\le 1} \frac{1}{y^{\alpha-1/p'}}\left|\int_0^y \frac{f(s)}{(y-s)^{1-\alpha}}\,ds\right| \\
&= \sup_{y\in[0,1]} \frac{1}{y^{\alpha-1/p'}}\left|\int_0^y \frac{f(s)}{(y-s)^{1-\alpha}}\,ds\right| \\
&= \sup_{y\in[0,1]} \frac{1}{y^{\alpha-1/p'}}\left|\int_0^y \frac{\sum_{k=0}^\infty a_k e^{\pi iks}}{(y-s)^{1-\alpha}}\,ds\right|
\end{aligned}$$

Using Lemma 3.2 we get

$$\begin{aligned}
\|f\|_{N_{p',\infty;\alpha}} &\le \sup_{y\in[0,1]} \frac{1}{y^{\alpha-1/p'}} \sum_{k=0}^\infty |a_k| \left|\int_0^y \frac{e^{\pi iks}}{(y-s)^{1-\alpha}}\,ds\right| \\
&\le C \sup_{y\in[0,1]} \frac{1}{y^{\alpha-1/p'}} \sum_{k=0}^\infty |a_k| \min\left(y^\alpha, \frac{1}{\widetilde{k}^\alpha}\right).
\end{aligned}$$

Let $y \in [0,1]$ and $1/k < y$. From $p \le 1/(1-\alpha)$ it follows that $\alpha \ge 1/p'$. Then

$$\begin{aligned}
\frac{1}{y^{\alpha-1/p'}}|a_k| \min\left(y^\alpha, \frac{1}{\widetilde{k}^\alpha}\right) &= \frac{1}{y^{\alpha-1/p'}}|a_k|\frac{1}{k^\alpha} = \frac{|a_k|}{(ky)^{\alpha-1/p'}}\frac{1}{k^{1/p'}} \\
&\le |a_k|k^{-1/p'} = |a_k|k^{1/p-1}.
\end{aligned}$$

Now let $1/k \ge y$ and $k \ne 0$. We obtain

$$\frac{1}{y^{\alpha-1/p'}}|a_k| \min\left(y^\alpha, \frac{1}{\widetilde{k}^\alpha}\right) = \frac{1}{y^{\alpha-1/p'}}|a_k|y^\alpha = |a_k|y^{1/p'} \le |a_k|k^{-1/p'}.$$

Let $k = 0$. Then

$$\frac{1}{y^{\alpha-1/p'}}|a_k| \min\left(y^\alpha, \frac{1}{\widetilde{k}^\alpha}\right) = \frac{1}{y^{\alpha-1/p'}}|a_k|y^\alpha = |a_0|y^{1/p'} \le |a_0|.$$

Now by the Hardy-Littlewood inequality (3.1) we get

$$\|f\|_{N_{p',\infty;\alpha}} \le C\left(|a_0| + \sum_{k=1}^\infty |a_k|k^{1/p-1}\right) \le C\|a\|_{\ell_{p1}}.$$

Finally, using the interpolation theorem for the $\ell_{p,q}$ spaces (see [2], p. 113) and Theorem 2.4 we obtain inequality (3.2).

THEOREM 3.7. *Let* $0 < \alpha \leq 1$, $1/\alpha < p < \infty$, $1 \leq q \leq \infty$ *and* f *be an integrable function with the trigonometric series* $\sum_{k=0}^{\infty} a_k \cos \pi k x$ *such that* $f \in M_\alpha$. *If*

$$\left(\sum_{k=0}^{\infty}\left[2^{(k+1)/p}\int_{2^{-(k+1)}}^{2^{-k}} t^{\alpha}\left|mathcal D^{\alpha} f(t)\right| dt\right]^{q}\right)^{1/q} + f_{1-\alpha}(1) \leq B < \infty,$$

then the following inequality holds

$$\|a\|_{\ell_{pq}} \leq C(p,q,\alpha)B.$$

PROOF. Let $N \in \mathbb{N}$, consider the sequence $a^N = \{a_k^N\}_{k=0}^{\infty}$, where

$$a_k^N = \begin{cases} a_k, & \text{for } 0 \leq k \leq N, \\ 0, & \text{for } k > N. \end{cases}$$

Using the dual representation of the norm in the space ℓ_{pq}

$$\|a^N\|_{\ell_{pq}} \leq C \sup_{\|b\|_{\ell_{p'q'}}=1} \sum_{k=0}^{N} a_k^N b_k = C \sup_{\|b\|_{\ell_{p'q'}}=1} \int_0^1 f(x)g(x)\,dx,$$

where $g(x) = \sum_{k=0}^{N} b_k \cos \pi k x$.

