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A DEFORMATION OF THE ORLIK-SOLOMON ALGEBRA

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Abstract

A deformation of the Orlik-Solomon algebra of a matroid $\mathfrak{M}$ is defined as a quotient of the free associative algebra over a commutative ring R with 1. It is shown that the given generators form a Gröbner basis and that after suitable homogenization the deformation and the Orlik-Solomon have the same Hilbert series as R-algebras. For supersolvable matroids, equivalently fiber type arrangements, there is a quadratic Gröbner basis and hence the algebra is Koszul.

1. Introduction and statement of results

In this paper we introduce and study a deformation of the Orlik-Solomon algebra of a matroid $\mathfrak{M}$. We refer the reader to [6] and [12] for general facts about the classical Orlik-Solomon algebra. Our deformation, which is different from the one in [10], is presented as a quotient of the free associative algebra over some commutative ring R with 1 by an ideal $I_q(\mathfrak{M})$ whose generators are deformations of the classical generators of the defining ideal of the Orlik-Solomon algebra by a parameter $q \in R$ in terms of the combinatorics of circuits of the matroid. Choosing $q = 0$ yields the Orlik-Solomon algebra over R. Our main result, Theorem 1.1, states that the given generators of $I_q(\mathfrak{M})$ are a Gröbner basis of the ideal. As a consequence it is shown in Corollary 1.3 that the deformation with q regarded as a degree 2 element is a standard graded R-algebra which has the same Hilbert series as the Orlik-Solomon algebra. It can be shown that under weak assumptions our deformation is the only deformation of the Orlik-Solomon algebra with this property in general. Moreover, the fact that the parameter q must be chosen of degree 2 reveals interesting combinatorics. For supersolvable matroids, equivalently fiber type arrangements, the existence of a quadratic Gröbner basis is shown which implies that the algebra is Koszul generalizing known facts for the classical Orlik-Solomon algebra. As further consequences we obtain in Corollary 1.5 a known Gröbner basis for the Orlik-Solomon algebra as a quotient of the free and the exterior algebra. The remaining part of the introduction is devoted to the basic definitions and statement of results. In Section 2 basic facts about non-commutative

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Gröbner basis theory are given. Section 3 provides technical lemmas needed for the proof of the main result. Finally in Section 4 the missing proofs are given and an independence statement is presented.

Let S be a finite set and fix a total order $<$ on S. Let R be a commutative ring with unit 1 and let $q \in R$. For an arbitrary set system $\mathfrak{M} \subseteq 2^S$, we define a two-sided ideal $I_q(\mathfrak{M})$ in the ring

$$A := R\langle t_s \mid s \in S\rangle$$

of non-commutative polynomials in the variables t_s, $s \in S$, with coefficients in R.

Let $\mathscr{I}_q(\mathfrak{M})$ denote the subset of A consisting of the elements

$$t_s^2 - q, \qquad s \in S, \tag{1.1}$$

$$t_r t_s + t_s t_r - 2q, \qquad r, s \in S, s < r, \tag{1.2}$$

$$t_J^- := \sum_{I \subseteq J, 2 \nmid \# I} (-1)^{\ell_J(I)} (-q)^{(\# I - 1)/2} t_{J \setminus I}, \qquad \text{for all} \quad J \in \mathfrak{M}. \tag{1.3}$$

Here, for a subset $I = \{j_{\alpha_1} < \cdots < j_{\alpha_{\# I}}\} \subseteq J = \{j_1 < \cdots < j_{\# J}\}$ we set

$$\ell_J(I) = \sum_{\nu=1}^{\# I} (\alpha_\nu - \nu).$$

The two-sided ideal of A generated by $\mathscr{I}_q(\mathfrak{M})$ will be denoted by $I_q(\mathfrak{M})$. We write $\mathrm{OS}_q(\mathfrak{M})$ for the quotient $A/I_q(\mathfrak{M})$. Our main motivation comes from the situation when $q = 0$ and $\mathfrak{M}$ is indeed the set of circuits of a loopless matroid without parallel elements. We refer the reader to the books [11] and [8] as a general reference for matroid theory and recall that for loopless matroids without parallel elements the set of circuits $\mathfrak{M}$ is characterized by the following three axioms:

- (C1) $J \in \mathfrak{M} \Rightarrow \# J > 2$.
- (C2) $J, K \in \mathfrak{M}$, $J \subseteq K \Rightarrow J = K$.
- (C3) For any $J, K \in \mathfrak{M}$ such that $J \neq K$ and for any $x \in J \cap K$ there exists $L \in \mathfrak{M}$ such that $L \subseteq (J \cup K) \setminus \{x\}$.

We will refer to (C3) also by the name *circuit axiom*.

Now if $\mathfrak{M}$ is the set of circuits of a loopless matroid without parallel elements then for $q = 0$ the algebra $\mathrm{OS}_q(\mathfrak{M})$ is the *Orlik-Solomon algebra* of $\mathfrak{M}$. In case $\mathfrak{M}$ is realizable as the set of circuits of a finite set of hyperplanes in $\mathbb{C}^d$ then the Orlik-Solomon algebra of $\mathfrak{M}$ is known to be the cohomology algebra of the set-theoretic complement of the union of the hyperplanes [7, (5.2)].

In general, for $q \neq 0$ the algebra $\mathrm{OS}_q(\mathfrak{M})$ is not isomorphic to the Orlik-Solomon algebra of $\mathfrak{M}$, take for example S with $\#S = 1$. On the other hand, if q is a square and a unit in R, then $\mathrm{OS}_q(\mathfrak{M})$ is easily seen to be isomorphic to $\mathrm{OS}_1(\mathfrak{M})$.

Before we can proceed to the statement of our main results we need some more definitions. Throughout this paper we will use the *degree lexicographic* order $<_{\mathrm{dlex}}$ (see Section 2) on the monomials in A induced by the total order $<$ on S as a term order for the monomials in A. We enumerate the elements of any subset $J \subseteq S$ by $j_1, \ldots, j_{\#J}$ such that $j_1 < \cdots < j_{\#J}$. For any $J \subseteq S$ we write t_J for the monomial $t_{j_1} \cdots t_{j_{\#J}} \in A$. The degree lexicographic order on the monomials t_J induces the degree lexicographic order on subsets of S; that is for subsets $J, K \subseteq S$ we set $J <_{\mathrm{dlex}} K$ if and only if $\#J < \#K$ or $\#J = \#K$ and the minimum of the symmetric difference of J and K is contained in J. In addition, for two subsets $K, J \subseteq S$ we say that K is a *convex subset of* J if $K \subseteq J$ and if $j \in K$ for all $j \in J$ with $k < j < k'$ for some $k, k' \in K$. We write $K \sqsubseteq J$ in this situation. Recall that a subset $J \subseteq S$ is called *dependent* in $\mathfrak{M}$ if it contains a circuit from $\mathfrak{M}$. Let $\overline{\mathfrak{M}}$ be the set of dependent subsets of S defined recursively as follows:

(GC) A dependent set $J \subseteq S$ belongs to $\overline{\mathfrak{M}}$ if and only if $K \in \overline{\mathfrak{M}}$ with $K <_{\mathrm{dlex}} J$ implies that $K \setminus \{k_1\} \not\sqsubseteq J \setminus \{j_1\}$.

Then we call $\overline{\mathfrak{M}}$ the set of *Gröbner circuits* of $\mathfrak{M}$.

Observe that the definition of $\overline{\mathfrak{M}}$ depends crucially on the chosen total order on S. For example, if $S = \{1, 2, 3, 4\}$ and $\mathfrak{M} = \{\{1, 2, 3\}\}$ then we obtain that $\overline{\mathfrak{M}} = \{\{1, 2, 3\}\}$. However, if $\mathfrak{M} = \{\{1, 2, 4\}\}$ then $\overline{\mathfrak{M}} = \{\{1, 2, 4\}, \{1, 2, 3, 4\}\}$ since $\{2, 4\} \not\sqsubseteq \{2, 3, 4\}$.

