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ON SEGRE NUMBERS OF HOMOGENEOUS MAP GERMS

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Abstract

Segre numbers and Segre cycles of ideals were independently introduced by Tworzewski, by Achilles and Manaresi and by Gaffney and Gassler. They are generalization of the Lê numbers and Lê cycles, introduced by Massey. In this article we give Lê-Iomdine type formulas for these cycles and numbers of arbitrary ideals. As a consequence we give a Plücker type formula for the Segre numbers of ideals generated by weighted homogeneous functions, in terms of their weights and degree. As an application of these results, we compute, in a purely combinatorial manner, the Segre numbers of the ideal which defines the critical loci of a map germ defined by a sequence of central hyperplane arrangements in C^{n+1}.

Introduction

Let $\mathscr{O}_{n+1}$ be the ring of holomorphic function germs in C^{n+1} at the origin, let I be an ideal of $\mathscr{O}_{n+1}$ and set $s = \dim V(I)$. Tworzewski in [15], Achilles and Manaresi in [1] (see also [2]) and Gaffney and Gassler in [8] have independently introduced a sequence of cycles and numbers, $\Lambda_I^0, \dots, \Lambda_I^s$ and $\lambda_I^0(0), \dots, \lambda_I^s(0)$ respectively, that we, following [8], call the *Segre cycles* and *Segre numbers* of I at the origin. When $I = \left(\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_n}\right)$ is the Jacobian ideal of a function germ $f\colon(\mathsf{C}^{n+1}, 0) \to (\mathsf{C}, 0)$, these Segre cycles and Segre numbers coincide with the Lê cycles and Lê numbers introduced by Massey in [11]. The definition of the Segre numbers given in [1] and [2] is of an algebraic nature, which could be seen as a generalization of the classical Hilbert-Samuel multiplicity of an ideal in a local ring $(R, \mathfrak{m})$.

The importance of the Lê numbers introduced by Massey [11] can not be underestimated: they generalize the well-known Milnor number of an isolated hypersurface singularity; they allow one to describe a handle decomposition of the Milnor fiber of an hypersurface with arbitrary singular locus; the constancy of the Lê numbers in a multi-parameter family of hypersurfaces implies the Thom's a_f condition for the ambient space along the parameters of the family.

On the other hand, Andersson, Samuelsson, Wulcan and Yger in [3] describe

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the Segre numbers of an ideal I as the Lelong numbers of certain positive currents (see also [4]). More precisely, if $\underline{f} = (f_0, \ldots, f_n)$ is a tuple of generators of I and Z is the variety of I then $\lambda_I^k(0) = \ell_0(\mathbf{1}_Z(dd^c \log|\underline{f}|^2)^{n+1-k})$, where ℓ_0 denotes the Lelong number at 0 (see [10]) and $\mathbf{1}_Z$ is the characteristic function for Z.

Also, Gaffney and Gassler [8] obtained the following generalization of Rees's Theorem (see [14]): if $I \subseteq J$ are ideals of $\mathcal{O}_{n+1}$ (or more generally, of $\mathcal{O}_{X,0}$, where $X \subseteq (\mathbb{C}^{n+1}, 0)$ is an analytic germ of pure dimension), then I and J have the same integral closure if and only if $\lambda_I^k(0) = \lambda_J^k(0)$ for all $k = 0, \ldots, n$. The proof of this theorem is complex analytic in nature since it depends heavily on the so-called principle of specialization of the integral closure. Nonetheless, Dunn [6] extended the above result to any pair of ideals in a formally equidimensional local ring, given a complete numerical characterization of the integral closure of ideals, generalizing the above mentioned Rees's Theorem to this context.

Massey in [11, Chapter 4] proved a Lê-Iomdine type formula for functions with arbitrary singularities. These formulas relate the Lê numbers of a hypersurface singularity to the Lê numbers of a sequence of hypersurface singularities, which approach the original one, but having smaller dimensional loci. This formula has a large number of applications. For example, Massey in [11, Corollary 4.7] proved a Plücker formula for a homogeneous polynomial function h of degree d in $(n+1)$-variables. This formula says that if $s = \dim_0 \Sigma(h)$, then for a generic coordinate system

$$\sum_{i=0}^{s}(d-1)^i \lambda_h^i(0) = (d-1)^{n+1}.$$

Massey used this formula to compute the Lê numbers of a central hyperplane arrangement in $\mathbb{C}^{n+1}$ in a purely combinatorial manner (see [11, Example 5.1]). In [13], two of the authors extended this result computing the Lê numbers of a semi-weighted homogeneous arrangement in $\mathbb{C}^{n+1}$.

The Lê-Iomdine type formulas were generalized by Massey in [12, Theorem 3.4] for any sequence $\underline{f} = (f_0, \ldots, f_n)$ of analytic functions defined on an analytic variety X of dimension $n+1$, such that the Vogel cycle $\Lambda_{\underline{f}}^i$ is defined (see [12, Definition 2.14]).

In this work we generalize all the above mentioned results, as we describe next. In section 1 we recall the notion of Segre cycles and Segre numbers of ideals with respect to a Vogel sequence. In section 2, in order to fix notation and for completeness, we state and prove the Lê-Iomdine type formula for the Segre numbers of arbitrary ideals in $\mathcal{O}_X$, as described in [12, Theorem 3.4], but in the context we will need for applications, that is when $X = \mathbb{C}^{n+1}$. In section 3 we prove a Plücker formula for the Segre numbers of ideals generated by weighted homogeneous functions, in terms of their weights and degree. In section 4 we study the Segre numbers associated to a map germ defined by homogeneous polynomials, which are the Segre numbers of ideals defined by the critical loci of this germ. In this context, we show how these Segre numbers behave under hyperplane sections, which is a fundamental result. In section 5 we compute, in a purely combinatorial manner, the Segre numbers of the ideal which defines the critical loci of a map germ defined by a sequence of central hyperplane arrangements in $\mathbb{C}^{n+1}$.

1. Segre numbers

We assume that the reader is familiar with the notion of gap sheaves [11]. For the purpose of fixing the notation, for a sheaf α and an analytic subset W in an affine space, we denote by α/W the corresponding gap sheaf and by $V(\alpha)/W$ the scheme associated with the sheaf α/W. We shall at times enclose cycles in square brackets, $[\cdot]$, and their supports in bars, $|\cdot|$.

Let $\mathcal{O}_{n+1}$ be the ring of holomorphic function germs in $\mathbb{C}^{n+1}$ at the origin and let I be an ideal of $\mathcal{O}_{n+1}$. A sequence $\underline{f} = (f_0, \dots, f_n)$ of elements of I is called a *Vogel sequence of I at the origin* if there is a neighborhood $\mathcal{U} \subseteq \mathbb{C}^{n+1}$ of the origin, where the f_j are defined, such that

$$\dim\big((\mathcal{U} \setminus V(I)) \cap V(f_{k+1}, \dots, f_n)\big) \leq k, \qquad \text{for all} \quad k. \tag{1}$$

Here the left-hand side should be understood as $-\infty$ for empty intersection.