Using Lemma 1.5,

$$\int_0^1 f(x)g(x)\,dx = \int_0^1 \left[I^{\alpha}\mathscr{D}^{\alpha} f(x) + \frac{f_{1-\alpha}(1)}{\Gamma(\alpha)(1-x)^{1-\alpha}}\right] g(x)\,dx = I_1 + I_2.$$

For I_1 we have

$$\begin{aligned} I_1 &= \frac{1}{\Gamma(\alpha)} \int_0^1 \int_x^1 \frac{g(x)}{(t-x)^{1-\alpha}} (\mathscr{D}^{\alpha} f)(t)\,dt\,dx \\ &= \frac{1}{\Gamma(\alpha)} \int_0^1 \left[\int_0^t \frac{g(x)}{(t-x)^{1-\alpha}} (\mathscr{D}^{\alpha} f)(t)\,dx\right] dt \\ &\leq C \int_0^1 \left|\int_0^t \frac{g(x)}{(t-x)^{1-\alpha}}\,dx\right| \left|(\mathscr{D}^{\alpha} f)(t)\right| dt \\ &= C \sum_{k=0}^{\infty} \int_{2^{-(k+1)}}^{2^{-k}} \left|\int_0^t \frac{g(x)}{(t-x)^{1-\alpha}}\,dx\right| \left|(\mathscr{D}^{\alpha} f)(t)\right| dt \end{aligned}$$

$$= C\sum_{k=0}^{\infty}\int_{2^{-(k+1)}}^{2^{-k}}\frac{1}{t^{\alpha}}\left|\int_0^t\frac{g(x)}{(t-x)^{1-\alpha}}\,dx\right|t^{\alpha}\left|(\mathscr{D}^{\alpha}f)(t)\right|dt$$

$$\le C\sum_{k=0}^{\infty}\sup_{t\ge 2^{-(k+1)}}\frac{1}{t^{\alpha}}\left|\int_0^t\frac{g(x)}{(t-x)^{1-\alpha}}\,dx\right|\int_{2^{-(k+1)}}^{2^{-k}}t^{\alpha}\left|(\mathscr{D}^{\alpha}f)(t)\right|dt$$

$$= C\sum_{k=0}^{\infty}\widetilde{g}(2^{-(k+1)},\alpha)\int_{2^{-(k+1)}}^{2^{-k}}t^{\alpha}\left|(\mathscr{D}^{\alpha}f)(t)\right|dt$$

$$= C\sum_{k=0}^{\infty}2^{-(k+1)/p}\widetilde{g}(2^{-(k+1)},\alpha)2^{(k+1)/p}\int_{2^{-(k+1)}}^{2^{-k}}t^{\alpha}\left|(\mathscr{D}^{\alpha}f)(t)\right|dt.$$

By the Hölder inequality, we get

$$I_1 \le C\left(\sum_{k=0}^{\infty}\left[2^{-(k+1)/p}\widetilde{g}(2^{-(k+1)},\alpha)\right]^{q'}\right)^{1/q'} \times\left(\sum_{k=0}^{\infty}\left[2^{(k+1)/p}\int_{2^{-(k+1)}}^{2^{-k}}t^{\alpha}\left|(\mathscr{D}^{\alpha}f)(t)\right|dt\right]^{q}\right)^{1/q}$$

$$\le C\|g\|_{N_{p,q';\alpha}}\left(\sum_{k=0}^{\infty}\left[2^{(k+1)/p}\int_{2^{-(k+1)}}^{2^{-k}}t^{\alpha}\left|(\mathscr{D}^{\alpha}f)(t)\right|dt\right]^{q}\right)^{1/q}$$

$$\le C\|b\|_{\ell_{p'q'}}\left(\sum_{k=0}^{\infty}\left[2^{(k+1)/p}\int_{2^{-(k+1)}}^{2^{-k}}t^{\alpha}\left|(\mathscr{D}^{\alpha}f)(t)\right|dt\right]^{q}\right)^{1/q}$$

$$\le C\left(\sum_{k=0}^{\infty}\left[2^{\frac{(k+1)}{p}}\int_{2^{-(k+1)}}^{2^{-k}}t^{\alpha}\left|(\mathscr{D}^{\alpha}f)(t)\right|dt\right]^{q}\right)^{1/q}.$$