Our requirement that $\mathfrak{M}$ is a loopless matroid without parallel elements is of purely technical nature. If $\mathfrak{M}$ contains loops then $\overline{\mathfrak{M}}$ will contain exactly one loop. For this loop (1.3) reduces to 1 and $I_q(\mathfrak{M}) = I_q(\overline{\mathfrak{M}})$ is the full polynomial ring. If $\mathfrak{M}$ contains parallel elements then (1.3) implies that the corresponding variables are identified modulo $I_q(\mathfrak{M})$. Note that then $\mathrm{OS}_q(\mathfrak{M})$ is isomorphic to the algebra corresponding to the matroid obtained from $\mathfrak{M}$ by deleting all but one from a maximal set of pairwise parallel elements. If $\mathfrak{M}$ contains no loops then (GC) guarantees that $\overline{\mathfrak{M}}$ contains sufficient two element subsets to force that variables corresponding to parallel elements are also identified modulo $I_q(\overline{\mathfrak{M}})$. Thus Gröbner bases for $I_q(\mathfrak{M})$ for matroids with loops or parallel elements are either trivial or can be found by simple extensions of Gröbner bases for loopless matroid without parallel elements. Since our proofs will require (C1) at several points we therefore confine ourselves to this situation.

THEOREM 1.1. *Let $\overline{\mathfrak{M}}$ be a set of Gröbner circuits of a loopless matroid $\mathfrak{M}$ without parallel elements. Then $I_q(\mathfrak{M}) = I_q(\overline{\mathfrak{M}})$ and the set $\mathcal{I}_q(\overline{\mathfrak{M}})$ is a Gröbner basis of the ideal $I_q(\mathfrak{M})$ with respect to the degree lexicographic order for all choices of q.*

The Gröbner basis is easily seen to depend on the total order chosen on S but in Theorem 4.1 we show that $I_q(\mathfrak{M})$ and hence $\mathrm{OS}_q(\mathfrak{M})$ is independent of the order on S. Simple inspection shows that the leading monomial of (1.1) is t_s^2, of (1.2) is $t_r t_s$ for $s < r$ and of (1.3) is $t_{J \setminus \{j_1\}}$. Thus all leading coefficients of $\mathcal{I}_q(\overline{\mathfrak{M}})$ are 1. The standard monomials with respect to the Gröbner basis $\mathcal{I}_q(\overline{\mathfrak{M}})$ are the monomials $m = t_J$ for $J \subseteq S$ for which there is no factorization $m = m_1 t_{K'} m_2$ for monomials m_1, m_2 where K' is a *broken Gröbner circuit*; that is there is a Gröbner circuit K for which $K' = K \setminus \{k_1\}$. By the definition of a Gröbner circuit it then follows that the standard monomials with respect to the Gröbner basis $\mathcal{I}_q(\overline{\mathfrak{M}})$ are the monomials $m = t_J$ such that J does not contain a *broken circuit* of $\mathfrak{M}$; that is a circuit with its least element removed. These facts immediately imply:

COROLLARY 1.2. *Let $\mathfrak{M}$ be a loopless matroid without parallel elements. Then the algebra $\mathrm{OS}_q(\mathfrak{M})$ is a free R-module whose rank is independent of q.*

The standard monomials with respect to the Gröbner basis $\mathcal{I}_q(\overline{\mathfrak{M}})$ are the monomials $m = t_J$ for $J \subseteq S$ for which J does not contain a broken circuit.

The algebra $\mathrm{OS}_q(\mathfrak{M})$ is $\mathsf{Z}/2\mathsf{Z}$-graded for the grading induced by $\deg t_s = 1$ for all $s \in S$. If R is Z-graded and $t_0 \in R$ is homogeneous of degree one then the Z-grading of R extends to a Z-grading of $\mathrm{OS}_q(\mathfrak{M})$ for $q = t_0^2$. We consider the case $R = Q[t_0]$ for some commutative ring Q with 1 and extend the total order on the variables by setting t_0 to be the least variable. Then we consider $I_q(\mathfrak{M})$ as an ideal in $Q\langle t_s \mid s \in S\rangle[t_0]$. We deduce from Theorem 1.1, standard facts about homogenizing Gröbner bases (see [4, Thm. 3.7]) and Corollary 1.2 the following corollary.

COROLLARY 1.3. *Let $\mathfrak{M}$ be a loopless matroid without parallel elements, $R = Q[t_0]$ for a commutative ring Q with 1 and t_0 a degree one variable. If we set $q = t_0^2$ then*

(1) *the set $\mathcal{I}_q(\overline{\mathfrak{M}})$ is a Gröbner basis of $I_q(\mathfrak{M})$ for the degree lexicographic order with t_0 being the least variable. The algebra $\mathrm{OS}_q(\mathfrak{M})$ is a free Q-module and a standard graded Q-algebra.*

(2) *The Hilbert series of $\mathrm{OS}_q(\mathfrak{M})$ as a Q-algebra is*

$$\frac{1 + c_1 z + \cdots + c_{\mathrm{rk}(\mathfrak{M})} z^{\mathrm{rk}(\mathfrak{M})}}{1 - z} = \mathrm{Hilb}_{\mathfrak{M}}(z) \cdot \frac{1}{1 - z},$$

where $\mathrm{rk}(\mathfrak{M})$ *is the rank of* $\mathfrak{M}$, c_i *is the number of subsets of* S *of cardinality* i *not containing a broken circuit and* $\mathrm{Hilb}_{\mathfrak{M}}(z)$ *the Hilbert series of the Orlik-Solomon algebra of* $\mathfrak{M}$ *as an algebra over a field* k.

Note that the first part together with the second part of Corollary 1.2 implies that the standard monomials of the Gröbner basis are $q^i t_J$ for $i \geq 0$ and $J \subset S$ such that J does not contain a broken circuit. Part (2) of Corollary 1.3 hence follows by a simple counting argument and standard facts about Orlik-Solomon algebras.

We note that experiments suggest that the generators (1.1), (1.2) and (1.3) are the unique deformations of the corresponding polynomials for $q = 0$ by variables of degree ≥ 1 satisfying Corollary 1.3(2).

Using results from matroid theory [1] we obtain the following results extending results from [10] (Koszul property) and [9] (quadratic Gröbner basis and Koszul property) for Orlik-Solomon algebras to our deformation. We refer the reader to [3] for basic facts about Koszul algebras.

COROLLARY 1.4. *Let* $\mathfrak{M}$ *be a supersolvable loopless matroid without parallel elements,* $R = Q[t_0]$ *for a commutative ring* Q *with* 1 *and* t_0 *a degree one variable. If we set* $q = t_0^2$ *then* $\mathcal{I}_q(\overline{\mathfrak{M}})$ *is a quadratic Gröbner basis of* $I_q(\mathfrak{M})$ *and in particular* $\mathrm{OS}_q(\mathfrak{M})$ *is a standard graded Koszul algebra.*

We postpone the derivation of this corollary till Section 4 and note that for the classical Orlik-Solomon algebra of a hyperplane arrangement it is an open question if Koszulness is equivalent to the arrangement being supersolvable. We do not see an obstruction to the validity of this equivalence in the setting of 1.4, but at the moment we consider the positive evidence as too weak to make a corresponding conjecture.

For $q = 0$, Theorem 1.1 states

COROLLARY 1.5. *Let* $\overline{\mathfrak{M}}$ *be the set of Gröbner circuits of a loopless matroid* $\mathfrak{M}$ *without parallel elements. Then the polynomials* t_s^2 *with* $s \in S$, $t_r t_s + t_s t_r$ *with* $r, s \in S$, $s < r$ *and* $\sum_{\nu=1}^{\#J} (-1)^{\nu-1} t_{J \setminus \{j_\nu\}}$, *where* $J \in \overline{\mathfrak{M}}$, *form a Gröbner basis of the defining ideal* $I_0(\mathfrak{M})$ *of the Orlik-Solomon algebra of* $\mathfrak{M}$ *with respect to the degree lexicographic order.*

Since for $q = 0$ the quotient of A by t_s^2 with $s \in S$, $t_r t_s + t_s t_r$ with $r, s \in S$, $s < r$ is the exterior algebra E we also get the following corollary from [5, Prop. 9.3]. For its formulation we identify $t_J \in A$ for $J \subseteq S$ with its image in E.