REMARK 1.1. One way to construct Vogel sequences of I at the origin is as follows: let $g_1, \dots, g_N$ be an ordered generating system of I and let A be a generic $(n+1) \times N$ matrix with complex coefficients. Here we assume that $N \geq n+1$. Set

$$\begin{pmatrix} f_0 \\ \vdots \\ f_n \end{pmatrix} := A \cdot \begin{pmatrix} g_1 \\ \vdots \\ g_N \end{pmatrix}.$$

Then the sequence $\underline{f} = (f_0, \dots, f_n)$ is a Vogel sequence at the origin. Also, the ideal $I_A := (f_0, \dots, f_n)\mathcal{O}_{n+1}$ is such that $V(I_A) = V(I)$, as sets.

Based on Massey's work [11] (see also [8]), we introduce some concepts associated to I via a Vogel sequence.

DEFINITION 1.2. Let $\underline{f} = (f_0, \dots, f_n)$ be a Vogel sequence of I at the origin. The *k-th polar variety* of I with respect to $\underline{f}$, denoted by $\Gamma^k_{I,\underline{f}}$, is defined as the scheme $V(f_k, \dots, f_n)/V(I)$.

Notice that by equation (1) each $\Gamma^k_{I,\underline{f}}$ is k-dimensional. Denote the corresponding cycle by $[\Gamma^k_{I,\underline{f}}]$.

DEFINITION 1.3. The *k-th Segre cycle* of I with respect to $\underline{f}$, denoted by $\Lambda^k_{I,\underline{f}}$, is defined as the cycle

$$\left[\Gamma^{k+1}_{I,\underline{f}} \cap V(f_k)\right] - \left[\Gamma^k_{I,\underline{f}}\right].$$

Gaffney and Gassler proved in [8, Lemma 2.2] (see also [9, Theorem 3.3]) that the Segre cycles are representatives of the Segre classes of $V(I)$, as defined in [7, §4.2]. For this reason, we have the following definition:

DEFINITION 1.4. The *k-th Segre number* of I with respect to $\underline{f}$ at the origin, denoted by $\lambda^k_{I,\underline{f}}(0)$, is the multiplicity of the k-th Segre cycle of I with respect to $\underline{f}$ at the origin, that is,

$$\lambda^k_{I,\underline{f}}(0) = \mathrm{mult}_0\, \Lambda^k_{I,\underline{f}}.$$

REMARK 1.5. Consider a function germ $f\colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ and let $I = \left(\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_n}\right)\mathcal{O}_{n+1}$ be the ideal generated by the $(n+1)$-tuple of the partial derivatives of f with respect to the coordinate system $\underline{x} = (x_0, \dots, x_n)$. Suppose $\underline{z} = (z_0, \dots, z_n)$ is a generic linear system of coordinates of $\mathbb{C}^{n+1}$ around the origin. Then, there is a $n + 1$ invertible matrix A such that $\underline{x} = \underline{z} \cdot A$. Then

$$\begin{pmatrix} \dfrac{\partial f}{\partial z_0} \\ \vdots \\ \dfrac{\partial f}{\partial z_n} \end{pmatrix} = A^t \cdot \begin{pmatrix} \dfrac{\partial f}{\partial x_0} \\ \vdots \\ \dfrac{\partial f}{\partial x_n} \end{pmatrix}.$$

By this choice of A we have that $\underline{h} = \left(\frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_n}\right)$ is a Vogel sequence of I. In this case, the k-th Segre cycle of I with respect to $\underline{h}$ is called the k-th *Lê cycle* of f with respect to the coordinates $\underline{z} = (z_0, \dots, z_n)$ and its multiplicity at the origin is called the k-th generic *Lê number* of f at 0 with respect to the coordinates $\underline{z} = (z_0, \dots, z_n)$ (see [11]).

2. Generalized Lê-Iomdine formulas for ideals

We describe a technique that reduces an s-dimensional variety $V(I)$ to an $(s-1)$-dimensional variety $V(J)$ and relates the Segre numbers of the first variety to the Segre numbers of the second one, under certain conditions. This technique was introduced in [11, Theorem 4.5] for hypersurfaces with arbitrary singularities, and called *Lê-Iomdine formulas*. These formulas were generalized by Massey in [12, Theorem 3.4] for any sequence $\underline{f} = (f_0, \ldots, f_n)$ of analytic functions defined on an analytic variety X of dimension $n+1$, such that the Vogel cycles $\Lambda^i_{\underline{f}}$ are defined (see [12, Definition 2.14]). These Vogel cycles agree with the Segre cycles when $X = \mathbb{C}^{n+1}$.

In order to fix notation and for completeness, we state and prove the Lê-Iomdine type formula for the Segre numbers of arbitrary ideals in $\mathcal{O}_X$, as described in [12, Theorem 3.4], but in the context we will need for applications, that is when $X = \mathbb{C}^{n+1}$. An important concept involved in this technique is the following number.

Definition 2.1. Let I be an ideal of $\mathcal{O}_{n+1}$ and let $\underline{f} = (f_0, \ldots, f_n)$ be a Vogel sequence of I at the origin. Let η be an irreducible component (with its reduced structure) of $\Gamma^1_{I,\underline{f}}$ which passes through the origin. If $\eta \cap V(x_0)$ is zero-dimensional at the origin, the *polar ratio* of η at 0 (for I with respect to $\underline{f}$) is defined as $(\eta \cdot V(f_0))_0/(\eta \cdot V(x_0))_0$, where $(\eta \cdot V(f_0))_0$ is the proper intersection multiplicity of η and $V(f_0)$ at the origin, which is well defined by equation (1). Otherwise, we say that the *polar ratio* of η equals 0. The maximum of these polar ratio over all possible η is called the *maximum polar ratio* for I with respect to $\underline{f}$.

