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REVERSE LEXICOGRAPHIC GRÖBNER BASES AND STRONGLY KOSZUL TORIC RINGS

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Abstract

Restuccia and Rinaldo proved that a standard graded K-algebra $K[x_1, \dots, x_n]/I$ is strongly Koszul if the reduced Gröbner basis of I with respect to any reverse lexicographic order is quadratic. In this paper, we give a sufficient condition for a toric ring $K[A]$ to be strongly Koszul in terms of the reverse lexicographic Gröbner bases of its toric ideal I_A. This is a partial extension of a result given by Restuccia and Rinaldo.

In addition, we show that any strongly Koszul toric ring generated by squarefree monomials is compressed. Using this fact, we show that our sufficient condition for $K[A]$ to be strongly Koszul is both necessary and sufficient when $K[A]$ is generated by squarefree monomials.

Introduction

Herzog, Hibi, and Restuccia [9] introduced the notion of strongly Koszul algebras. Let R be a standard graded K-algebra with the graded maximal ideal $\mathfrak{m}$. Then R is said to be *strongly Koszul* if $\mathfrak{m}$ admits a minimal system of generators $u_1, \dots, u_n$ of the same degree such that for any $1 \le i_1 < \cdots < i_r \le n$ and for all $j = 1, 2, \dots, r$, the colon ideal $(u_{i_1}, \dots, u_{i_{j-1}}) : u_{i_j}$ of R is generated by a subset of $\{u_1, \dots, u_n\}$. Inspired by this notion, Conca, Trung, and Valla [4] introduced the notion of Koszul filtrations. A family $\mathscr{F}$ of ideals of R is called a *Koszul filtration* if $\mathscr{F}$ satisfies (i) every $I \in \mathscr{F}$ is generated by linear forms; (ii) (0) and $\mathfrak{m}$ are in $\mathscr{F}$; and (iii) for each non-zero ideal $I \in \mathscr{F}$, there exists $J \in \mathscr{F}$ with $J \subset I$ such that I/J is cyclic and $J : I \in \mathscr{F}$. For example, if R is strongly Koszul, then $\mathscr{F} = \{(0)\} \cup \{\, (u_{i_1}, \dots, u_{i_r}) \mid 1 \le i_1 < \cdots < i_r \le n, 1 \le r \le n \,\}$ is a Koszul filtration of R. The existence of a Koszul filtration of R is an effective sufficient condition for R to be Koszul. Some classes of Koszul algebras which have special Koszul filtrations have been studied, e.g., universally Koszul algebras [2] and initially Koszul algebras [1].

On the other hand, it is important to characterize the Koszulness in terms of the Gröbner bases of its defining ideal. It is a well-known fact that if R is G-quadratic (i.e., its defining ideal has a quadratic Gröbner basis) then R is

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Koszul. Conca, Rossi, and Valla [3] proved that, if R is initially Koszul, then R is G-quadratic. Moreover, they and Blum gave a necessary and sufficient condition for R to be initially Koszul in terms of initial ideals of toric ideals ([1], [3]).

Let $A = \{u_1, \dots, u_n\}$ be a set of monomials of the same degree in a polynomial ring $K[T] = K[t_1, \dots, t_d]$ in d variables over a field K. Then the *toric ring* $K[A] \subset K[T]$ is a semigroup ring generated by the set A over K. Let $K[X] = K[x_1, \dots, x_n]$ be a polynomial ring in n variables over K. The *toric ideal* I_A of $K[A]$ is the kernel of the surjective homomorphism $\pi\colon K[X] \to K[A]$ defined by $\pi(x_i) = u_i$ for each $1 \le i \le n$. Then we have $K[A] \simeq K[X]/I_A$. A toric ring $K[A]$ is called *compressed* [16] if $\sqrt{\mathrm{in}_<(I_A)} = \mathrm{in}_<(I_A)$ for any reverse lexicographic order $<$.