COROLLARY 1.6. *Let $\overline{\mathfrak{M}}$ be the set of Gröbner circuits of a loopless matroid $\mathfrak{M}$ without parallel elements. Then the polynomials*

$$\sum_{\nu=1}^{\#J}(-1)^{\nu-1}t_{J\setminus\{j_\nu\}},$$

where $J \in \mathfrak{M} \cap \overline{\mathfrak{M}}$, form a Gröbner basis of the defining ideal of the Orlik-Solomon algebra in E.

Gröbner bases of the defining ideal of the Orlik-Solomon algebra inside the exterior algebra have been described previously (see for example [12, Thm. 2.8], [2], [9]).

2. Non-commutative Gröbner basics

Recall that the *degree lexicographic order* or *deglex order* on the monomials in A is the total order $<_{\text{dlex}}$ such that for two monomials $t_{i_1}\cdots t_{i_k}$ and $t_{j_1}\cdots t_{j_l}$ in A we have $t_{i_1}\cdots t_{i_k} <_{\text{dlex}} t_{j_1}\cdots t_{j_l}$ if and only if either $k < l$ or $k = l$ and for some $0 \le h < k$ we have $i_1 = j_1, \dots, i_h = j_h$ and $i_{h+1} < j_{h+1}$. Any $\xi \in A \setminus \{0\}$ can uniquely be written as a polynomial of the form $f = c_1m_1 + \cdots + c_jm_j$, for non-commutative monomials $m_j <_{\text{dlex}} \cdots m_2 <_{\text{dlex}} m_1$ in the variables t_s, $s \in S$, and ring elements $c_1, \dots, c_j \in R\setminus\{0\}$. In this polynomial, c_1m_1 is called the *leading term*, c_1 the *leading coefficient* and m_1 the *leading monomial* of f. We write $\text{lt}(f)$ for the leading term, $\text{lm}(f)$ for the leading monomial and $\text{lc}(f)$ for the leading coefficient of f. The $m_1, \dots, m_k$ are called the *monomials of* f. In other words, the leading monomial is the largest monomial among all monomials of f with respect to the deglex order. Further, for any monomial $m \in A$ there are only finitely many monomials $m' \in A$ such that $m' <_{\text{dlex}} m$.

Let $\mathscr{I}$ be a set of elements of A with leading coefficient 1. A *reduction* of a polynomial $f \in A$ modulo $\mathscr{I}$ is an expression obtained from f by replacing the leading monomial m of an element $g \in \mathscr{I}$, appearing as a subword of one of the monomials of f, by $m - g$. By construction, a reduction does not have monomials larger than the leading monomial of f. For any $f, g \in A$ we say that f *reduces to* g (modulo $\mathscr{I}$) and write

$$f \overset{\mathscr{I}}{\searrow} g \tag{2.1}$$

if there is a sequence of expressions $f = f_0, f_1, \dots, f_k = g$, where $k \in \mathsf{N}_0$, such that f_{i+1} is a reduction of f_i for all $i \in \{0, 1, \dots, k-1\}$.

A subset G of a two-sided ideal I in A is called a *Gröbner basis* of I if the two-sided ideal generated by $\{\text{lt}(g) \mid g \in G\}$ coincides with the two-sided ideal generated by $\{\text{lt}(f) \mid f \in I\}$.

For two polynomials f, g in A with $\mathrm{lc}(f) = \mathrm{lc}(g) = 1$ an *S-polynomial of* (f, g) is any non-zero expression $m_1 f m_2 - n_1 g n_2 \in A$ for monomials m_1, m_2, n_1, n_2 such that

$$m_1 \,\mathrm{lm}(f) m_2 = n_1 \,\mathrm{lm}(g) n_2. \tag{2.2}$$

Let $J_{f,g}$ be the submodule of the A-bimodule $(A \oplus A) \otimes_R (A \oplus A)$ generated by the tensors $(m_1, n_1) \otimes (m_2, n_2)$ for monomials m_1, m_2, n_1, n_2 for which (2.2) holds. Being generated by tensors of pairs of monomials there is a unique inclusionwise minimal set of generators of $J_{f,g}$ consisting of tensors of pairs of monomials. It is easily seen that any of the generators will be of the form $(m, 1) \otimes (1, n)$, $(1, n) \otimes (m, 1)$, $(1, n_1) \otimes (1, n_2)$ or $(m_1, 1) \otimes (m_2, 1)$. The criterion from the following theorem will be employed in order to derive Theorem 1.1.

THEOREM 2.1. *Let R be a field and I a two-sided ideal of A. A set $\mathscr{I} := \{f_1, \ldots, f_r\} \subseteq I$ is a Gröbner basis for I if and only if for all $1 \leq i \leq j \leq r$ and for any minimal generator $(m_1, n_1) \otimes (m_2, n_2)$ of J_{f_i, f_j} the corresponding S-polynomial of (f_i, f_j) reduces to 0 modulo $\mathscr{I}$.*

It is possible to simplify the Gröbner basis criterion in Theorem 2.1 by using the following fact [5, Cor. 5.8].

LEMMA 2.2. *Let $f, g \in \mathscr{I}$. Then the S-polynomials of (f, g) corresponding to the generators $(\mathrm{lm}(g)\, m, 1) \otimes (1, m \;\mathrm{lm}(f))$ and $(1, \mathrm{lm}(f)\, m) \otimes (m \;\mathrm{lm}(g), 1)$ of $J_{f,g}$, where m is an arbitrary monomial, reduce to 0 modulo $\mathscr{I}$.*

We will apply Theorem 2.1 and Lemma 2.2 in a situation where R is not necessarily a field. But since all our polynomials have leading coefficient 1 and since all reductions only use coefficients ± 1 the assertions remain valid.

3. Technical lemmas

3.1. General set systems

In this section we collect some useful formulas which are valid for arbitrary set systems $\mathfrak{M}$ over S.

Generalizing the notation in the introduction, for all $J \subseteq S$ let

$$\begin{aligned} t_J^- &= \sum_{I \subseteq J, 2 \nmid \# I} (-1)^{\ell_J(I)} (-q)^{(\# I - 1)/2} t_{J \setminus I}, \; t_J^+ \\ &= \sum_{I \subseteq J, 2 \mid \# I} (-1)^{\ell_J(I)} (-q)^{\# I/2} t_{J \setminus I}. \end{aligned}$$

We start with deriving formulas which are valid in A.

LEMMA 3.1. *Let $J, J', J'' \subseteq S$ such that $J = J' \cup J''$ and $j' < j''$ for all $j' \in J'$, $j'' \in J''$. Then in A the equations*

$$t_J^+ = t_{J'}^+ t_{J''}^+ + (-1)^{\#J'} q t_{J'}^- t_{J''}^-, \tag{3.1}$$

$$t_J^- = t_{J'}^- t_{J''}^+ + (-1)^{\#J'} t_{J'}^+ t_{J''}^- \tag{3.2}$$

hold.

PROOF. We proceed by induction on $\#J'$. Since $t_\emptyset^+ = 1$ and $t_\emptyset^- = 0$, the claim holds for $J' = \emptyset$. Assume now that $\#J \geq 1$, $J' = \{j_1\}$ and $J'' = J \setminus \{j_1\}$. Then

$$\begin{aligned}
t_J^+ &= \sum_{I \subseteq J, 2 \mid \#I} (-1)^{\ell_J(I)} (-q)^{\#I/2} t_{J \setminus I} \\
&= \sum_{I \subseteq J, 2 \mid \#I, j_1 \in I} (-1)^{\ell_J(I)} (-q)^{\#I/2} t_{J \setminus I} \\
&\quad + \sum_{I \subseteq J, 2 \mid \#I, j_1 \notin I} (-1)^{\ell_J(I)} (-q)^{\#I/2} t_{J \setminus I} \\
&= \sum_{L \subseteq J'', 2 \nmid \#L} (-1)^{\ell_{J''}(L)} (-q)^{(\#L+1)/2} t_{J'' \setminus L} \\
&\quad + \sum_{L \subseteq J'', 2 \mid \#L} (-1)^{\ell_{J''}(L) + \#L} (-q)^{\#L/2} t_{j_1} t_{J'' \setminus L} \\
&= -q t_{J''}^- + t_{j_1} t_{J''}^+ = t_{J'}^+ t_{J''}^+ - q t_{J'}^- t_{J''}^-.
\end{aligned}$$

Similarly,

$$t_J^- = t_{J''}^+ - t_{j_1} t_{J''}^- = t_{J'}^- t_{J''}^+ - t_{J'}^+ t_{J''}^-.$$

The rest follows from the induction hypothesis and the associativity law of A.