Theorem 2.2. *Let I be an ideal of $\mathcal{O}_{n+1}$ and let $\underline{f} = (f_0, \ldots, f_n)$ be a Vogel sequence of I at the origin. For $a \in \mathbb{C} \setminus \{0\}$ and $j \geq 1$ an integer, let $\underline{f}^{(0)} = (f_1, \ldots, f_n, f_0 + ax_0^j)$ and let $I_0 = (f_1, \ldots, f_n, f_0 + ax_0^j)\mathcal{O}_{n+1}$.*

Suppose that $V(x_0)$ intersects $V(I)$ and each $\Lambda^i_{I,\underline{f}}$ at the origin transversely, for all $i \geq 1$. If j is greater or equal than the maximum polar ratio at 0 for I with respect to $\underline{f}$, then for all but (possibly) a finite number of complex a, in a neighborhood of 0:

(i) *there is an equality of sets given by* $V(I_0) = V(I) \cap V(x_0)$;

(ii) $\dim_0 V(I_0) = \dim_0 V(I) - 1$, *provided that* $\dim_0 V(I) \geq 1$;

(iii) $\left[\Gamma^0_{I_0,\underline{f}^{(0)}}\right] = \left[\Gamma^0_{I,\underline{f}}\right] + j\left(\left[\Gamma^1_{I,\underline{f}}\right] \cdot [V(x_0)]\right)$ *and* $\left[\Gamma^i_{I_0,\underline{f}^{(0)}}\right] = j\left(\left[\Gamma^{i+1}_{I,\underline{f}}\right] \cdot [V(x_0)]\right)$, *for* $1 \leq i \leq n-1$;

(iv) $\Lambda^0_{I_0,\underline{f}^{(0)}} = \Lambda^0_{I,\underline{f}} + j\left(\Lambda^1_{I,\underline{f}} \cdot [V(x_0)]\right)$ *and* $\Lambda^i_{I_0,\underline{f}^{(0)}} = j\left(\Lambda^{i+1}_{I,\underline{f}} \cdot [V(x_0)]\right)$, *for* $1 \leq i \leq n-1$;

(v) $\lambda^0_{I_0,\underline{f}^{(0)}}(0) = \lambda^0_{I,\underline{f}}(0) + j\lambda^1_{I,\underline{f}}(0)$ *and* $\lambda^i_{I_0,\underline{f}^{(0)}}(0) = j\lambda^{i+1}_{I,\underline{f}}(0)$, *for* $1 \leq i \leq n-1$.

Proof. By definition $\Gamma^1_{I,\underline{f}}$ is 1-dimensional at 0. If we write the cycle

$\Gamma^1_{I,\underline{f}} = \sum_\eta k_\eta[\eta]$, where η are the irreducible components of $\Gamma^1_{I,\underline{f}}$ then

$$\big(\Gamma^1_{I,\underline{f}} \cdot V(f_0 + ax_0^j)\big)_0 = \sum_\eta k_\eta\big(\eta \cdot V(f_0 + ax_0^j)\big)_0.$$

Let $\alpha_\eta(t)$ be a parametrization of η. Denoting by $\operatorname{mult}_t g(t)$ the lowest degree term of $g(t)$, we have

$$\begin{aligned}\big(\eta \cdot V(f_0 + ax_0^j)\big)_0 &= \operatorname{mult}_t(f_0 + ax_0^j)(\alpha_\eta(t))\\ &= \min\big\{\operatorname{mult}_t f_0(\alpha_\eta(t)), \operatorname{mult}_t ax_0^j(\alpha_\eta(t))\big\}\\ &= \min\big\{(\eta \cdot V(f_0))_0, (\eta \cdot V(x_0^j))_0\big\}.\end{aligned}$$

The second equality holds with the exception of, possibly, the single value of a which makes the lowest degree terms of $f_0(\alpha_\eta(t))$ and $ax_0^j(\alpha_\eta(t))$ add up to zero.

On the other hand, using that $(\eta \cdot V(x_0^j))_0 = j(\eta \cdot V(x_0))_0$, and since $j \geq (\eta \cdot V(f_0))_0/(\eta \cdot V(x_0))_0$, we have that $(\eta \cdot V(x_0^j))_0 \geq (\eta \cdot V(f_0))_0$.

Hence, $\big(\eta \cdot V(f_0 + ax_0^j)\big)_0 = (\eta \cdot V(f_0))_0$ and we conclude that

(2) $$\big(\Gamma^1_{I,\underline{f}} \cdot V(f_0 + ax_0^j)\big)_0 = \big(\Gamma^1_{I,\underline{f}} \cdot V(f_0)\big)_0 = \lambda^0_{I,\underline{f}}(0).$$

As sets,

$$\begin{aligned}V(I_0) = V(f_0 + ax_0^j, f_1, \dots, f_n) &= V(f_0 + ax_0^j) \cap V(f_1, \dots, f_n)\\ &= V(f_0 + ax_0^j) \cap \big(\Gamma^1_{I,\underline{f}} \cup V(I)\big)\\ &= \big(V(f_0 + ax_0^j) \cap \Gamma^1_{I,\underline{f}}\big) \cup \big(V(f_0 + ax_0^j) \cap V(I)\big)\\ &= \big(V(f_0 + ax_0^j) \cap \Gamma^1_{I,\underline{f}}\big) \cup (V(x_0) \cap V(I)).\end{aligned}$$

By equation (2), $V(f_0 + ax_0^j) \cap \Gamma^1_{I,\underline{f}}$ is 0-dimensional near the origin and, hence, near the origin,

(3) $$V(I_0) = V(x_0) \cap V(I).$$

Since $V(x_0)$ intersects $V(I)$ transversely at the origin,

$$\dim_0 V(I_0) = \dim_0 V(I) - 1.$$

Using the equation (3), we have that

$$\begin{aligned}\Gamma^i_{I_0,\underline{f}^{(0)}} &= V(f_{i+1},\dots,f_n,f_0+ax_0^j)/V(I_0)\\ &= V(f_{i+1},\dots,f_n)\cap V(f_0+ax_0^j)/(V(I)\cap V(x_0))\\ &= (\Gamma^{i+1}_{I,\underline{f}}\cup R)\cap V(f_0+ax_0^j)/(V(I)\cap V(x_0)),\end{aligned}$$

where the ideal defining the scheme R consists of the intersection of those primary components $\mathfrak{q}$ of any primary decomposition of the ideal $(f_{i+1},\dots,f_n)\mathcal{O}_{n+1}$, such that $|V(\mathfrak{q})|\subseteq|V(I)|$. Regardless of the primary decomposition, $|R|\subseteq|V(I)|$ and so $|R\cap V(f_0+ax_0^j)|\subseteq|V(I)\cap V(x_0)|$. Therefore,

$$\Gamma^i_{I_0,\underline{f}^{(0)}} = \big(\Gamma^{i+1}_{I,\underline{f}}\cap V(f_0+ax_0^j)\big)/(V(I)\cap V(x_0)).$$

By the number of equations, the dimension of any component of $\Gamma^{i+1}_{I,\underline{f}}\cap V(f_0+ax_0^j)$ is at least i.