In this paper, we study Gröbner bases of toric ideals of strongly Koszul toric rings. First, in Section 1, we give a sufficient condition for $K[A]$ to be strongly Koszul in terms of the Gröbner bases of I_A (Theorem 1.2). We then have Corollary 1.3, i.e., if the reduced Gröbner basis of I_A with respect to any reverse lexicographic order is quadratic, then $K[A]$ is strongly Koszul [15, Theorem 2.7]. On the other hand, Examples 1.6 and 1.7 are counterexamples of [15, Conjecture 3.11] (i.e., counterexamples of the converse of Corollary 1.3). In Section 2, we discuss strongly Koszul toric rings generated by squarefree monomials. We show that such toric rings are compressed (Theorem 2.1). Using this fact, we show that the sufficient condition for $K[A]$ to be strongly Koszul in Theorem 1.2 is both necessary and sufficient when the toric rings are generated by squarefree monomials (Theorem 2.3).

1. Gröbner bases and strong Koszulness

First, we give a sufficient condition for toric rings to be strongly Koszul in terms of the reverse lexicographic Gröbner bases. We need the following lemma:

Lemma 1.1. *Suppose that, for each $1 \le i < j \le n$, there exists a monomial order $\prec$ such that, with respect to $\prec$, an arbitrary binomial g in the reduced Gröbner basis of I_A satisfies the following conditions:*

(i) $x_i \mid \mathrm{in}_\prec(g)$ *and* $x_j \nmid \mathrm{in}_\prec(g) \Longrightarrow g = x_i x_k - x_j x_\ell$ *for some* $1 \le k, \ell \le n$,

(ii) $x_j \mid \mathrm{in}_\prec(g)$ *and* $x_i \nmid \mathrm{in}_\prec(g) \Longrightarrow g = x_j x_\ell - x_i x_k$ *for some* $1 \le k, \ell \le n$.

Then, $K[A]$ is strongly Koszul.

Proof. Suppose that $K[A]$ is not strongly Koszul. By [9, Proposition 1.4], there exists a monomial $u_{k_1} \cdots u_{k_s}$ of a minimal set of generators of $(u_i) \cap (u_j)$ such that $s \ge 3$. Since $u_{k_1} \cdots u_{k_s}$ belongs to $(u_i) \cap (u_j)$, there exist binomials $x_{k_1} \cdots x_{k_s} - x_i X^\alpha$ and $x_{k_1} \cdots x_{k_s} - x_j X^\beta$ in I_A. Let $\mathcal{G}$ be the reduced Gröbner basis of I_A with respect to $\prec$. Since $x_i X^\alpha - x_j X^\beta \in I_A$ is reduced to 0 with respect to $\mathscr{G}$, it follows that both $x_i X^\alpha$ and $x_j X^\beta$ are reduced to the same monomial m with respect to $\mathscr{G}$.

Suppose that $g \in \mathscr{G}$ is used in the computation $x_i X^\alpha \xrightarrow{\mathscr{G}} m$ and that x_i divides $\text{in}_\prec(g)$. If x_j divides $\text{in}_\prec(g)$, then it follows that $x_{k_1} \cdots x_{k_s} - x_i x_j X^\gamma$ belongs to I_A. Thus, $u_i u_j$ divides $u_{k_1} \cdots u_{k_s}$. This contradicts that $u_{k_1} \cdots u_{k_s}$ belongs to a minimal set of generators of $(u_i) \cap (u_j)$. If x_j does not divide $\text{in}_\prec(g)$, then $g = x_i x_k - x_j x_\ell$ by assumption (i). Hence, $u_i u_k \in (u_i) \cap (u_j)$ divides $u_{k_1} \cdots u_{k_s}$. This contradicts that $u_{k_1} \cdots u_{k_s}$ belongs to a minimal set of generators of $(u_i) \cap (u_j)$.