LEMMA 3.2. *Let $K, L \subseteq S$ such that $k < l$ for all $k \in K$, $l \in L$. Then in A we have*

$$t_K^- t_L^- = \sum_{i=1}^{\#K} (-1)^{\#K-i} t_{L \cup K \setminus \{k_i\}}^- = \sum_{i=1}^{\#L} (-1)^{i-1} t_{K \cup L \setminus \{l_i\}}^-. \tag{3.3}$$

PROOF. We prove the first equality by induction on $\#K$. If $K = \emptyset$ then both sides of the equality are zero. Assume now that $K \neq \emptyset$ and let $n = \#K$, $K' = K \setminus \{k_n\}$ and $L' = \{k_n\} \cup L$. Then Lemma 3.1 applied three times implies that

$$\begin{aligned}
t_K^- t_L^- &= (t_{K'}^- t_{k_n} + (-1)^{n-1} t_{K'}^+) t_L^- \\
&= t_{K'}^- (-t_{L'}^- + t_L^+) + (-1)^{n-1} t_{K'}^+ t_L^- \\
&= -t_{K'}^- t_{L'}^- + t_{L \cup K \setminus \{k_n\}}^-
\end{aligned}$$

from which the first equation follows from the induction hypothesis. In particular, for $L = \emptyset$ we obtain that $0 = \sum_{i=1}^{\#J}(-1)^{\#J-i} t^-_{J\setminus\{j_i\}}$ for all $J \subseteq S$. This implies that the second and the third expression in (3.3) coincide.

LEMMA 3.3. *Let $J, L \subseteq S$ such that $n := \#J \geq 2$, $J \setminus \{j_1, j_n\} = L \setminus \{l_1\}$ and $j_1 < l_1$. Then in A we have*

$$t^-_J - t^-_L t_{j_n} - \sum_{i=2}^{n-1}(-1)^i t^-_{(J\cup L)\setminus\{j_i\}} + (-1)^n t_{j_1} t^-_L = 0. \tag{3.4}$$

PROOF. Let $J' = J \setminus \{j_1\}$ and $L' = L \setminus \{l_1\}$. Then $t^-_J = t^+_{J'} - t_{j_1} t^-_{J'}$ by Lemma 3.1 and

$$\sum_{i=2}^{n-1}(-1)^i t^-_{(J\cup L)\setminus\{j_i\}} = (t_{l_1} - t_{j_1}) t^-_{J'} - (-1)^n t^-_{(J\cup L)\setminus\{j_n\}}$$

by Lemma 3.2. Thus Equation (3.4) is equivalent to

$$t^+_{J'} - t^-_L t_{j_n} - t_{l_1} t^-_{J'} + (-1)^n t^-_{(J\cup L)\setminus\{j_n\}} + (-1)^n t_{j_1} t^-_L = 0.$$

By Lemma 3.1, the left hand side of the latter equation can be written as

$$\begin{aligned}
&(t^+_{L'} t_{j_n} + (-1)^{n-2} q t^-_{L'}) - (t^+_{L'} - t_{l_1} t^-_{L'}) t_{j_n} \\
&\quad - t_{l_1}(t^-_{L'} t_{j_n} + (-1)^{n-2} t^+_{L'}) + (-1)^n (t^+_L - t_{j_1} t^-_L) + (-1)^n t_{j_1} t^-_L \\
&= (-1)^{n-2} q t^-_{L'} + (-1)^{n-1} t_{l_1} t^+_{L'} + (-1)^n t^+_L \\
&= 0
\end{aligned}$$

which proves the claim.

LEMMA 3.4. *Let $J, L \subseteq S$ such that $n := \#J \geq 2$, $L \setminus \{l_1\} = J \setminus \{j_1, j_2\}$ and $j_1 < l_1 < j_2$. Then in A we have*

$$t^-_J - (t_{j_2} t_{l_1} + t_{l_1} t_{j_2} - 2q) t^-_{L\setminus\{l_1\}} - t^-_{j_1 j_2} t^-_L + \sum_{i=3}^{n}(-1)^{i+1} t^-_{(J\cup L)\setminus\{j_i\}} = 0. \tag{3.5}$$

PROOF. Let $J' = J \setminus \{j_1\}$ and $L' = L \setminus \{l_1\}$. Then

$$\sum_{i=3}^{n}(-1)^{i+1} t^-_{(J\cup L)\setminus\{j_i\}} = t^-_{\{j_1, l_1, j_2\}} t^-_{L'}$$

by Lemma 3.2. Applying this and Lemma 3.1 repeatedly, the left hand side of Equation (3.4) becomes

$$\begin{aligned} & t_J^- - (t_{j_2}t_{l_1} + t_{l_1}t_{j_2} - 2q)t_{L'}^- \\ & \quad - (t_{j_2} - t_{j_1})(t_{L'}^+ - t_{l_1}t_{L'}^-) + (t_{l_1}t_{j_2} - t_{j_1}t_{j_2} + t_{j_1}t_{l_1} - q)t_{L'}^- \\ & = t_J^- + (q - t_{j_1}t_{j_2})t_{L'}^- - (t_{j_2} - t_{j_1})t_{L'}^+ \\ & = t_{J'}^+ - t_{j_1}t_{J'}^- + (q - t_{j_1}t_{j_2})t_{L'}^- - (t_{j_2} - t_{j_1})t_{L'}^+ \\ & = 0 \end{aligned}$$

which proves the claim.

Now we turn to reductions. Observe that if $\mathfrak{N} \subseteq \mathfrak{M}$ are sets of subsets of S then for $t \in A$ we have that if t reduces to zero modulo $\mathscr{I}_q(\mathfrak{N})$ then t reduces to zero modulo $\mathscr{I}_q(\mathfrak{M})$. In particular, if t reduces to zero modulo $\mathscr{I}_q(\emptyset)$ then t reduces to zero modulo $\mathscr{I}_q(\mathfrak{M})$. Note that in the reduction modulo $\mathscr{I}_q(\emptyset)$ only relations (1.1) and (1.2) are involved.

LEMMA 3.5. *Let $J \subseteq S$ and let $s \in S$.*

(1) *Assume that $s \in J$. Then*

$$t_s t_J^+ \searrow^{\mathscr{I}_q(\emptyset)} q t_J^-, \qquad t_s t_J^- \searrow^{\mathscr{I}_q(\emptyset)} t_J^+.$$

(2) *Assume that $s \notin J$. Let $J' = \{j \in J \mid j < s\}$. Then*

$$t_s t_J^+ \searrow^{\mathscr{I}_q(\emptyset)} (-1)^{\#J'} t_{J\cup\{s\}}^+ + q t_J^-, \qquad t_s t_J^- \searrow^{\mathscr{I}_q(\emptyset)} (-1)^{\#J'-1} t_{J\cup\{s\}}^- + t_J^+.$$

REMARK 3.6. If $s < j_1$ then in the last expression of Lemma 3.5(2) the leading term of $t_s t_J^-$ is $t_s t_{j_2} \cdots t_{j_{\#J}}$. On the other hand, the leading term of both $t_{J\cup\{s\}}^-$ and t_J^+ is t_J which is larger than $t_s t_{j_2} \cdots t_{j_{\#J}}$ with respect to the deglex order. The reduction formula means that these two leading terms cancel and $t_s t_J^-$ reduces modulo $\mathscr{I}_q(\emptyset)$ to the remaining expression. In fact, due to Equation (3.2) for $t_{J\cup\{s\}}^-$, no reduction is needed to obtain the result.

PROOF. We proceed by induction on $\#J$. Assume first that $s < j$ for all $j \in J$. (This holds in particular if $J = \emptyset$.) Then $t_{J\cup\{s\}}^+ = t_s t_J^+ - q t_J^-$ and $t_{J\cup\{s\}}^- = t_J^+ - t_s t_J^-$ in A by Lemma 3.1 and hence (2) holds in this case.