On the other hand, as sets we have

$$\begin{aligned}\Gamma^{k+1}_{I,\underline{f}}\cap V(I) = \Gamma^{k+1}_{I,\underline{f}}\cap V(f_k)\cap V(I) &= \big(\Gamma^k_{I,\underline{f}}\cup|\Lambda^k_{I,\underline{f}}|\big)\cap V(I)\\ &= \big(\Gamma^k_{I,\underline{f}}\cap V(I)\big)\cup|\Lambda^k_{I,\underline{f}}|.\end{aligned}$$

By induction, this gives $\Gamma^{i+1}_{I,\underline{f}}\cap V(I)=\bigcup_{k\le i}|\Lambda^k_{I,\underline{f}}|$. Then, we have

$$\begin{aligned}\Gamma^{i+1}_{I,\underline{f}}\cap V(f_0+ax_0^j)\cap V(I)\cap V(x_0) &= \Gamma^{i+1}_{I,\underline{f}}\cap V(I)\cap V(x_0)\\ &= \bigcup_{k\le i}|\Lambda^k_{I,\underline{f}}|\cap V(x_0).\end{aligned}$$

By definition each $\Lambda^k_{I,\underline{f}}$ is k-dimensional at the origin and we have assumed $V(x_0)$ intersects Λ^k_I transversely at origin. Hence the dimension of $\bigcup_{k\le i}|\Lambda^k_{I,\underline{f}}|\cap V(x_0)$ is at most $i-1$.

Therefore, by [11, Lemma 1.5], $\Gamma^{i+1}_{I,\underline{f}}\cap V(f_0+ax_0^j)$ has no component contained in $V(I)\cap V(x_0)$ and we conclude that

$$\Gamma^i_{I_0,\underline{f}^{(0)}} = \Gamma^{i+1}_{I,\underline{f}}\cdot V(f_0+ax_0^j). \tag{4}$$

Therefore,

$$\begin{aligned}\lambda^0_{I_0,\underline{f}^{(0)}}(0) = \big(\Gamma^1_{I_0,\underline{f}^{(0)}}\cdot V(f_1)\big)_0 &= \big(\Gamma^2_{I,\underline{f}}\cdot V(f_0+ax_0^j)\cdot V(f_1)\big)_0\\ &= \big((\Gamma^1_{I,\underline{f}}+\Lambda^1_{I,\underline{f}})\cdot V(f_0+ax_0^j)\big)_0\\ &= \big(\Gamma^1_{I,\underline{f}}\cdot V(f_0+ax_0^j)\big)_0 + j\big(\Lambda^1_{I,\underline{f}}\cdot V(x_0)\big)_0\end{aligned}$$

and so, using equation (2),

$$\lambda^0_{I_0,\underline{f}^{(0)}}(0) = \lambda^0_{I,\underline{f}}(0) + j\lambda^1_{I,\underline{f}}(0).$$

Using equation (4), we have that

$$\begin{aligned}
\Gamma^i_{I_0,\underline{f}^{(0)}} + \Lambda^i_{I_0,\underline{f}^{(0)}} &= \Gamma^{i+1}_{I_0,\underline{f}^{(0)}} \cdot V(f_{i+1}) = \Gamma^{i+2}_{I,\underline{f}} \cdot V(f_0 + ax_0^j) \cdot V(f_{i+1}) \\
&= \big(\Gamma^{i+1}_{I,\underline{f}} + \Lambda^{i+1}_{I,\underline{f}}\big) \cdot V(f_0 + ax_0^j) \\
&= \big(\Gamma^{i+1}_{I,\underline{f}} \cdot V(f_0 + ax_0^j)\big) + j(\Lambda^{i+1}_{I,\underline{f}} \cdot V(x_0)) \\
&= \Gamma^i_{I_0,\underline{f}^{(0)}} + j\big(\Lambda^{i+1}_{I,\underline{f}} \cdot V(x_0)\big).
\end{aligned}$$

Cancelling $\Gamma^i_{I_0,\underline{f}^{(0)}}$ on each side of the equation, we have that

$$\Lambda^i_{I_0,\underline{f}^{(0)}} = j\big(\Lambda^{i+1}_{I,\underline{f}} \cdot V(x_0)\big).$$

Therefore,

$$\begin{aligned}
\lambda^i_{I_0,\underline{f}^{(0)}}(0) &= \big(\Lambda^i_{I_0,\underline{f}^{(0)}} \cdot V(x_1, \dots, x_i)\big)_0 \\
&= j\big(\Lambda^{i+1}_{I,\underline{f}} \cdot V(x_0) \cdot V(x_1, \dots, x_i)\big)_0 \\
&= j\lambda^{i+1}_{I,\underline{f}}.
\end{aligned}$$

Let I be an ideal of $\mathcal{O}_{n+1}$ and let $\underline{f} = (f_0, \dots, f_n)$ be a Vogel sequence of I at the origin. Set $s := \dim V(I)$. Suppose that $V(x_0)$ intersects transversely $V(I)$ and each $\Lambda^k_{I,\underline{f}}$ at the origin, for all $0 \le k \le s$. Then, by Theorem 2.2 we can choose a complex number a_0 and a positive integer j_0 such that the sequence $\underline{f}^{(0)} := (f_1, \dots, f_n, f_0 + a_0x_0^{j_0})$ is a Vogel sequence for the ideal $I_0 := (f_1, \dots, f_n, f_0 + a_0x_0^{j_0})\mathcal{O}_{n+1}$ and satisfies all properties of Theorem 2.2. Inductively, suppose we have found complex numbers $a_0, \dots, a_{i-1}$ and positive integers $j_0, \dots, j_{i-1}$ such that the sequence

$$\underline{f}^{(i-1)} := (f_i, \dots, f_n, f_0 + a_0x_0^{j_0}, \dots, f_{i-1} + a_{i-1}x_{i-1}^{j_{i-1}})$$

is a Vogel sequence for the ideal

$$I_{i-1} := (f_i, \dots, f_n, f_0 + a_0x_0^{j_0}, \dots, f_{i-1} + a_{i-1}x_{i-1}^{j_{i-1}})\mathcal{O}_{n+1}.$$

Then, supposing that $V(x_i)$ intersects $V(I_{i-1})$ and each $\Lambda^k_{I_{i-1},\underline{f}^{(i-1)}}$ at the origin transversely, for all $0 \le k \le s - i$, we can find a complex number a_i and a positive integer j_i such that the sequence $\underline{f}^{(i)} := (f_{i+1}, \dots, f_n, f_0 +$

$a_0x_0^{j_0}, \ldots, f_i+a_ix_i^{j_i})$ is a Vogel sequence for the ideal $I_i := (f_{i+1}, \ldots, f_n, f_0+a_0x_0^{j_0}, \ldots, f_i + a_ix_i^{j_i})\mathscr{O}_{n+1}$. We set $\underline{f}^{(-1)} = \underline{f}$ and $I_{-1} = I$.

