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ESSENTIAL SPECTRUM AND FREDHOLM INDEX FOR CERTAIN COMPOSITION OPERATORS

CHRISTOPHER J. YAKES*

Abstract

We investigate a composition operator on $H^\infty(U)$, U a subdomain of the open unit disk, for which the essential resolvent has infinitely many components, and for which the Fredholm index of the resolvent operator attains all nonnegative integer values.

1. Introduction

The spectra and essential spectra of composition operators on spaces of analytic functions on the open unit disk have been studied by a number of authors (see [10], [9], [8], [3], [4]). Composition operators have also been studied in the context of uniform algebras (see [7], [5], [6]), where they arise as the unital homomorphisms of the algebras.

Let $\mathsf{D} = \{|z| < 1\}$ be the open unit disk in the complex plane. An analytic function $\psi : \mathsf{D} \to \mathsf{D}$ determines the composition operator C_ψ on $H^\infty(\mathsf{D})$ by

$$(C_\psi f)(z) = f(\psi(z)), \qquad z \in \mathsf{D},\ f \in H^\infty(\mathsf{D}).$$

The eigenvalue equation for this composition operator is $C_\psi(f) = \lambda f$. This is *Schröder's equation*, which arises in a number of contexts in analysis.

If $\psi^{\circ n}(\mathsf{D})$ is a relatively compact subset of D for some iterate $\psi^{\circ n}$ of ψ, then the iterates of ψ converge to a fixed point $z_0 \in \mathsf{D}$ of ψ. Further, the composition operator C_ψ is power compact, so the essential spectrum of C_ψ consists of the singleton $\{0\}$. If $\psi'(z_0) \neq 0$, then the point spectrum of C_ψ consists of a sequence of simple eigenvalues $\{\psi'(z_0)^n\}_{n=0}^\infty$. If $\psi'(z_0) = 0$, then the only point in the spectrum other than 0 is the simple eigenvalue 1 corresponding to the constant functions.

In contrast to this situation, L. Zheng [12] has shown that if ψ has a fixed point in D but C_ψ is not power compact, then the spectrum of C_ψ coincides with the closed unit disk: $\sigma(C_\psi) = \overline{\mathsf{D}}$. In this case it is not known whether

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the essential spectrum $\sigma_e(C_\psi)$ coincides also with the closed unit disk. Less is known in the case that ψ has its (Denjoy-Wolff) fixed point z_0 on $\partial\mathbb{D}$. Theorem 7.21 of [4], which applies to $H^\infty(\mathbb{D})$, shows that if $\psi'(z_0) < 1$, the spectrum of C_ψ is circular, that is, rotation-invariant. U. Gül [8] has shown that, under certain conditions on the boundary fixed point, the spectrum of C_ψ is a shrinking tube that spirals toward the origin.

Our aim is to investigate the spectral properties of a composition operator on an infinitely connected subdomain U of $\mathbb{D}$ for which the essential resolvent has infinitely many components, and for which the Fredholm index of the resolvent operator attains all nonnegative integer values. In Section 2 we introduce the domain U and we describe the Mittag-Leffler decomposition of analytic functions on U. In Section 3 we introduce the composition operator C_φ and describe the null space of $\lambda I - C_\varphi$. In Section 4 we determine the spectrum of C_φ. In Section 5 we determine the essential spectrum of C_φ and the Fredholm index of $\lambda I - C_\varphi$ for λ in the essential resolvent set.

These results were obtained by the author in his thesis [11] by a different method, which depended on the isomorphism used in [1] to find an infinitely connected domain in the plane for which the corona conjecture fails.

2. The Domain *U*

Fix $0 < \alpha < 1$, $0 < \sigma < 1$, and c_1 such that $\alpha < c_1 < 1$. Let $\gamma = \sigma\alpha$. We consider the domain U obtained from the punctured open unit disk $\mathbb{D} \setminus \{0\}$ by excising the closed subdisks $D_n = \{|z - c_n| \le \gamma^n\}$, $n \ge 1$, with centers $c_n = \alpha^{n-1}c_1$ and radii γ^n tending geometrically to 0,

$$U = (\mathbb{D} \setminus \{0\}) \setminus \cup_{n\ge1} D_n.$$

We choose the parameters α and c_1 so that $c_1 + \gamma < 1$, $\gamma(1+\gamma) < (1-\alpha)c_1$, and $\alpha + \gamma < c_1$. We define $c_0 = c_1/\alpha > 1$.