For I_2 we have

$$|I_2| = \frac{f_{1-\alpha}(1)}{\Gamma(\alpha)}\left|\int_0^1\frac{g(x)}{(1-x)^{1-\alpha}}\,dx\right| = \frac{f_{1-\alpha}(1)}{\Gamma(\alpha)}\left|\sum_{k=0}^{N}b_k\int_0^1\frac{\cos\pi kx}{(1-x)^{1-\alpha}}\,dx\right|.$$

Using Corollary 3.3 and the Hardy-Littlewood inequality (3.1),

$$|I_2| \le \frac{f_{1-\alpha}(1)}{\Gamma(\alpha)}\sum_{k=0}^{\infty}|b_k|\left|\int_0^1\frac{\cos\pi kx}{(1-x)^{1-\alpha}}\,dx\right| \le Cf_{1-\alpha}(1)\sum_{k=0}^{\infty}|b_k|\widetilde{k}^{-\alpha}$$

$$\le Cf_{1-\alpha}(1)\left(b_0^{*}+\sum_{k=1}^{\infty}b_k^{*}k^{-\alpha}\right) = Cf_{1-\alpha}(1)\sum_{k=0}^{\infty}b_k^{*}k^{1/p'-1/q'}k^{1/p-1/q}k^{-\alpha}$$

$$\leq Cf_{1-\alpha}(1)\left(\sum_{k=0}^{\infty}\big(b_k^* k^{1/p'-1/q'}\big)^{q'}\right)^{1/q'}\left(\sum_{k=0}^{\infty}\big(k^{1/p-1/q}k^{-\alpha}\big)^{q}\right)^{1/q}$$

$$= Cf_{1-\alpha}(1)\|b\|_{\ell_{p'q'}}\left(\sum_{k=0}^{\infty}k^{(1/p-\alpha)q-1}\right)^{1/q} \leq Cf_{1-\alpha}(1).$$

4. Proof of Theorem 1.6

Let $\alpha = 1$ and $a = \{a_k\} \in \ell_{pq}$, then

$$\left(\int_0^1 \big(t^{1/p'}\widetilde{f}(t,1)\big)^q \frac{dt}{t}\right)^{1/q} \leq C\|a\|_{\ell_{pq}}.$$

Hence,

$$\left(\int_0^1 \big(t^{1/p'}\overline{f}(t)\big)^q \frac{dt}{t}\right)^{1/q} \leq C\|a\|_{\ell_{pq}}.$$

Now we show the reverse inequality. Let $1/\alpha < p < \infty$, $f \in M_\alpha$, and assume that the integral $\int_0^1 \big(t^{1/p'}\overline{f}(t)\big)^q \frac{dt}{t}$ converges. Then, Lemma 2.3 implies the convergence of the series

$$\sum_{k=1}^{\infty}\big(2^{-k/p'}\widetilde{f}(2^{-k},1)\big)^q.$$

Therefore,

$$\begin{aligned}
&\widetilde{f}(2^{-k},1)\\
&\quad\geq 2^k\int_0^{2^{-k}} f(s)\,ds\\
&\quad= \frac{2^k}{\Gamma(\alpha)}\int_0^{2^{-k}}\int_s^1 \frac{(\mathscr{D}^\alpha f)(t)}{(t-s)^{1-\alpha}}\,dt\,ds + 2^k\frac{f_{1-\alpha}(1)}{\Gamma(\alpha)}\int_0^{2^{-k}}(1-x)^{\alpha-1}\,dx\\
&\quad= \frac{2^k}{\Gamma(\alpha)}\int_0^{2^{-k}}\int_s^{2^{-k}} \frac{(\mathscr{D}^\alpha f)(t)}{(t-s)^{1-\alpha}}\,dt\,ds\\
&\qquad+ \frac{2^k}{\Gamma(\alpha)}\int_0^{2^{-k}}\int_{2^{-k}}^1 \frac{(\mathscr{D}^\alpha f)(t)}{(t-s)^{1-\alpha}}\,dt\,ds + 2^k\frac{f_{1-\alpha}(1)}{\alpha\Gamma(\alpha)}\big(1-(1-2^{-k})^\alpha\big).
\end{aligned}$$