Therefore, x_i never appears in the initial monomials of $g \in \mathscr{G}$ which are used in the computation $x_i X^\alpha \xrightarrow{\mathscr{G}} m$. Hence, x_i divides m. By the same argument, it follows that x_j never appears in the initial monomials of $g \in \mathscr{G}$ which are used in the computation $x_j X^\beta \xrightarrow{\mathscr{G}} m$, and hence, x_j divides m. Thus, $x_i x_j$ divides m, which means that $u_i u_j$ divides $u_{k_1} \cdots u_{k_s}$. This contradicts that $u_{k_1} \cdots u_{k_s}$ belongs to a minimal set of generators of $(u_i) \cap (u_j)$.

Let $G(I)$ denote the (unique) minimal set of monomial generators of a monomial ideal I. Given an ordering $x_{i_1} < x_{i_2} < \cdots < x_{i_n}$ of variables $\{x_1, \ldots, x_n\}$, let $<_{\text{rlex}}$ denote the reverse lexicographic order induced by the ordering $<$.

THEOREM 1.2. *Suppose that, for each $1 \leq i < j \leq n$, there exists an ordering $x_{i_1} < x_{i_2} < \cdots < x_{i_n}$ with $\{i_1, i_2\} = \{i, j\}$, such that any monomial in $G(\text{in}_{<_{\text{rlex}}}(I_A)) \cap (x_{i_2})$ is quadratic. Then, $K[A]$ is strongly Koszul.*

PROOF. We may assume that $x_j < x_i$. By Lemma 1.1, it is enough to show that $<_{\text{rlex}}$ satisfies conditions (i) and (ii) in Lemma 1.1. Let g be an arbitrary (irreducible) binomial in the reduced Gröbner basis of I_A with respect to $<_{\text{rlex}}$.

Since x_j is the smallest variable, x_j does not divide $\text{in}_{<_{\text{rlex}}}(g)$. Hence, $<_{\text{rlex}}$ satisfies condition (ii). Suppose that x_i divides $\text{in}_{<_{\text{rlex}}}(g)$. By the assumption for $<$, $\deg(\text{in}_{<_{\text{rlex}}}(g)) = 2$. Hence, $g = x_i x_p - x_q x_r$ for some $1 \leq p, q, r \leq n$. Since $x_q x_r <_{\text{rlex}} x_i x_p$, we have $j \in \{q, r\}$, and hence, $<_{\text{rlex}}$ satisfies condition (i).

As a corollary, in case of toric rings, we have a result of Restuccia and Rinaldo [15, Theorem 2.7]:

COROLLARY 1.3. *Suppose that the reduced Gröbner basis of I_A is quadratic with respect to any reverse lexicographic order. Then, $K[A]$ is strongly Koszul.*

EXAMPLE 1.4. Let $K[A_n] = K[s, t_1 s, \ldots, t_n s, t_1^{-1} s, \ldots, t_n^{-1} s]$. Then, I_{A_n} is the kernel of the surjective homomorphism $\pi\colon K[X] \to K[A_n]$ defined by

$\pi(z) = s$, $\pi(x_i) = t_i s$, and $\pi(y_i) = t_i^{-1} s$. It is easy to see that $K[A_n]$ is isomorphic to

$$K[A_G^{\pm}] = K\big[s, t_1 t_{n+1} s, \ldots, t_n t_{n+1} s, t_1^{-1} t_{n+1}^{-1} s, \ldots, t_n^{-1} t_{n+1}^{-1} s\big],$$

where $A_G^{\pm}$ is the *centrally symmetric configuration* [13] of A_G associated with the star graph $G = K_{1,n}$ with $n+1$ vertices. By [13, Theorem 4.4], I_{A_n} is generated by $\mathscr{F} = \{ x_i y_i - z^2 \mid i = 1, 2, \ldots, n \}$. Then, the Buchberger criterion tells us that the set $\mathscr{F} \cup \{ x_i y_i - x_j y_j \mid 1 \le i < j \le n \}$ is a Gröbner basis of I_{A_n} with respect to any monomial order (i.e., a universal Gröbner basis of I_{A_n}). Thus, by Corollary 1.3, $K[A_n]$ is strongly Koszul for all $n \in \mathbb{N}$.