COROLLARY 2.3. *Under the above construction, we have that* $\dim_0 V(I_{s-1}) = 0$ *and*

$$\dim_{\mathbb{C}} \frac{\mathscr{O}_{n+1}}{I_{s-1}} = \lambda^0_{I_{s-1},\underline{f}^{(s-1)}}(0) = \sum_{i=0}^{s}\left(\prod_{k=0}^{i-1} j_k\right)\lambda^i_{I,\underline{f}}(0).$$

PROOF. The second equality follows by recursive application of Theorem 2.2(v). Since $\dim_0 V(I_{s-1}) = 0$ and I_{s-1} is generated by $n+1$ elements in $\mathscr{O}_{n+1}$, which is an $n+1$-dimensional local Cohen-Macaulay ring, we have that

$$\dim_{\mathbb{C}} \frac{\mathscr{O}_{n+1}}{I_{s-1}} = e(I_{s-1}),$$

where $e(I_{s-1})$ is the Hilbert-Samuel multiplicity of the ideal I_{s-1}. The result follows since $e(I_{s-1}) = \lambda^0_{I_{s-1},\underline{f}^{(s-1)}}(0)$.

3. Ideals generated by weighted homogeneous germs

From now on, we focus on ideals generated by a specific class of germs called weighted homogeneous. We recall some basic definitions and results we will need in the sequel.

DEFINITION 3.1. Let $g\colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ in the coordinates $x_0, \ldots, x_n$. The germ g is *weighted homogeneous* if there exist positive integers $r_0, \ldots, r_n$, called the weights of g, and an integer d, called the degree of g, such that

$$g(\lambda^{r_0}x_0, \ldots, \lambda^{r_n}x_n) = \lambda^d g(x_0, \ldots, x_n).$$

In this case, we say that g is weighted homogeneous of type $(r_0, \ldots, r_n; d)$.

REMARK 3.2. Arnold, Gusein-Zade and Varchenko proved in [5, Theorem 12.3] the following result. Let $J = (h_0, \ldots, h_n)\mathscr{O}_{n+1}$, where $h_i\colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ are weighted homogeneous of type $(r_0, \ldots, r_n; d_i)$. Assume $\dim_{\mathbb{C}} \mathscr{O}_{n+1}/J < \infty$. Then,

$$\dim_{\mathbb{C}} \frac{\mathscr{O}_{n+1}}{J} = \frac{d_0 \cdots d_n}{r_0 \cdots r_n}.$$

We will determine the polar ratio of ideals generated by weighted homogeneous map germs.

In this section we let I be an ideal of $\mathscr{O}_{n+1}$ and let $\underline{f} = (f_0, \ldots, f_n)$ be a Vogel sequence of I at the origin, where each f_i is weighted homogeneous of type $(r_0, \ldots, r_n; D)$.

Let η be an irreducible component of $\Gamma^1_{I,\underline{f}}$. It is well known that any irreducible curve defined over a field of characteristic zero has a local parametrization. In this case, a local parametrization of η is of the form $\phi(t) = (\phi_0(t), \dots, \phi_n(t))$, where $\phi_j(t) = c_j t^{r_j}$, $c_j \neq 0$.

PROPOSITION 3.3. *The maximum polar ratio of I at 0 with respect to $\underline{f}$ is equal to D/r_0.*

PROOF. We may suppose that η intersects $V(x_0)$ transversely, because otherwise the polar ratio is 0. By [7, Example 7.1.17], $(\eta \cdot V(f_0))_0 = \operatorname{mult}_t f_0(\phi(t)) = D$ and $(\eta \cdot V(x_0))_0 = \operatorname{mult}_t x_0(\phi(t)) = r_0$. Hence, we have that

$$\frac{(\eta \cdot V(f_0))_0}{(\eta \cdot V(x_0))_0} = \frac{D}{r_0}.$$

Hence the maximum polar ratio of I at 0 with respect to $\underline{f}$ is equal to D/r_0.

PROPOSITION 3.4. *Assume that r_i divides D for each $i = 0, \dots, s-1$, where $s = \dim_0 V(I)$. Let $\underline{f}^{(i)} = (f_{i+1}, \dots, f_n, f_0 + a_0 x_0^{D/r_0}, \dots, f_i + a_i x_i^{D/r_i})$ and let*

$$I_i = (f_{i+1}, \dots, f_n, f_0 + a_0 x_0^{D/r_0}, \dots, f_i + a_i x_i^{D/r_i})\mathcal{O}_{n+1}.$$

Suppose that $V(x_i)$ intersects $V(I_{i-1})$ and each $\Lambda^k_{I_{i-1},\underline{f}^{(i-1)}}$ at 0 transversely, for all $0 \le i \le s-1$ and $k \le s-1$. Then, the maximum polar ratio of I_i at 0 with respect to $\underline{f}^{(i)}$ is equal to D/r_i.

PROOF. Since $\Gamma^i_{I,\underline{f}}$ is i-dimensional and $V(x_0, \dots, x_{i-1})$ intersects $\Gamma^{i+1}_{I,\underline{f}}$ transversally for $i \ge 1$, by Theorem 2.2, $\Gamma^1_{I_i,\underline{f}^{(i)}}$ is 1-dimensional and the proof follows similarly to that of the previous result.

3.1. A Plücker type formula

In this section we let I be an ideal of $\mathcal{O}_{n+1}$ which admits a Vogel sequence $\underline{f} = (f_0, \dots, f_n)$ at the origin, with each f_i being weighted homogeneous of type $(r_0, \dots, r_n; D)$, where r_i divide D for $i = 0, \dots, n$. We shall show how to describe a Plücker type formula associated to the Segre numbers at the origin of I in terms of their weights and degree.

THEOREM 3.5. *Keeping the notation of Proposition 3.4, suppose that $V(x_i)$ intersects $V(I_{i-1})$ and each $\Lambda^k_{I_{i-1},\underline{f}^{(i-1)}}$ transversely at 0 for all $0 \le i \le s-1$ and $k \le s-1$, where $\dim_0 V(I) = s$. Then*

$$\frac{D^{n+1}}{r_0 r_1 \cdots r_n} = \sum_{i=0}^{s} \left(\frac{D^i}{r_0 r_1 \cdots r_{i-1}} \right) \lambda^i_{I,\underline{f}}(0).$$

PROOF. We apply the Lê-Iomdine formulas (Theorem 2.2) for I and use that the maximal polar ratio of I is $j = D/r_0$, by Proposition 3.3. This gives an ideal

$$I_0 = \langle f_1, \dots, f_n, f_0 + a_0 x_0^{D/r_0} \rangle$$

for which $V(I_0)$ has dimension one less than the dimension of $V(I)$. Similarly, applying again Lê-Iomdine formulas inductively for each I_i, $i \geq 1$, using $j = D/r_i$, we get ideals

$$I_{i+1} = \langle f_{i+1}, \dots, f_n, f_0 + a_0 x_0^{D/r_0}, \dots, f_i + a_i x_i^{D/r_i} \rangle$$

whose $V(I_{i+1})$ have dimension one less than the dimension of $V(I_i)$.