Using the non-negativity of the $\mathscr{D}^\alpha f(x)$ we have

$$\begin{aligned}
\widetilde{f}(2^{-k},1) &\geq \frac{2^k}{\Gamma(\alpha)}\int_0^{2^{-k}}\int_s^{2^{-k}}\frac{(\mathscr{D}^\alpha f)(t)}{(t-s)^{1-\alpha}}\,dt\,ds + 2^k\frac{f_{1-\alpha}(1)}{\alpha\Gamma(\alpha)}2^{-\alpha k}\\
&= \frac{2^k}{\Gamma(\alpha)}\int_0^{2^{-k}}\int_0^{t}\frac{(\mathscr{D}^\alpha f)(t)}{(t-s)^{1-\alpha}}\,ds\,dt + \frac{f_{1-\alpha}(1)}{\alpha\Gamma(\alpha)}2^{(1-\alpha)k}\\
&= \frac{2^k}{\Gamma(\alpha)\alpha}\int_0^{2^{-k}} t^\alpha(\mathscr{D}^\alpha f)(t)\,dt + \frac{f_{1-\alpha}(1)}{\alpha\Gamma(\alpha)}2^{(1-\alpha)k}\\
&= \frac{2^k}{\alpha\Gamma(\alpha)}\int_{2^{-(k+1)}}^{2^{-k}} t^\alpha\big|(\mathscr{D}^\alpha f)(t)\big|\,dt + \frac{f_{1-\alpha}(1)}{\alpha\Gamma(\alpha)}2^{(1-\alpha)k}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&I(f)\\
&:=\Big(\sum_{k=0}^\infty\big(2^{-k/p'}\widetilde{f}(2^{-k},1)\big)^q\Big)^{1/q}\\
&\geq\Bigg(\sum_{k=0}^\infty\bigg(2^{-k/p'}\bigg(\frac{2^k}{\alpha\Gamma(\alpha)}\int_{2^{-(k+1)}}^{2^{-k}} t^\alpha\big|(\mathscr{D}^\alpha f)(t)\big|\,dt + \frac{f_{1-\alpha}(1)}{\alpha\Gamma(\alpha)}2^{(1-\alpha)k}\bigg)\bigg)^q\Bigg)^{1/q}\\
&\geq C\Bigg(\sum_{k=0}^\infty\bigg(\frac{2^{k/p}}{\alpha\Gamma(\alpha)}\int_{2^{-(k+1)}}^{2^{-k}} t^\alpha\big|(\mathscr{D}^\alpha f)(t)\big|\,dt\bigg)^q + \sum_{k=0}^\infty\bigg(\frac{f_{1-\alpha}(1)}{\alpha\Gamma(\alpha)}2^{(1/p-\alpha)k}\bigg)^q\Bigg)^{1/q}\\
&\geq C\Bigg(\sum_{k=0}^\infty\bigg(2^{k/p}\int_{2^{-(k+1)}}^{2^{-k}} t^\alpha\big|(\mathscr{D}^\alpha f)(t)\big|\,dt\bigg)^q\Bigg)^{1/q} + C\Big(\sum_{k=0}^\infty\big(f_{1-\alpha}(1)2^{(1/p-\alpha)k}\big)^q\Big)^{1/q}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
I(f) &\geq C\Bigg(\sum_{k=0}^\infty\bigg(2^{k/p}\int_{2^{-(k+1)}}^{2^{-k}} t^\alpha\big|(\mathscr{D}^\alpha f)(t)\big|\,dt\bigg)^q\Bigg)^{1/q}\\
&\quad + C\Big(\sum_{k=0}^\infty\big(f_{1-\alpha}(1)2^{(1/p-\alpha)k}\big)^q\Big)^{1/q}\\
&\geq C\Bigg(\Bigg(\sum_{k=0}^\infty\bigg[2^{(k+1)/p}\int_{2^{-(k+1)}}^{2^{-k}} t^\alpha\big|(\mathscr{D}^\alpha f)(t)\big|\,dt\bigg]^q\Bigg)^{1/q} + f_{1-\alpha}(1)\Bigg).
\end{aligned}$$

Now Theorem 3.7 implies the reverse inequality.

5. A counterexample in the case $p < 1/\alpha$

LEMMA 5.1. *Let* $\beta \in (0, 1)$, $y \in (0, 1)$, *and*

$$a_n = \int_0^y \frac{\cos \pi n x}{x^\beta}\, dx, \quad b_n = \int_0^y \frac{\sin \pi n x}{x^\beta}\, dx$$

for $n \in \mathbb{N}$. *Then*

$$a_n \sim \frac{C_1(\beta)}{n^{1-\beta}}, \;\; b_n \sim \frac{C_2(\beta)}{n^{1-\beta}} \quad \text{as } n \to +\infty,$$

where $C_1(\beta), C_2(\beta) \neq 0$.