Eliminating the variable z from $\mathscr{F}$, by the same argument above, it follows that the toric ring $K[B_n] = K[t_1 s, \ldots, t_n s, t_1^{-1} s, \ldots, t_n^{-1} s]$ is strongly Koszul for all $n \in \mathbb{N}$. Note that $K[B_n]$ is isomorphic to some toric ring generated by squarefree monomials.

REMARK 1.5. A standard graded K-algebra R is said to be *c-universally Koszul* [6] if the set of all ideals of R which are generated by subsets of the variables is a Koszul filtration of R. Ene, Herzog, and Hibi proved that a toric ring $K[A]$ is c-universally Koszul if the reduced Gröbner basis of I_A is quadratic with respect to any reverse lexicographic order [6, Corollary 1.4]. However, it is known that a toric ring $K[A]$ is c-universally Koszul if and only if $K[A]$ is strongly Koszul. See [14, Definition 7.2] or [11, Lemma 3.18]. So, [6, Corollary 1.4] is equivalent to Corollary 1.3.

In Section 2, we will show that the converse of Theorem 1.2 holds when $K[A]$ is generated by squarefree monomials. However, the converse does not hold in general.

EXAMPLE 1.6. It is known [9] that any Veronese subring of a polynomial ring is strongly Koszul. Let $K[A]$ be the fourth Veronese subring of $K[t_1, t_2]$, i.e., $K[A] = K[t_1^4, t_1^3 t_2, t_1^2 t_2^2, t_1 t_2^3, t_2^4]$. Then I_A is generated by the binomials

$$x_3 x_5 - x_4^2,\ x_1 x_3 - x_2^2,\ x_3^2 - x_2 x_4,\ x_1 x_5 - x_2 x_4,\ x_2 x_3 - x_1 x_4,\ x_3 x_4 - x_2 x_5.$$

Let $<$ be an ordering of variables such that $x_{i_1} < x_{i_2} < x_{i_3} < x_{i_4} < x_{i_5}$ with $\{i_1, i_2\} = \{2, 4\}$. Since both $x_2^3 - x_1^2 x_4$ and $x_4^3 - x_2 x_5^2$ belong to I_A, it is easy to see that either x_2^3 or x_4^3 belongs to $G(\mathrm{in}_{<_{\mathrm{rlex}}}(I_A)) \cap (x_{i_2})$. Thus, I_A does not satisfy the hypothesis of Theorem 1.2.

On the other hand, the converse of Corollary 1.3 does not hold even if $K[A]$ is generated by squarefree monomials. Note that Examples 1.6 and 1.7 are counterexamples to [15, Conjecture 3.11].

EXAMPLE 1.7. Let $K[A]=K[t_4, t_1t_4, t_2t_4, t_3t_4, t_1t_2t_4, t_2t_3t_4, t_1t_3t_4, t_1t_2t_3t_4]$, which is the toric ring of the stable set polytope of the empty graph with three vertices. Since any empty graph is *trivially perfect* (see also Example 2.2), $K[A]$ is strongly Koszul. See [10] for the details. The toric ideal I_A is generated by the binomials

$$x_1x_5 - x_2x_3,\ x_1x_6 - x_3x_4,\ x_1x_7 - x_2x_4,\ x_5x_6 - x_3x_8,\ x_6x_7 - x_4x_8,$$
$$x_5x_7 - x_2x_8,\ x_1x_8 - x_4x_5,\ x_2x_6 - x_4x_5,\ x_3x_7 - x_4x_5.$$

Let $<$ be an ordering $x_4 < x_3 < x_2 < x_1 < x_8 < x_7 < x_6 < x_5$. Since, with respect to $<_{\text{rlex}}$, the initial monomial (i.e., the first monomial) of any quadratic binomial above does not divide the initial monomial $x_2x_3x_8$ of $x_4x_5^2 - x_2x_3x_8 \in I_A$, we have $x_2x_3x_8 \in G(\text{in}_{<_{\text{rlex}}}(I_A))$. Thus, the reduced Gröbner basis of I_A with respect to $<_{\text{rlex}}$ is not quadratic. Below, we show that Theorem 2.3 guarantees that I_A satisfies the hypothesis of Theorem 1.2. See also Example 2.2.