Since $V(I_{s-1})$ is 0-dimensional at 0, using Remark 3.2 and Corollary 2.3 for I_{s-1}, we have

$$\frac{D^{n+1}}{r_0 r_1 \cdots r_n} = \sum_{i=0}^{s} \left(\frac{D^i}{r_0 r_1 \cdots r_{i-1}} \right) \lambda^i_{I,\underline{f}}(0).$$

4. The homogeneous map germ case

Let $h\colon \mathbb{C}^{n+1} \to \mathbb{C}^p$ given by $h = (h_1, \cdots, h_p)$, where $h_i \in \mathbb{C}\{x_0, \cdots, x_n\}$ are homogeneous map germs of degree d_i. We assume $p \leq n$.

Consider J the Jacobian matrix of h, that is,

$$J = \begin{pmatrix} \dfrac{\partial h_1}{\partial x_0} & \cdots & \dfrac{\partial h_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial h_p}{\partial x_0} & \cdots & \dfrac{\partial h_p}{\partial x_n} \end{pmatrix}$$

Denote by I the ideal generated by the $p \times p$ minors of the matrix J. Note that $V(I)$ is the critical set $\Sigma(h)$ of h.

REMARK 4.1. Let $(g_1, \dots, g_N)$ be a sequence of generators of I, where $N = \binom{n+1}{p}$ and g_i are the $p \times p$ minors of the matrix J chosen in a suitable order. Each g_i is written as the sum, up to sign, of the monomials:

$$\frac{\partial h_1}{\partial x_{i_1}} \frac{\partial h_2}{\partial x_{i_2}} \cdots \frac{\partial h_p}{\partial x_{i_p}}.$$

Then g_i is homogeneous of degree $D = \left(\sum_{i=1}^{p} d_i\right) - p$.

Let $\underline{f} = (f_0, \ldots, f_n)$ be a Vogel sequence for I at the origin, constructed via a generic $(n+1) \times N$ matrix A, as in Remark 1.1. We denote $\Gamma^k_{I,\underline{f}}$ and $\Lambda^k_{I,\underline{f}}$ by $\Gamma^k_{h,\underline{f}}$ and $\Lambda^k_{h,\underline{f}}$ respectively.

In general, the Segre cycles depends on the choice of the Vogel sequence $\underline{f}$. However, their multiplicities at 0 is independent of the choice of $\underline{f}$. In fact, Gaffney and Gassler in [8, (3.2)] gave intersection formulas for the Segre numbers that are independent of the choice of a generic Vogel sequence $\underline{f}$. This motivates the following definition.

DEFINITION 4.2. The *i-th Segre number* of the map germ h at the origin, denoted by $\lambda^i_h(0)$, is defined as the i-th Segre number of I at the origin with respect to a generic Vogel sequence $\underline{f}$. That is, $\lambda^i_h(0) = \text{mult}_0\, \Lambda^k_{h,\underline{f}}$

Notice that, when $p = 1$ and h has an isolated singularity at the origin, the Segre number $\lambda^0_h(0)$ is the Milnor number of h.

Since each generator g_i of I is homogeneous of degree $D = \left(\sum_{i=1}^{p} d_i\right) - p$, we have a Plücker type formula for the Segre numbers $\lambda^i_h(0)$, given by Theorem 3.5. Precisely we have

$$D^{n+1} = \sum_{i=0}^{s} D^i \lambda^i_h(0).$$

The next proposition describes how the Segre numbers behave under hyperplane sections, which is a fundamental result.

PROPOSITION 4.3. *Suppose $\Sigma(h) \cap V(x_0) = \Sigma(h|_{V(x_0)})$ and use the coordinates $\tilde{x} = (x_1, \ldots, x_n)$ for $h|_{V(x_0)}$. Then $\lambda^k_{h|_{V(x_0)}}(0) = \lambda^{k+1}_h(0)$.*

PROOF. Write $h_j(x_0, \ldots, x_n) = x_0 H_j(x_0, \ldots, x_n) + \tilde{h}_j(x_1, \ldots, x_n)$ for $j = 0, \ldots, p$. Then, $h|_{V(x_0)} \colon \mathbb{C}^n \to \mathbb{C}^p$ is given by $h|_{V(x_0)} = (\tilde{h}_1, \ldots, \tilde{h}_p)$. Furthermore, the Jacobian matrices of h and of $h|_{V(x_0)}$ are given by

$$J(h) = \begin{pmatrix} H_1 + x_0\dfrac{\partial H_1}{\partial x_0} & x_0\dfrac{\partial H_1}{\partial x_1} + \dfrac{\partial \tilde{h}_1}{\partial x_1} & \cdots & x_0\dfrac{\partial H_1}{\partial x_n} + \dfrac{\partial \tilde{h}_1}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ H_p + x_0\dfrac{\partial H_p}{\partial x_0} & x_0\dfrac{\partial H_p}{\partial x_1} + \dfrac{\partial \tilde{h}_p}{\partial x_1} & \cdots & x_0\dfrac{\partial H_p}{\partial x_n} + \dfrac{\partial \tilde{h}_p}{\partial x_n} \end{pmatrix}$$

and

$$J(h|_{V(x_0)}) = \begin{pmatrix} \frac{\partial \tilde{h}_1}{\partial x_1} & \cdots & \frac{\partial \tilde{h}_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial \tilde{h}_p}{\partial x_1} & \cdots & \frac{\partial \tilde{h}_p}{\partial x_n} \end{pmatrix}.$$

Let $g_1, \ldots, g_m$ be all the $p \times p$ minors of $J(h)$ which involve its first column, where $m = \binom{n}{p-1}$, and let $g_{m+1}, \ldots, g_N$ be the remaining $p \times p$ minors of $J(h)$, with $N = \binom{n+1}{p}$. We use here the lexicographical order for the sequence $(g_0, \ldots, g_m, g_{m+1}, \ldots, g_N)$.