LEMMA 2.1. *With this choice of the parameters c_1 and γ, the closed disks D_n are disjoint subdisks of $\mathbb{D}$, $D_{n+1} \subset \alpha D_n$, and $\alpha U \subset U$. Further, if $\rho > \gamma$ is sufficiently close to γ, the annuli*

$$A_n = \{z : \gamma^n < |z - c_n| < \rho^n\}, \qquad n \ge 1,$$

form disjoint collars in U around the D_n's.

PROOF. The condition $c_1 + \gamma < 1$ guarantees that $D_1 \subset \mathbb{D}$. The condition for the D_n's to be disjoint is that $c_{n+1} + \gamma^{n+1} < c_n - \gamma^n$, and this follows from the condition on $\gamma(1+\gamma)$. One checks, using the condition $\alpha + \gamma < c_1$, that $\alpha U \subset U$. If $\rho > \gamma$ satisfies the same conditions as γ above, then the annular collars A_n are disjoint.

The radii defining the annular collars A_n satisfy $\sum \gamma^n/\rho^n < \infty$. Consequently U is a Behrens L-domain (see [1], [2]). We will use several of the estimates for Behrens L-domains appearing in [1] and [2].

Let $f \in H^\infty(U)$. For $n \geq 1$, we define $P_n f$ to be the unique function such that $P_n f$ is analytic outside D_n, $P_n f$ tends to 0 at ∞, and $f - P_n f$ extends to be analytic for $|z - c_n| < \rho^n$. Thus $f = P_n f + [f - P_n f]$ is the Laurent decomposition of f with respect to the annular collar A_n. In particular, $P_n f \in H^\infty(U)$. Each $P_n f$ has an expansion in powers of $1/(z - c_n)$:

$$(P_n f)(z) = \sum_{k=1}^{\infty} \frac{a_{nk}}{(z - c_n)^k}, \qquad |z - c_n| > \gamma^n, n \geq 1.$$

Similarly, we define $P_0 f \in H^\infty(\mathsf{D})$ to be the principal part of the Laurent expansion about the most external collar. The operators P_n are orthogonal projections, in the sense that $P_n^2 = P_n$ for $n \geq 0$, and $P_n P_m = 0$ for $n \neq m$. Each function $f \in H^\infty(U)$ has a Mittag-Leffler decomposition

$$f = \sum_{n=0}^{\infty} P_n f.$$

See [2], or the next lemma, for details about the convergence of this series.

For an integer $r \geq 1$, let M_r be the subspace of functions $f \in H^\infty(U)$ such that for $n \geq 1$, $(z - c_n)^r P_n f$ is bounded at ∞. In other words, M_r is the space of functions $f \in H^\infty(U)$ for which the coefficients in the Laurent expansion of $P_n f$ satisfy $a_{nk} = 0$ for $n \geq 1$ and $1 \leq k < r$. Thus M_1 coincides with $H^\infty(U)$.

LEMMA 2.2. *Fix an integer* $r \geq 1$. *If* $g_0 \in H^\infty(\mathsf{D})$, *and for* $1 \leq n < \infty$, $g_n \in H^\infty(D_n^c)$ *with* $\sup_{n\geq 0} \|g_n\| < \infty$, *then*

$$G = g_0 + \sum_{n=1}^{\infty} \frac{\gamma^{nr}}{(z - c_n)^r} g_n \tag{2.1}$$

converges boundedly on U *and uniformly on each subset of* U *at a positive distance from* 0, *and the function* G *is in* M_r. *Further, there are constants* C_0 *and* C_1, *independent of* r, *such that*

$$C_0\|G\| \leq \sup_{n\geq 0} \|g_n\| \leq C_1\|G\|.$$

Conversely, if $G \in M_r$, *then* G *has the above form for functions* g_n *as above.*

PROOF. Suppose $\sup \|g_n\| \leq 1$. Then $G_n = \gamma^{nr} g_n/(z - c_n)^r$ is analytic for $|z - c_n| > \gamma^n$ and at ∞, and it is bounded by $(\gamma/\rho)^{rn}$ in modulus for

$|z - c_n| = \rho^n$. By the maximum principle, $|G_n| \le (\gamma/\rho)^{rn}$ for $|z - c_n| \ge \rho^n$. Since the collars A_n are disjoint, we obtain for fixed $m \ge 1$ that

$$|G_n(z)| \le (\gamma/\rho)^{rn}, \qquad z \in A_m, n \ne m.$$

Thus the sum for $G(z)$ converges absolutely on U, it converges uniformly on each collar, and $|G(z)| \le 2 + \sum(\gamma/\rho)^{rn}$. For the converse, we define $G_n = P_n G$ for $n \ge 0$, $g_0 = G_0$, and $g_n = \gamma^{-nr}/(z - c_n)^r G_n$ for $n \ge 1$, and we make the usual estimates for G_n.