PROOF. We prove the statement of the lemma for the sequence a_n. The proof for the sequence b_n is similar. Substituting nx for t we have,

$$a_n = n^{\beta-1} \int_0^{ny} \frac{\cos \pi t}{t^\beta}\, dt.$$

Using the well-known formula

$$\int_0^\infty \frac{\cos \pi t}{t^\beta}\, dt = \frac{\pi^\beta}{2\Gamma(\beta) \cos \frac{\pi\beta}{2}},$$

we take $C_1(\beta) := \pi^\beta / \big(2\Gamma(\beta)\cos(\pi\beta/2)\big)$.

LEMMA 5.2. *Let* $\alpha \in (0, 1)$ *and the function* $g(t) \in L(0, 1)$ *be such that* $\operatorname{supp} g \subseteq [1/2, 1]$. *Also, let*

$$f(x) = \int_x^1 \frac{g(t)}{(t - x)^\alpha}\, dt \quad \text{and} \quad a_n(f) = \int_0^1 f(x) \cos \pi n x\, dx.$$

Then

$$a_n(f) = a_n(g)\gamma_n + b_n(g)\delta_n + \xi_n + \zeta_n,$$

where $\gamma_n \sim C_1(\alpha)/n^{1-\alpha}$, $\delta_n \sim C_2(\alpha)/n^{1-\alpha}$ *as* $n \to \infty$, *and* $|\xi_n + \zeta_n| \leq C(g, \alpha)/n$.

PROOF. We have,

$$
\begin{aligned}
a_n(f) &= \int_0^1 \cos \pi n x \int_x^1 \frac{g(t)}{(t-x)^\alpha}\, dt\, dx \\
&= \int_0^1 g(t) \int_0^t \frac{\cos \pi n x}{(t-x)^\alpha}\, dx\, dt = \int_0^1 g(t) \int_0^t \frac{\cos \pi n (t-y)}{y^\alpha}\, dy\, dt \\
&= \int_0^1 g(t) \cos \pi n t \int_0^t \frac{\cos \pi n y}{y^\alpha}\, dy\, dt \\
&\quad + \int_0^1 g(t) \sin \pi n t \int_0^t \frac{\sin \pi n y}{y^\alpha}\, dy\, dt \\
&= \int_0^1 g(t) \cos \pi n t \int_0^1 \frac{\cos \pi n y}{y^\alpha}\, dy\, dt \\
&\quad - \int_0^1 g(t) \cos \pi n t \int_t^1 \frac{\cos \pi n y}{y^\alpha}\, dy\, dt \\
&\quad + \int_0^1 g(t) \sin \pi n t \int_0^1 \frac{\sin \pi n y}{y^\alpha}\, dy\, dt \\
&\quad - \int_0^1 g(t) \sin \pi n t \int_t^1 \frac{\sin \pi n y}{y^\alpha}\, dy\, dt \\
&= a_n(g)\gamma_n + \xi_n + b_n(g)\delta_n + \zeta_n.
\end{aligned}
$$

From Lemma 5.1 we get $\gamma_n \sim C_1(\alpha)/n^{1-\alpha}$, $\delta_n \sim C_2(\alpha)/n^{1-\alpha}$ as $n \to \infty$. Now since $\operatorname{supp} g \subseteq [1/2, 1]$, we obtain

$$
\begin{aligned}
|\xi_n| &= \left| \int_{1/2}^1 g(t) \cos \pi n t \int_t^1 \frac{\cos \pi n y}{y^\alpha}\, dy\, dt \right| \\
&\le \int_{1/2}^1 |g(t)| \left| \int_t^1 \frac{\cos \pi n y}{y^\alpha}\, dy \right| dt \\
&\le \frac{C(\alpha)}{n} \int_{1/2}^1 |g(t)|\, dt = \frac{C(g,\alpha)}{n}.
\end{aligned}
$$

The same inequality holds for ζ_n. This completes the proof of Lemma 5.2.