2. Strongly Koszul toric rings generated by squarefree monomials

In this section, we consider the case when $K[A]$ is *squarefree*, i.e., $K[A]$ is isomorphic to a semigroup ring generated by squarefree monomials. A toric ring $K[A]$ is called *compressed* [16] if $\sqrt{\text{in}_<(I_A)} = \text{in}_<(I_A)$ for any reverse lexicographic order $<$. It is known that $K[A]$ is normal if it is compressed.

THEOREM 2.1. *Suppose that $K[A]$ is strongly Koszul. Then, the following conditions are equivalent:*

(i) $K[A]$ *is squarefree;*

(ii) I_A *has no quadratic binomial of the form* $x_i^2 - x_jx_k$*;*

(iii) $K[A]$ *is compressed.*

In particular, any squarefree strongly Koszul toric ring is compressed.

PROOF. First, (i) ⇒ (ii) is trivial. By [16, Theorem 2.4], we have (iii) ⇒ (i). Thus it is enough to show (ii) ⇒ (iii).

Let $K[A]$ be a strongly Koszul toric ring such that I_A has no quadratic binomial of the form $x_i^2 - x_jx_k$. Suppose that an irreducible binomial $f = x_i^2X^\alpha - x_jX^\beta$ belongs to the reduced Gröbner basis of I_A with respect to a reverse lexicographic order $<_{\text{rlex}}$ and that x_j is the smallest variable in f. Then, $u_i^2U^\alpha$ belongs to $(U^\alpha) \cap (u_j)$. Since $K[A]$ is strongly Koszul, by [9, Corollary 1.5], $(U^\alpha) \cap (u_j)$ is generated by the element in $(U^\alpha) \cap (u_j)$ of degree $\le \deg(X^\alpha) + 1$. Hence, $u_i^2U^\alpha$ is generated by such elements. Thus, there exist binomials $x_i^2X^\alpha - X^\alpha x_kx_\ell$ and $x_i^2X^\alpha - x_jX^\gamma x_\ell$ in I_A. Then, we have $x_i^2 - x_kx_\ell \in I_A$. By assumption, we have $x_i^2 - x_kx_\ell = 0$, and hence,

$i = k = \ell$. Thus, the binomial $g = x_i X^\alpha - x_j X^\gamma$ belongs to I_A. Since x_j is the smallest variable in f, it follows that $x_i X^\alpha$ is the initial monomial of g. This contradicts that f appears in the reduced Gröbner basis of I_A with respect to $<_{\rm rlex}$. Hence, $K[A]$ is compressed.

EXAMPLE 2.2. Let G be a simple graph on the vertex set $V(G) = \{1, \ldots, d\}$ with the edge set $E(G)$. A subset $S \subset V(G)$ is said to be *stable* if $\{i, j\} \notin E(G)$ for all $i, j \in S$. For each stable set S of G, we define the monomial $u_S = t_{d+1} \prod_{i \in S} t_i$ in $K[t_1, \ldots, t_{d+1}]$. Then the toric ring $K[Q_G]$ generated by $\{ u_S \mid S \text{ is a stable set of } G \}$ over a field K is called the toric ring of the *stable set polytope* of G. It is known that

- $K[Q_G]$ is compressed $\Longleftrightarrow$ G is perfect ([12, Example 1.3 (c)], [8]).
- $K[Q_G]$ is strongly Koszul $\Longleftrightarrow$ G is trivially perfect ([10, Theorem 5.1]).