Let I be the ideal of $\mathcal{O}_{n+1}$ generated by $g_0, \ldots, g_m, g_{m+1}, \ldots, g_N$. Let $\underline{f} = (f_0, \ldots, f_n)$ be a generic Vogel sequence for the ideal I at the origin. Write $f_j = \sum_{r=1}^m a_{j,k} g_k + \sum_{q=1}^M b_{j,q} g_{m+q}$, where $M = \binom{n}{p}$. Then, each $\Gamma^k_{h,\underline{f}}$ is purely k-dimensional at the origin and thus has no embedded components (see [11, Proposition 1.7]).

For the same reason, $\Gamma^{k+1}_{h,\underline{f}} \cap V(f_k)$ is purely k-dimensional at the origin and thus, by [11, Proposition 1.16] or [7, 7.1b], we have an equality of cycles

$$\left[\Gamma^{k+1}_{h,\underline{f}} \cap V(f_k)\right] = \Gamma^{k+1}_{h,\underline{f}} \cdot V(f_k) = \Gamma^k_{h,\underline{f}} + \Lambda^k_{h,\underline{f}}.$$

In addition, using the transversely condition of $V(x_0)$, we see that $\Gamma^{k+1}_{h,\underline{f}} \cap V(f_k) \cap V(x_0)$ is purely $(k-1)$-dimensional at the origin. Hence

$$\dim_0\bigl(\Gamma^{k+1}_{h,\underline{f}} \cap \Sigma(h) \cap V(x_0)\bigr) \le k-1.$$

Let $(\tilde{g}_1, \ldots, \tilde{g}_M)$ be all the $p \times p$ minors of the matrix $J(h|_{V(x_0)})$, ordered lexicographically. Let $\tilde{I}$ be the ideal of $\mathcal{O}_n$ generated by $\tilde{g}_1, \ldots, \tilde{g}_M$, which we may assume to be reduced. Notice that

$$\tilde{g}_q = g_{m+q}(0, x_1, \ldots, x_n)$$

for all $q = 1, \ldots, M$.

Since by assumption $V(\tilde{I}) = V(I, x_0)$ as sets, and $V(\tilde{I})$ is reduced, we have that $g_1(0, x_1, \ldots, x_n), \ldots, g_m(0, x_1, \ldots, x_n) \in \sqrt{\tilde{I}} = \tilde{I}$. Hence,

$$g_r(0, x_1, \ldots, x_n) = \sum_{q=1}^{M} c_{r,q} \tilde{g}_q,$$

for all $r = 1, \ldots, m$, where $c_{r,q}$ are complex numbers, since all the g_i and $\widetilde{g_q}$ are homogeneous of the same degree D.

Define

$$\tilde{f}_j = \sum_{q=1}^{M}\Big[\Big(\sum_{r=1}^{m} a_{j,r}c_{r,q}\Big) + b_{j,q}\Big]\tilde{g}_q,$$

for all $j = 1, \dots, n$. Then, by Remark 1.1, $\underline{\tilde{f}} := (\tilde{f}_1, \dots, \tilde{f}_n)$ is a generic Vogel sequence for $\tilde{I}$ at the origin.

Now, consider the cycle $\Gamma^k_{h|_{V(x_0)},\underline{\tilde{f}}}$. By definition,

$$\Gamma^k_{h|_{V(x_0)},\underline{\tilde{f}}} = V(x_0, \tilde{f}_{k+1}, \dots, \tilde{f}_n)/\Sigma(h_{|_{V(x_0)}}).$$

Using [11, Lemma 1.2 i)] and the assumption that $\Sigma(h) \cap V(x_0) = \Sigma(h_{|_{V(x_0)}})$, we have

$$\Gamma^k_{h|_{V(x_0)},\underline{\tilde{f}}} = \big(V(x_0) \cap \Gamma^{k+1}_{h,\underline{f}}\big)/(\Sigma(h) \cap V(x_0)) = \big(V(x_0) \cap \Gamma^{k+1}_{h,\underline{f}}\big)/\Sigma(h).$$

However, $V(x_0) \cap \Gamma^{k+1}_{h,\underline{f}}$ is purely k-dimensional at the origin and $\dim_0\big(\Gamma^{k+1}_{h,\underline{f}} \cap \Sigma(h) \cap V(x_0)\big) \le k-1$. Hence, $\Sigma(h)$ has no isolated components of $V(x_0) \cap \Gamma^{k+1}_{h,\underline{f}}$ and thus, as cycles,

$$\begin{aligned}
\Gamma^k_{h|_{V(x_0)},\underline{\tilde{f}}} &= \Gamma^{k+1}_{h,\underline{f}} \cap V(x_0) = \Gamma^{k+1}_{h,\underline{f}} \cdot V(x_0),\\
\Gamma^{k-1}_{h|_{V(x_0)},\underline{\tilde{f}}} + \Lambda^{k-1}_{h|_{V(x_0)},\underline{\tilde{f}}} &= \Gamma^k_{h|_{V(x_0)},\underline{\tilde{f}}} \cdot V(f_k) = \Gamma^{k+1}_{h,\underline{f}} \cdot V(x_0) \cdot V(f_k)\\
&= (\Gamma^k_{h,\underline{f}} + \Lambda^k_{h,\underline{f}}) \cdot V(x_0).
\end{aligned}$$

Since Λ^i and Γ^i are disjoint for all i, we have that $\Lambda^{k-1}_{h|_{V(x_0)},\underline{\tilde{f}}} = \Lambda^k_{h,\underline{f}} \cdot V(x_0)$ and we obtain the result.

5. Segre numbers of hyperplane arrangement map germs

Consider $h\colon \mathbb{C}^{n+1} \to \mathbb{C}^p$ with $p \le n$, $h = (h_1, \dots, h_p)$, where $h_i \in \mathbb{C}\{x_0, \dots, x_n\}$ are an hyperplane arrangements, that is, each $h_i = f_{1,i}^{m_{1,i}} \cdots f_{r_i,i}^{m_{r_i,i}}$ with $H_{j,i} = V(f_{j,i})$ a hyperplane in $\mathbb{C}^{n+1}$.

Suppose that $V(h)$ is a complete intersection. Since $V(h) = \bigcap_{i=1}^p V(h_i)$, a Whitney stratification of $V(h)$ is given as follows: for each $\underline{J} = (J_1, \dots, J_p)$, with $J_i \subseteq \{1, \dots, r_i\}$, set $w_{\underline{J}} = \bigcap_{i=1}^p w_{J_i}$, where $w_{J_i} = \bigcap_{j\in J_i} H_{j,i}$, and put $S_{\underline{J}} = w_{\underline{J}} \setminus \bigcup_{\underline{J}<\underline{K}} w_K$, where $\underline{J} < \underline{K}$ means that $J_i \subseteq K_i$ for all $i = 1, \dots, p$ but $\underline{J} \neq \underline{K}$ as p-tuples.