THEOREM 5.3. *Let $\alpha \in (0,1)$ and $1 < p < 1/\alpha$. Then there exists a function $f \in M_\alpha$ such that*

$$
\left(\int_0^1 \left(t^{1/p'} \overline{f}(t) \right)^q \frac{dt}{t} \right)^{1/q} < \infty,
$$

and $a = \{a_n\}_{n=0}^{\infty} \notin \ell_{p,q}$, *where*

$$a_n(f) = \int_0^1 f(x) \cos \pi n x \, dx.$$

PROOF. By $g(x)$ we denote the following 1-periodic function

$$g(x) = \begin{cases} 0 & \text{for } x \in [0, 1/2], \\ \dfrac{1}{\left(x - \frac{1}{2}\right)^{1-\delta}} & \text{for } x \in (1/2, 1), \end{cases}$$

where $\delta \in (0, 1/p - \alpha)$. Let us define $f(x) = \int_x^1 \frac{g(t)}{(t-x)^{1-\alpha}} \, dt$ for $x \in [0, 1]$. Then, using the non-negativity of $g(t)$ and the following equality

$$\begin{aligned} \int_u^1 \frac{f(x)}{(x-u)^{\alpha}} \, dx &= \int_u^1 g(t) \int_u^t \frac{dx}{(x-u)^{\alpha}(t-x)^{1-\alpha}} \, dt \\ &= \int_u^1 g(t) \int_0^1 \frac{dv}{v^{\alpha}(1-v)^{1-\alpha}} \, dt = C(\alpha) \int_u^1 g(t) \, dt, \end{aligned}$$

we get that $I^{1-\alpha} f(x)$ is a non-increasing function. Also from [21, p. 43, T. 2.3.] we obtain that $I^{1-\alpha} f(x)$ is an absolutely continuous function. Hence, $f(x) \in M_{\alpha}$. From Lemma 5.2 it follows that $a_n(f) = a_n(g)\gamma_n + b_n(g)\delta_n + \xi_n + \zeta_n$, where $\gamma_n \sim C_1(\alpha)/n^{\alpha}$, $\delta_n \sim C_2(\alpha)/n^{\alpha}$ as $n \to \infty$, and $|\xi_n + \zeta_n| \leq C(g, \alpha)/n$.

Now using Lemma 5.1, we get

$$\begin{aligned} a_n(g) &= \int_{1/2}^1 \cos \pi n t \frac{dt}{\left(t - \frac{1}{2}\right)^{1-\delta}} = \int_0^{1/2} \cos \pi n \left(u + \frac{1}{2}\right) \frac{du}{u^{1-\delta}} \\ &= \cos \frac{\pi n}{2} \int_0^{1/2} \frac{\cos \pi n u}{u^{1-\delta}} \, du - \sin \frac{\pi n}{2} \int_0^{1/2} \frac{\sin \pi n u}{u^{1-\delta}} \, du \sim \frac{C(\alpha)}{n^{\delta}} \end{aligned}$$

as $n \to \infty$.

Hence, if $\alpha + \delta < 1$, then $|a_n(f)| \geq \frac{C}{n^{\alpha+\delta}}$ for sufficiently large n. Now we choose $\delta \in (0, 1)$ such that $p < 1/(\alpha + \delta)$. Then

$$\begin{aligned} \sum_{n=1}^{\infty} \left(n^{1/p} |a_n(f)|\right)^q \frac{1}{n} &\geq C(p, q, \alpha) \sum_{n=n_0}^{\infty} n^{q(1/p-\alpha-\delta)-1} \\ &\geq C(p, q, \alpha) \sum_{n=n_0}^{\infty} n^{-1} = \infty, \end{aligned}$$

i.e., $\{a_n(f)\} \notin \ell_{pq}$.

On the other hand, $f(x)$ is a bounded function. Indeed, let $0 \le x \le 1/4$ and then

$$0 \le f(x) = \int_x^1 \frac{g(t)}{(t-x)^{1-\alpha}}\,dt \le \int_{1/2}^1 \frac{g(t)}{(t-x)^{1-\alpha}}\,dt$$
$$\le 4^{1-\alpha}\int_{1/2}^1 g(t)\,dt = C(\alpha, g).$$

Hence, the following inequality

$$\overline{f}(t) = \sup_{y\ge t}\frac{1}{y}\left|\int_0^y f(x)\,dx\right| \le C(g,\alpha),$$

holds for all $t \in [0, 1]$. Therefore,

$$\left(\int_0^1 \left(t^{1/p'}\overline{f}(t)\right)^q \frac{dt}{t}\right)^{1/q} \le C(g,\alpha)\left(\int_0^1 t^{q/p'-1}\,dt\right)^{1/q} < \infty.$$

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