Here, a graph G is said to be *perfect* if the size of maximal clique of G_W equals to the chromatic number of G_W for any induced subgraph G_W of G, and a graph G is said to be *trivially perfect* if the size of maximal stable set of G_W equals to the number of maximal cliques of G_W for any induced subgraph G_W of G. (For the standard terminologies of graph theory, see [5].) Since any trivially perfect graph is perfect [7], these facts are consistent with Theorem 2.1. On the other hand, with respect to some *lexicographic order*, the initial ideal of the toric ideal in Example 1.7 is not generated by squarefree monomials.

Using Theorem 2.1, we now show that the converse of Theorem 1.2 holds when $K[A]$ is squarefree.

THEOREM 2.3. *Suppose that $K[A]$ is squarefree and strongly Koszul. Let $1 \le i < j \le n$, and let $<$ be any ordering of variables satisfying*

$$x_j < x_i < \{ x_k \mid u_i u_k / u_j \in K[A], k \neq j \} < \textit{other variables}.$$

Then, any monomial in $G(\mathrm{in}_{<_{\rm rlex}}(I_A)) \cap (x_i)$ is quadratic.

PROOF. Let $\mathcal{G}$ be the reduced Gröbner basis of I_A with respect to $<_{\rm rlex}$. Suppose that $x_i X^\alpha \in G(\mathrm{in}_{<_{\rm rlex}}(I_A)) \cap (x_i)$ is not quadratic. Then, there exists a binomial $g = x_i X^\alpha - x_j X^\beta$ in $\mathcal{G}$. Note that $\mathrm{in}_{<_{\rm rlex}}(g) = x_i X^\alpha$ is squarefree by Theorem 2.1. Hence, X^α is not divisible by x_i. Moreover, since $\mathcal{G}$ is reduced, X^α is not divisible by x_j.

Since g belongs to I_A, it follows that $u_i U^\alpha = u_j U^\beta$ belongs to the ideal $(u_i) \cap (u_j)$. Then, $u_i U^\alpha$ is generated by $u_i u_k = u_j u_\ell \in (u_i) \cap (u_j)$ for some $1 \le k, \ell \le n$. Thus, there exist binomials $x_i X^\alpha - x_i x_k X^\gamma$ and $x_i X^\alpha - x_j x_\ell X^\gamma$ in I_A. Then, we have $X^\alpha - x_k X^\gamma \in I_A$. If $k \in \{i, j\}$, then $X^\alpha \in \mathrm{in}_{<_{\rm rlex}}(I_A)$. This contradicts $x_i X^\alpha \in G(\mathrm{in}_{<_{\rm rlex}}(I_A))$. Hence, $k \notin \{i, j\}$. Then, $0 \neq x_i x_k - x_j x_\ell \in I_A$ and $\mathrm{in}_{<_{\rm rlex}}(x_i x_k - x_j x_\ell) = x_i x_k$. Since $x_i X^\alpha$ is not divisible by $x_i x_k$, X^α is not divisible by x_k. In particular, $0 \neq X^\alpha - x_k X^\gamma \in I_A$ and $X^\alpha <_{\text{rlex}} x_k X^\gamma$. Since $u_i u_k / u_j = u_\ell$, x_k belongs to $\{ x_k \mid u_i u_k / u_j \in K[A], k \neq j \}$. Thus, the smallest variable x_m appearing in X^α belongs to $\{ x_k \mid u_i u_k / u_j \in K[A], k \neq j \}$. Let $u_{m'} = u_i u_m / u_j$. Then, $x_i x_m - x_j x_{m'}$ ($\neq 0$) belongs to I_A. Therefore, $x_i x_m$ belongs to $\text{in}_{<_{\text{rlex}}}(I_A)$ and divides $x_i X^\alpha$, which is a contradiction.

By Theorem 2.3, we can check whether a squarefree toric ring $K[A] = K[u_1, \ldots, u_n]$ is strongly Koszul by computing the reverse lexicographic Gröbner bases of I_A at most $n(n-1)/2$ times.

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