Assume now that $s \in J$ and $s \leq j$ for all $j \in J$. Let $K = J \setminus \{s\}$. Then

$$t_J^+ = t_s t_K^+ - q t_K^-, \quad t_J^- = t_K^+ - t_s t_K^-$$

in A by Lemma 3.1. Since $t_s^2 - q \in \mathscr{I}_q(\emptyset)$, it follows that

$$\begin{aligned} t_s t_J^+ &= t_s^2 t_K^+ - q t_s t_K^- \searrow^{\mathscr{I}_q(\emptyset)} q t_K^+ - q t_s t_K^- = q t_J^-, \\ t_s t_J^- &= t_s t_K^+ - t_s^2 t_K^- \searrow^{\mathscr{I}_q(\emptyset)} t_s t_K^+ - q t_K^- = t_J^+ \end{aligned}$$

by Lemma 3.1. Hence (1) holds in this case.

Finally, assume that $J \neq \emptyset$ and that $s > j_1$. Let $K = J \setminus \{j_1\}$. We assume first that $s \in J$ and prove (1). Since $t_s t_{j_1} + t_{j_1} t_s - 2q \in \mathscr{I}_q(\emptyset)$, by Lemma 3.1 and by induction hypothesis we obtain that

$$\begin{aligned} t_s t_J^+ = t_s(t_{j_1} t_K^+ - q t_K^-) &\searrow^{\mathscr{I}_q(\emptyset)} (-t_{j_1} t_s + 2q) t_K^+ - q t_s t_K^- \\ &\searrow^{\mathscr{I}_q(\emptyset)} -t_{j_1}(q t_K^-) + q t_K^+ = q t_J^-. \end{aligned}$$

Similarly,

$$\begin{aligned} t_s t_J^- = t_s(t_K^+ - t_{j_1} t_K^-) &\searrow^{\mathscr{I}_q(\emptyset)} q t_K^- - (-t_{j_1} t_s + 2q) t_K^- \\ &\searrow^{\mathscr{I}_q(\emptyset)} t_{j_1} t_K^+ - q t_K^- = t_J^+. \end{aligned}$$

A similar argument proves (2).

The following lemma is a right-handed analogue of the previous result.

LEMMA 3.7. *Let $J \subseteq S$ and let $s \in S$.*

(1) *Assume that $s \in J$. Then*

$$t_J^+ t_s \searrow^{\mathscr{I}_q(\emptyset)} (-1)^{\#J+1} q t_J^-, \qquad t_J^- t_s \searrow^{\mathscr{I}_q(\emptyset)} (-1)^{\#J+1} t_J^+.$$

(2) *Assume that $s \notin J$. Let $J'' = \{j \in J \mid s < j\}$. Then*

$$\begin{aligned} t_J^+ t_s &\searrow^{\mathscr{I}_q(\emptyset)} (-1)^{\#J''} t_{J\cup\{s\}}^+ + (-1)^{\#J+1} q t_J^-, \\ t_J^- t_s &\searrow^{\mathscr{I}_q(\emptyset)} (-1)^{\#J''} t_{J\cup\{s\}}^- + (-1)^{\#J+1} t_J^+. \end{aligned}$$

PROOF. See the proof of Lemma 3.5.

LEMMA 3.8. *Let $J \subseteq S$ with $J \neq \emptyset$. If t_J^- reduces to zero modulo $\mathscr{I}_q(\mathfrak{M})$, then t_J^+ reduces to zero modulo $\mathscr{I}_q(\mathfrak{M})$.*

PROOF. Let $K = J \setminus \{j_1\}$. Lemma 3.1 gives that

$$t_J^+ = t_{j_1} t_K^+ - q t_K^-, \qquad t_J^- = t_K^+ - t_{j_1} t_K^-$$

in A and the leading term of t_J^- is the leading term of t_K^+. Thus, since $t_J^- \searrow^{\mathscr{I}_q(\mathfrak{M})} 0$, it follows that $t_K^+ \searrow^{\mathscr{I}_q(\mathfrak{M})} t_{j_1} t_K^-$. Hence

$$t_J^+ = t_{j_1} t_K^+ - q t_K^- \searrow^{\mathscr{I}_q(\mathfrak{M})} t_{j_1}^2 t_K^- - q t_K^-$$

which reduces to zero modulo $\mathscr{I}_q(\mathfrak{M})$ since $t_{j_1}^2 - q \in \mathscr{I}_q(\mathfrak{M})$.

LEMMA 3.9. *Let $J \subseteq S$ and let $s \in S$. Assume that t_J^- reduces to zero modulo $\mathcal{I}_q(\mathfrak{M})$. If $J \neq \emptyset$, $r < s$ for all $r \in J$ or $\#J \geq 2$, $s < j_2$ then $t_{J\cup\{s\}}^-$ reduces to zero modulo $\mathcal{I}_q(\mathfrak{M})$.*

PROOF. If $r < s$ for all $r \in J$ then $t_{J\cup\{s\}}^- = t_J^- t_s - t_J^+$ by Lemma 3.1. Thus the first half of the claim holds by Lemma 3.8.

If $\#J \geq 2$, $s < j_1$ then $t_{J\cup\{s\}}^- = t_J^+ - t_s t_J^-$ by Lemma 3.1 and again the claim holds. If $s = j_1$ then there is nothing to prove. Finally, if $j_1 < s < j_2$ then let $J' = J \setminus \{j_1\}$. Lemma 3.1 gives that

$$
\begin{aligned}
t_{J\cup\{s\}}^- &= t_{\{j_1,s\}}^- t_{J'}^+ + t_{\{j_1,s\}}^+ t_{J'}^- = (t_s - t_{j_1})t_{J'}^+ + (t_{j_1}t_s - q)t_{J'}^- \\
&= (t_s - t_{j_1})(t_J^- + t_{j_1}t_{J'}^-) + (t_{j_1}t_s - q)t_{J'}^- \\
&= (t_s - t_{j_1})t_J^- + (t_s t_{j_1} + t_{j_1}t_s - 2q)t_{J'}^- - (t_{j_1}^2 - q)t_{J'}^-.
\end{aligned}
$$

This expression reduces to zero modulo $\mathcal{I}_q(\mathfrak{M})$ which proves the remaining claim.

3.2. Matroids

From now on let $\mathfrak{M}$ be the set of circuits of a loopless matroid without parallel elements on ground set S and let $\overline{\mathfrak{M}}$ be a set of Gröbner circuits of $\mathfrak{M}$.

EXAMPLE 3.10. A typical example where the set of circuits $\mathfrak{M}$ of a matroid is not sufficient to define a Gröbner basis of $\mathrm{OS}_q(\mathfrak{M})$ is the following.

Let $S = \{1, 2, 3, 4\}$ with the usual order and let $\mathfrak{M}$ be the set system consisting of $\{1, 2, 4\}$. Then

$$t_{1234}^- = t_2t_3t_4 - t_1t_3t_4 + t_1t_2t_4 - t_1t_2t_3 - qt_4 + qt_3 - qt_2 + qt_1$$

is zero in $\mathrm{OS}_q(\mathfrak{M})$ since

$$t_{1234}^- = -t_3t_{124}^- + t_{124}^+ = -t_3t_{124}^- + t_1t_{124}^-$$

by Lemma 3.5 and $t_{124}^- = 0$ in $\mathrm{OS}_q(\mathfrak{M})$. However, the leading term of t_{1234}^- cannot be reduced using the generators of $I_q(\mathfrak{M})$.

Before we prove that $\mathcal{I}_q(\overline{\mathfrak{M}})$ is a Gröbner basis of $\mathrm{OS}_q(\mathfrak{M})$, we show that t_J^- reduces to zero modulo $\mathcal{I}_q(\overline{\mathfrak{M}})$ for all $J \in \mathfrak{M}$.

LEMMA 3.11. *Let $J, K \subseteq S$ be two dependent sets such that $J \cap K$ is independent. Then for all $l \in J \cup K$ the set $(J \cup K) \setminus \{l\}$ is dependent.*

PROOF. Let $l \in J \cup K$. If there is a circuit contained in $J \setminus \{l\}$ or $K \setminus \{l\}$ then it is contained in $(J \cup K) \setminus \{l\}$. On the other hand, if $C \subseteq J$, $D \subseteq K$ are circuits containing l then $C \neq D$ since $J \cap K$ is independent. Hence by the circuit axiom there is a circuit contained in $(C \cup D) \setminus \{l\}$.