Define

$$e(w_{\underline{J}}) = \sum_{i=1}^{p}\sum_{j\in J_i} m_{j,i}.$$

Next we define the vanishing Möbius function, η, by downward induction on the dimension of $w_{\underline{J}}$.

For $\underline{J} = (\{j_1\}, \dots, \{j_p\})$, define

$$\eta(w_{\underline{J}}) = (e(w_{\underline{J}}) - p)^p.$$

For a smaller dimensional $w_{\underline{J}}$ define

$$\eta(w_{\underline{J}}) = (e(w_{\underline{J}}) - p)^{n+1-\dim w_{\underline{J}}} - \sum_{\underline{K}<\underline{J}} \eta(w_{\underline{K}}) \cdot (e(w_{\underline{J}}) - p)^{\dim(w_{\underline{K}})-\dim(w_{\underline{J}})}.$$

THEOREM 5.1. *$\lambda_h^k(0) = \sum_{\dim S_{\underline{J}}=k} \eta(w_{\underline{J}})$, for all $k = 0, \dots, n+1-p$.*

PROOF. Notice that as sets $\Gamma_{h,\underline{f}}^{k+1} \cap \Sigma(h) = \bigcup_{i\le k} |\Lambda_{h,\underline{f}}^i|$. In fact, we can prove this by induction using the formula

$$\begin{aligned}\Gamma_{h,\underline{f}}^{k+1} \cap \Sigma(h) &= \Gamma_{h,\underline{f}}^{k+1} \cap V(f_k) \cap \Sigma(h) = \left(\Gamma_{h,\underline{f}}^{k} \cup |\Lambda_{h,\underline{f}}^{k}|\right) \cap \Sigma(h)\\ &= \left(\Gamma_{h,\underline{f}}^{k} \cap \Sigma(h)\right) \cup |\Lambda_{h,\underline{f}}^{k}|.\end{aligned}$$

In particular, $\Sigma(h) = \bigcup_{i\le(n+1-p)} |\Lambda_{h,\underline{f}}^i|$.

On the other hand, as set at 0, $\Sigma(h) = \bigcup S_{\underline{J}} = \bigcup_{k=0}^{n+1-p}\left(\bigcup_{\dim w_{\underline{J}}=k} w_{\underline{J}}\right)$. Hence, as sets at 0, $|\Lambda_{h,\underline{f}}^k| = \bigcup_{\dim w_{\underline{J}}=k} w_{\underline{J}}$. Therefore, the Segre cycles are given by

$$\Lambda_{h,\underline{f}}^k = \sum_{\dim w_{\underline{J}}=k} a_{\underline{J}}[w_{\underline{J}}],$$

for some $a_{\underline{J}}$.

By Proposition 4.3, $a_{\underline{J}}$ may be calculated by taking any $q \in S_{\underline{J}}$ and a normal slice $N_{\underline{J}}$ to $S_{\underline{J}}$ in $\mathbb{C}^{n+1}$ at q, giving $a_{\underline{J}} = \lambda_{h|_{N_{\underline{J}}}}^0(q)$.

After a translation making the point q the origin, we see that $h|_{N_{\underline{J}}}$ at q is again (up to multiplication by units) a p-tuple of product of linear forms of degree $\sum_{j\in J_i} m_{j,i}$, for $i = 1, \dots, p$. Therefore

$$\lambda_h^{n+1-p}(0) = \sum_{\dim w_{\underline{J}}=n+1-p} a_J = \sum_{\dim w_{\underline{J}}=n+1-p} \lambda_{h|_{N_{\underline{J}}}}^0(0).$$

Notice that if $\dim w_{\underline{J}} = n+1-p$ then $\underline{J} = (\{j_1\}, \dots, \{j_p\})$. In this case, up to multiplication by units on each term, $h|_{N_{\underline{J}}} = (f_{j_1,1}^{m_{j_1,1}}, \dots, f_{j_p,p}^{m_{j_p,p}})$.

By Remark 4.1, $\Sigma(h|_{N_{\underline{J}}})$ is defined by homogeneous polynomials of degree $\left(\sum_{i=1}^p m_{j_i,i}\right) - p = e(w_{\underline{J}}) - p$.

Applying Proposition 3.5 for $h|_{N_{\underline{J}}}$ at the origin, we have $\lambda^0_{h|_{N_{\underline{J}}}}(0) = (e(w_{\underline{J}})-p)^p = \eta(w_{\underline{J}})$. Therefore

$$\lambda_h^{n+1-p}(0) = \sum_{\dim w_{\underline{J}}=n+1-p} \eta(w_{\underline{J}}).$$

Analogously $\lambda_h^{n-p}(0) = \sum_{\dim w_{\underline{J}}=n-p} a_J = \sum_{\dim w_{\underline{J}}=n-p} \lambda^0_{h|_{N_{\underline{J}}}}(0)$, where, up to multiplication by units on each term, $h|_{N_{\underline{J}}}$ is a p-tuple of polynomials in $p+1$ variables with 1-dimensional critical set. Also, by similar arguments, $\Sigma(h|_{N_{\underline{J}}})$ is defined by homogeneous polynomials of degree $e(w_{\underline{J}}) - p$.

Now if we again apply Proposition 3.5 for $h|_{N_{\underline{J}}}$ at the origin, then we obtain

$$\lambda^0_{h|_{N_{\underline{J}}}}(0) + (e(w_J) - p)\lambda^1_{h|_{N_{\underline{J}}}}(0) = (e(w_J) - p)^{p+1}.$$

By Proposition 4.3, $\lambda^1_{h|_{N_{\underline{J}}}}(0) = \lambda^0_{h|_{N_{\underline{K}}}}(0)$, where $\underline{K} = (J_1 \cup \{I_0\}, \ldots, J_p \cup \{i_p\})$. On the other hand, using the first step,

$$\lambda^0_{h|_{N_{\underline{K}}}}(0) = \sum_{\substack{w_K \supsetneq w_J \\ \dim w_K = n-p+1}} \eta(w_{\underline{K}})$$

Hence,

$$\lambda^0_{h|_{N_{\underline{J}}}}(0) = (e(w_{\underline{J}}) - p)^{p+1} - \sum_{\substack{w_K \supsetneq w_J \\ \dim w_K = n-p+1}} \eta(w_{\underline{K}}) \cdot (e(w_{\underline{J}}) - p) = \eta(w_{\underline{J}}).$$

Therefore

$$\lambda_h^{n-p}(0) = \sum_{\dim w_{\underline{J}}=n-p} \eta(w_{\underline{J}}).$$

Proceeding inductively we obtain

$$\lambda_h^k(0) = \sum_{\dim S_{\underline{J}}=k} \eta(w_{\underline{J}}), \qquad \text{for all} \quad k = 0, \ldots, n+1-p.$$

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