PROPOSITION 3.12. *For any dependent set J, t_J^- reduces to zero modulo $\mathscr{I}_q(\overline{\mathfrak{M}})$.*

PROOF. We proceed by induction with respect to deglex order.

Let J be a dependent set. Recall that $\#J \geq 3$. If $J \in \overline{\mathfrak{M}}$ then t_J^- reduces to zero. In particular, this holds if J is the smallest dependent set with respect to deglex order.

Assume now that $J \notin \overline{\mathfrak{M}}$. Then there exists $K \in \overline{\mathfrak{M}}$ such that $K <_{\mathrm{dlex}} J$ and $K \setminus \{k_1\} \sqsubseteq J \setminus \{j_1\}$. In particular, K is dependent. We now distinguish several cases according to the relations between k_1 and the elements of J.

If $k_1 < j_1$ then let $L = \{k_1\} \cup (J \setminus \{j_1\})$. In this case $L <_{\mathrm{dlex}} J$, $K \subset L$ and $K \setminus \{k_1\} \sqsubseteq L \setminus \{l_1\}$. Hence t_L^- reduces to zero modulo $\mathscr{I}_q(\overline{\mathfrak{M}})$ by induction hypothesis. If $L \cap J$ is dependent then $t_{L\cap J}^-$ reduces to zero by induction hypothesis and hence t_J^- reduces to zero by Lemma 3.9. Assume now that $L \cap J$ is independent. Then, by Lemma 3.11, $(J \cup L) \setminus \{j_i\}$ is dependent for all $i \in \{2, 3, \dots, \#J\}$ and is smaller than J with respect to $<_{\mathrm{dlex}}$. We conclude from Lemma 3.2 that

$$0 = t_J^- - t_L^- + \sum_{i=2}^{\#J} (-1)^i t_{(J\cup L)\setminus\{j_i\}}^-$$

in A and hence t_J^- reduces to zero by induction hypothesis.

If $k_1 = j_1$ then t_J^- reduces to zero by using that t_K^- reduces to zero and by repeatedly applying Lemma 3.9.

If $j_1 < k_1 < j_2$ then $\#K < \#J$ since $K <_{\mathrm{dlex}} J$. Since t_K^- reduces to zero, by repeatedly applying Lemma 3.9 we obtain a dependent set $L \subseteq S$ such that $\#L = \#J - 1$, $l_1 = k_1$, $L \setminus \{l_1\} \sqsubseteq J \setminus \{j_1\}$, and t_L^- reduces to zero. There are two cases: $j_{\#J} \notin L$ or $j_2 \notin L$. If $J \cap L$ is dependent then t_J^- reduces to zero by induction hypothesis and by Lemma 3.9. If $J \cap L$ is independent then $(J \cup L) \setminus \{s\}$ is dependent for all $s \in J \cap L$ by Lemma 3.11. Now if $j_{\#J} \notin L$ then t_J^- reduces to zero by induction hypothesis and by Lemma 3.3. Observe that in Lemma 3.3 t_J^- and $t_L^- t_{j_{\#J}}$ are the summands with the largest leading term. Similarly, if $j_2 \notin L$ then t_J^- reduces to zero by induction hypothesis and by Lemma 3.4.

Finally, assume that $j_2 = k_1$ or $j_2 < k_1$. By repeatedly applying Lemma 3.9 we obtain that the set $L = \{s \in J \mid k_1 = s \text{ or } k_1 < s\}$ is dependent and $L <_{\mathrm{dlex}} J$. If $l_1 \in J$ then t_J^- reduces to zero by Lemma 3.9. If $l_1 \notin J$ and $J \cap L$ is dependent then again J reduces to zero by induction hypothesis and Lemma 3.9. In the last case, if $J \cap L$ is independent then $(J \cup L) \setminus \{s\}$ is dependent for all $s \in L, l_1 < s$. In this case Lemma 3.2 applied to $J \setminus L$ and L implies that

$$t^-_{J\setminus L} t^-_L = t^-_J + \sum_{i=2}^{\#L} (-1)^{i-1} t^-_{(J\cup L)\setminus\{l_i\}}$$

and the leading term on both sides of the equation is the leading term of t^-_J. Thus t^-_J reduces to zero by induction hypothesis.

4. Independence statement and proofs

We first show that $I_q(\mathfrak{M})$ is independent of the chosen total order of S.

THEOREM 4.1. *For any set system $\mathfrak{M}$ over S, the ideal $I_q(\mathfrak{M})$ and hence the algebra $\mathrm{OS}_q(\mathfrak{M})$ are independent of the total order on S.*

PROOF. We have to show that for any two total orders on S the defining ideals of $\mathrm{OS}_q(\mathfrak{M})$ coincide. Relations (1.1), (1.2) are obviously independent of the chosen total order. Relations (1.2) can be used to reformulate a defining relation in (1.3) in terms of another order. We may simplify the problem by looking at orders $<, \ll$ which differ by exchanging two neighboring elements $a, b \in S$ with $a < b$, that is, $r \ll s$ for $r, s \in S$ if and only if either $(r, s) = (b, a)$ or $r < s$, $(r, s) \neq (a, b)$. We write $\ell^<_J(I)$ and $t^<_{J\setminus I}$ and similarly for $\ll$ to indicate the dependency on the order.

Let $J \in \mathfrak{M}$, $I \subseteq J$ with $2 \nmid \#I$. If $a \notin J$ or $b \notin J$ then

$$(-1)^{\ell^{\ll}_J(I)} = (-1)^{\ell^<_J(I)}, \qquad (-q)^{(\#I-1)/2} t^{\ll}_{J\setminus I} = (-q)^{(\#I-1)/2} t^<_{J\setminus I}$$

and hence (1.3) takes the same form with respect to $<$ and $\ll$. It remains to consider the case when $a, b \in J$. We prove that in this case the defining relations differ by a sign. Then the proof of the proposition is completed.

Assume that $a \in I, b \in J \setminus I$ or $a \in J \setminus I, b \in I$. Then

$$(-1)^{\ell^{\ll}_J(I)} = -(-1)^{\ell^<_J(I)}, \qquad t^{\ll}_{J\setminus I} = t^<_{J\setminus I}$$

and therefore $(-1)^{\ell^{\ll}_J(I)}(-q)^{(\#I-1)/2} t^{\ll}_{J\setminus I} = -(-1)^{\ell^<_J(I)}(-q)^{(\#I-1)/2} t^<_{J\setminus I}$. Assume now that $a, b \in J \setminus I$ and let $I' = I \cup \{a, b\}$. Then

$$\ell^{\ll}_J(I) = \ell^<_J(I), \qquad \ell^{\ll}_J(I') = \ell^<_J(I'), \qquad t^{\ll}_{J\setminus I'} = t^<_{J\setminus I'},$$
$$t^{\ll}_{J\setminus I} = t_{j_1} \cdots (t_b t_a) \cdots t_{j_{\#J}} = t_{j_1} \cdots (-t_a t_b + 2q) \cdots t_{j_{\#J}} = -t^<_{J\setminus I} + 2q t^<_{J\setminus I'},$$

and $(-1)^{\ell_J^{<}(I)} = (-1)^{\ell_J^{<}(I')}$. Hence

$$\begin{aligned}
&(-1)^{\ell_J^{\ll}(I)}(-q)^{(\#I-1)/2} t_{J\setminus I}^{\ll} + (-1)^{\ell_J^{\ll}(I')}(-q)^{(\#I'-1)/2} t_{J\setminus I'}^{\ll} \\
&\quad = (-1)^{\ell_J^{<}(I)}(-q)^{(\#I-1)/2}(-t_{J\setminus I}^{<} + 2q t_{J\setminus I'}^{<}) \\
&\qquad + (-1)^{\ell_J^{<}(I)}(-q)^{(\#I-1)/2}(-q) t_{J\setminus I'}^{<} \\
&\quad = -(-1)^{\ell_J^{<}(I)}(-q)^{(\#I-1)/2} t_{J\setminus I}^{<} + q(-1)^{\ell_J^{<}(I)}(-q)^{(\#I-1)/2} t_{J\setminus I'}^{<} \\
&\quad = -(-1)^{\ell_J^{<}(I)}(-q)^{(\#I-1)/2} t_{J\setminus I}^{<} - (-1)^{\ell_J^{<}(I')}(-q)^{(\#I'-1)/2} t_{J\setminus I'}^{<}.
\end{aligned}$$

This is what we wanted to show.

Next we provide the proof of the main result.

PROOF OF THEOREM 1.1.

CLAIM. *If $\mathfrak{M}$ is a loopless matroid without parallel elements then $I_q(\overline{\mathfrak{M}}) = I_q(\mathfrak{M})$.*

PROOF OF CLAIM. From Proposition 3.12 we deduce that $t_J^- \in I_q(\overline{\mathfrak{M}})$ for all circuits J. Hence $I_q(\mathfrak{M}) \subseteq I_q(\overline{\mathfrak{M}})$. To show equality it suffices to show that $t_J^- \in I_q(\mathfrak{M})$ for all dependent $J \subseteq S$. We prove the assertion by induction on the cardinality of the difference set of J and the circuit of largest cardinality contained in it. If the cardinality is 0 then J itself is a circuit and hence $t_J^- \in I_q(\mathfrak{M})$ by definition. If the cardinality is positive then there is an $s \in J$ such that $J \setminus \{s\}$ is dependent and by induction $t_{J\setminus\{s\}}^- \in I_q(\mathfrak{M})$. By Lemma 3.5(2) we can write t_J^- as an A linear combination of $t_{J\setminus\{s\}}^+$, $t_s t_{J\setminus\{s\}}^-$ and elements of $I_q(\emptyset) \subseteq I_q(\mathfrak{M})$. By Lemma 3.1 and since $t_{J\setminus\{s\}}^- \in I_q(\mathfrak{M})$ it follows that $t_{J\setminus\{s\}}^+ \in I_q(\mathfrak{M})$ and the claim follows.

We complete the proof of the theorem by showing that $\mathscr{I}_q(\mathfrak{M})$ is a Gröbner basis of $I_q(\overline{\mathfrak{M}}) = I_q(\mathfrak{M})$. For this we verify that the conditions of Theorem 2.1 under the simplification provided by Lemma 2.2 are fulfilled.

First we have to find minimal generators of the modules $J_{f,g}$, where f, g are polynomials (1.1), (1.2) or t_J^- with $J \in \overline{\mathfrak{M}}$. According to Lemma 2.2 we can ignore generators $(\mathrm{lm}(g)\, m, 1) \otimes (1, m\ \mathrm{lm}(f))$ and $(1, \mathrm{lm}(f)\, m) \otimes (m\ \mathrm{lm}(g), 1)$ of $J_{f,g}$, where m is an arbitrary monomial. Further, since we do not fix an order on the Gröbner basis, we may restrict ourselves to generators of $J_{f,g}$ of the form $(1, m) \otimes (n, 1)$ and $(1, m) \otimes (1, n)$. Therefore the following cases have to be considered.

Case 1. $f = t_s^2 - q$, $g = t_s^2 - q$, $s \in S$.

The remaining generator of $J_{f,g}$ is $(1, t_s) \otimes (t_s, 1)$. Then $f t_s - t_s g$ is obviously zero.

Case 2. $f = t_s^2 - q$, $g = t_s t_r + t_r t_s - 2q$, $r, s \in S$, $r < s$.

The remaining generator of $J_{f,g}$ is $(1, t_s) \otimes (t_r, 1)$. The corresponding S-polynomial is

$$
\begin{aligned}
f t_r - t_s g &= t_s^2 t_r - q t_r - (t_s^2 t_r + t_s t_r t_s - 2q t_s) \\
&\searrow^{\mathscr{I}_q(\emptyset)} -q t_r - (-t_r t_s + 2q) t_s + 2q t_s \\
&\searrow^{\mathscr{I}_q(\emptyset)} -q t_r + t_r q = 0.
\end{aligned}
$$

Case 3. $f = t_s t_r + t_r t_s - 2q$, $g = t_r^2 - q$, $r, s \in S$, $r < s$.

The remaining generator of $J_{f,g}$ is $(1, t_s) \otimes (t_r, 1)$. The corresponding S-polynomial is

$$
\begin{aligned}
f t_r - t_s g &= (t_s t_r + t_r t_s - 2q) t_r - t_s (t_r^2 - q) \\
&\searrow^{\mathscr{I}_q(\emptyset)} t_r(-t_r t_s + 2q) - 2q t_r + q t_s \\
&\searrow^{\mathscr{I}_q(\emptyset)} -q t_s + q t_s = 0.
\end{aligned}
$$

Case 4. $f = t_s^2 - q$, $g = t_J^-$, $J \in \overline{\mathfrak{M}}$, $s = j_2$.

Let $K = J \setminus \{j_1, j_2\}$. The remaining generator of $J_{f,g}$ is $(1, t_s) \otimes (t_K, 1)$. The corresponding S-polynomial is

$$
f t_K - t_s g = (t_s^2 - q) t_K - t_s t_J^- .
$$

By Lemma 3.5 the expression $t_s t_J^-$ reduces to t_J^+ modulo $\mathscr{I}_q(\emptyset)$. In this reduction the leading term $t_s^2 t_K$ of $t_s t_J^-$ has to be reduced at one moment to $q t_K$. Therefore $t_s t_J^- - (t_s^2 - q) t_K$ also reduces to t_J^+ modulo $\mathscr{I}_q(\emptyset)$. Thus $t_s t_J^- - (t_s^2 - q) t_K$ reduces to zero modulo $\mathscr{I}_q(\overline{\mathfrak{M}})$ by Lemma 3.8.

Case 5. $f = t_J^-$, $g = t_s^2 - q$, $J \in \overline{\mathfrak{M}}$, $s = j_{\#J}$.

Let $K = J \setminus \{j_1, j_{\#J}\}$. The remaining generator of $J_{f,g}$ is $(1, t_K) \otimes (t_s, 1)$. The corresponding S-polynomial is

$$
f t_s - t_K g = t_J^- t_s - t_K (t_s^2 - q).
$$

By Lemma 3.7 the expression $t_J^- t_s$ reduces to $(-1)^{\#J+1} t_J^+$ modulo $\mathscr{I}_q(\emptyset)$. In this reduction the leading term $t_K t_s^2$ of $t_J^- t_s$ has to be reduced at one moment to $q t_K$. Therefore $t_J^- t_s - t_K(t_s^2 - q)$ also reduces to $(-1)^{\#J+1} t_J^+$ modulo $\mathscr{I}_q(\emptyset)$. Thus $t_J^- t_s - t_K(t_s^2 - q)$ reduces to zero modulo $\mathscr{I}_q(\overline{\mathfrak{M}})$ by Lemma 3.8.

Case 6. $f = t_s t_r + t_r t_s - 2q$, $g = t_r t_p + t_p t_r - 2q$, $p, r, s \in S$, $p < r < s$.
The remaining generator of $J_{f,g}$ is $(1, t_s) \otimes (t_p, 1)$. The corresponding S-polynomial is

$$\begin{aligned} f t_p - t_s g &= (t_s t_r + t_r t_s - 2q) t_p - t_s (t_r t_p + t_p t_r - 2q) \\ &= t_r t_s t_p - 2q t_p - t_s t_p t_r + 2q t_s \\ &\searrow^{\mathscr{I}_q(\emptyset)} t_r(-t_p t_s + 2q) - (-t_p t_s + 2q) t_r - 2q t_p + 2q t_s \\ &\searrow^{\mathscr{I}_q(\emptyset)} -(-t_p t_r + 2q) t_s + t_p(-t_r t_s + 2q) - 2q t_p + 2q t_s = 0. \end{aligned}$$

Case 7. $f = t_s t_r + t_r t_s - 2q$, $g = t_J^-$, $r, s \in S$, $r < s$, $J \in \overline{\mathfrak{M}}$, $r = j_2$.
Let $K = J \setminus \{j_1, j_2\}$. The remaining generator of $J_{f,g}$ is $(1, t_s) \otimes (t_K, 1)$. The corresponding S-polynomial is

$$f t_K - t_s g = (t_s t_r + t_r t_s - 2q) t_K - t_s t_J^-.$$

By Lemma 3.5 the expression $t_s t_J^-$ reduces to $\pm t_{J \cup \{s\}}^- + t_J^+$ modulo $\mathscr{I}_q(\emptyset)$. In this reduction the leading term $t_s t_r t_K$ of $t_s t_J^-$ has to be reduced at one moment to $(-t_r t_s + 2q) t_K$. Therefore $t_s t_J^- - (t_s t_r + t_r t_s - 2q) t_K$ also reduces to $\pm t_{J \cup \{s\}}^- + t_J^+$ modulo $\mathscr{I}_q(\emptyset)$. Since t_J^+ reduces to zero modulo $\mathscr{I}_q(\overline{\mathfrak{M}})$ by Lemma 3.8, it suffices to prove that $t_{J \cup \{s\}}^-$ reduces to zero modulo $\mathscr{I}_q(\overline{\mathfrak{M}})$. The latter holds for $s \le j_2$ and for $j_{\#J} \le s$ by Lemma 3.9 and for $j_2 < s < j_{\#J}$ by (GC).

Case 8. $f = t_J^-$, $g = t_s t_r + t_r t_s - 2q$, $r, s \in S$, $r < s$, $J \in \overline{\mathfrak{M}}$, $s = j_{\#J}$.
Let $K = J \setminus \{j_1, s\}$. The remaining generator of $J_{f,g}$ is $(1, t_K) \otimes (t_r, 1)$. The corresponding S-polynomial is

$$f t_r - t_K g = t_J^- t_r - t_K (t_s t_r + t_r t_s - 2q).$$

The proof is similar to the one in Case 7 and uses Lemma 3.7.

Case 9. $f = t_J^-$, $g = t_K^-$, $J \setminus \{j_1\} \sqsubseteq K \setminus \{k_1\}$, $J, K \in \overline{\mathfrak{M}}$, $J \ne K$. This case does not appear by Condition (GC) on the elements of $\overline{\mathfrak{M}}$.

Case 10. $f = t_J^-$, $g = t_K^-$, $J, K \in \overline{\mathfrak{M}}$, there exists $i \in \{3, \ldots, \#J\}$ such that $j_i = k_2$, $j_{i+1} = k_3, \ldots, j_{\#J} = k_{\#J-i+2}$, $\#J - i + 2 < \#K$.
Let $n = \#J - i + 2$,

$$J' = J \setminus \{j_1\}, \qquad K' = K \setminus \{k_1\},$$
$$L = \{j \in J' \mid j < k_1\}, \qquad M = \{j \in J' \setminus K' \mid k_1 < j\}.$$

Thus the sets L, M and $J' \cap K'$ are pairwise disjoint and their union is J' (if $k_1 \notin J'$) or $J' \setminus \{k_1\}$ (if $k_1 \in J'$). We have to show that

$$t_J^- t_{K' \setminus J'} - t_{J' \setminus K'} t_K^- \searrow^{\mathscr{I}_q(\overline{\mathfrak{M}})} 0. \tag{4.1}$$

We will proceed in several steps, and at some moment we will have to distinguish the cases $k_1 \in J'$ and $k_1 \notin J'$.

Using Lemma 3.1 and Lemma 3.8 we observe first that

$$t_J^- t_{K'\setminus J'} = t_{J\cup K'}^- + \text{terms which reduce to zero modulo } \mathscr{I}_q(\overline{\mathfrak{M}}). \tag{4.2}$$

Further, by applying Lemma 3.5(2) and Lemma 3.8 we obtain that

$$t_M t_K^- = t_{M\cup K}^- + \text{terms which reduce to zero modulo } \mathscr{I}_q(\overline{\mathfrak{M}}).$$

In particular, if $j_1 = k_1$ then $M \cup K = J' \cup K = J \cup K'$ and hence (4.1) holds.

Assume now that $k_1 \in J'$. Then $t_{J'\setminus K'} = t_L t_{k_1} t_M$. Lemma 3.5(1) gives that

$$t_{k_1} t_{M\cup K}^- = t_{M\cup K}^+ + \text{terms which reduce to zero modulo } \mathscr{I}_q(\overline{\mathfrak{M}})$$

and Equation (3.1) implies that

$$t_L t_{M\cup K}^+ = t_{L\cup M\cup K}^+ + \text{terms which reduce to zero modulo } \mathscr{I}_q(\overline{\mathfrak{M}}).$$

Now, $L \cup M \cup K = (J \cup K) \setminus \{j_1\}$ and

$$t_{J\cup K'}^- = t_{J\cup K}^- = t_{(J\cup K)\setminus\{j_1\}}^+ - t_{j_1} t_{(J\cup K)\setminus\{j_1\}}^- \tag{4.3}$$

by Equation (3.2). Since $K \sqsubseteq (J \cup K) \setminus \{j_1\}$, Proposition 3.12 yields that $t_{(J\cup K)\setminus\{j_1\}}^-$ reduces to zero modulo $\mathscr{I}_q(\overline{\mathfrak{M}})$. Thus we conclude from (4.2) and (4.3) that (4.1) holds.

Finally, assume that $k_1 \notin J$. Then $t_{J'\setminus K'} = t_L t_M$. Further,

$$t_L t_{M\cup K}^- = t_{\{j_1\}\cup L}^- t_{M\cup K}^- + \text{terms which reduce to zero modulo } \mathscr{I}_q(\overline{\mathfrak{M}})$$

by definition of $t_{\{j_1\}\cup L}^-$, and hence

$$t_L t_{M\cup K}^- = \sum_{m=1}^{\#(M\cup K)} (-1)^{m-1} t_{(J\cup K)\setminus\{(M\cup K)_m\}}^- + \begin{matrix}\text{terms which}\\ \text{reduce to zero}\end{matrix} \tag{4.4}$$

by Lemma 3.2, where $(M \cup K)_m$ is the mth element of $M \cup K$. The summand containing the leading term of the last expression is $t_{(J\cup K)\setminus\{k_1\}}^-$ since $(M\cup K)_1 = k_1$. Since $J \cap K$ is independent by the assumptions $J \cap K = J \cap K' \sqsubseteq K'$ and $K \in \overline{\mathfrak{M}}$, Lemma 3.11 and Proposition 3.12 imply that all other summands in (4.4) reduce to zero modulo $\mathscr{I}_q(\overline{\mathfrak{M}})$. Thus (4.1) holds in this case.

It remains to provide the proof of Corollary 1.4.

PROOF OF COROLLARY 1.4.

CLAIM. *There is a total order on S for which all Gröbner circuits are of size* 3.

PROOF OF CLAIM. We recall the characterization of supersolvable matroids given in [1, Thm. 2.8 (5)]. There it is shown that for a supersolvable matroid $\mathfrak{M}$ on ground set S the set S can be partitioned into subsets $S = S_1 \cup \cdots \cup S_f$ such that for any $1 \leq h \leq f$ and two elements $x, y \in S_h$ there is an $1 \leq g < h$ and $z \in S_g$ such that $\{x, y, z\} \in \mathfrak{M}$ is a circuit. Now we choose a total order on S such that for $1 \leq g < h \leq f$ all elements from S_g come before S_h. Assume that J is a Gröbner circuit in this order. If $J \cap S_h \geq 2$ for some $1 \leq h \leq f$ then by [1, Thm. 2.8 (5)] for any two elements $x, y \in J \cap S_h$ there is a circuit K of size 3 such that $K \setminus \{k_1\} = \{x, y\}$. By choosing two elements $\{x, y\}$ from $J \cap S_h$ for which $\{x, y\} \sqsubseteq J$ we can choose K such that $K \leq_{\text{dlex}} J$ and $K \setminus \{k_1\} \sqsubseteq J \setminus \{j_1\}$. From this it follows by (GC) that $J = K$, giving the claim.

Now if all Gröbner circuits are of size 3 then the Gröbner basis from Corollary 1.3 is quadratic. Hence by well known facts (see [3, Sec. 4]) it follows that $\mathrm{OS}_q(\mathfrak{M})$ is Koszul.

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