3. The Null Space of $\lambda I - C_\varphi$

Define $\varphi(z) = \alpha z$. Since $D_{n+1} \subset \alpha D_n$, U is invariant under φ, and we may define the composition operator C_φ on $H^\infty(U)$ by

$$(C_\varphi f)(z) = f(\alpha z), \qquad z \in U.$$

LEMMA 3.1. *Let $f \in H^\infty(U)$, and let λ be complex. Then $C_\varphi f = \lambda f$ if and only*

$$(\lambda I - C_\varphi) P_0 f = C_\varphi P_1 f, \tag{3.1}$$

and

$$(P_n f)(\alpha^{n-1} z) = \lambda^{n-1} (P_1 f)(z), \qquad n \ge 2. \tag{3.2}$$

PROOF. One checks that $C_\varphi P_{n+1} = P_n C_\varphi$ for $n \ge 1$. Thus $C_\varphi f = \sum C_\varphi P_n f = C_\varphi P_0 f + C_\varphi P_1 f + \sum_{n\ge1} C_\varphi P_{n+1} f = C_\varphi P_0 f + C_\varphi P_1 f + \sum_{n\ge1} P_n C_\varphi f$. If we compare this with $\lambda f = \lambda P_0 f + \sum_{n\ge1} \lambda P_n f$ and note that $C_\varphi P_0 f$ and $C_\varphi P_1 f$ belong to $H^\infty(\mathsf{D})$, we obtain $C_\varphi P_0 f + C_\varphi P_1 f = \lambda P_0 f$, which yields the first identity of the lemma, and $C_\varphi P_{n+1} f = P_n C_\varphi f = \lambda P_n f$ for $n \ge 1$. Thus $C_\varphi P_2 f = \lambda P_1 f$, $C_\varphi^2 P_3 f = \lambda C_\varphi P_2 f = \lambda^2 P_1 f$, etc., which yields after iteration the second identity.

Since the functions $C_\varphi P_0 f$ and $C_\varphi P_1 f$ belong to $H^\infty(\mathsf{D})$, the first equation in the lemma can be viewed as an eigenvalue equation for the restriction composition operator $T = C_\varphi | H^\infty(\mathsf{D})$ of C_φ to $H^\infty(\mathsf{D})$. If $P_1 f$ is known, $P_0 f$ is obtained by setting $h = C_\varphi P_1 f$ and solving

$$(\lambda I - T) P_0 f = h. \tag{3.3}$$

We record the following result for future use. It can be easily verified directly; see also the introductory comments.

LEMMA 3.2. *Let $T = C_\varphi | H^\infty(\mathsf{D})$. Then $\sigma(T) = \{0, 1, \alpha, \alpha^2, \ldots\}$. Each of the values $\lambda = \alpha^j$, $j \ge 0$, is a simple eigenvalue of T with eigenfunction z^j. If*

$h \in H^\infty(\mathsf{D})$, the equation $(\alpha^j I - T)g = h$ has a solution $g \in H^\infty(\mathsf{D})$ if and only if $h^{(j)}(0) = 0$. Any solution g is unique, up to adding a constant multiple of z^j.

LEMMA 3.3. *Let $m \geq 0$, and suppose $|\lambda| > \sigma^{m+1}$. If f is an eigenfunction of C_φ with eigenvalue λ, then $P_1 f$ is a linear combination of the m functions $1/(z - c_1)^k$, $1 \leq k \leq m$.*

PROOF. Suppose $C_\varphi f = \lambda f$, and write $P_n f = \sum_k a_{nk}/(z - c_n)^k$ as before. It suffices to show that $a_{1k} = 0$ when $\sigma^k < |\lambda|$.

From equation (3.2) and $c_n = \alpha^{n-1} c_1$, we obtain for $n \geq 2$ that

$$\lambda^{n-1} \sum_{k=1}^{\infty} \frac{a_{1k}}{(z - c_1)^k} = \sum_{k=1}^{\infty} \frac{a_{nk}}{(\alpha^{n-1} z - c_n)^k} = \sum_{k=1}^{\infty} \frac{1}{\alpha^{(n-1)k}} \frac{a_{nk}}{(z - c_1)^k}.$$

Equating coefficients, we obtain

$$\lambda^{n-1} a_{1k} = \frac{a_{nk}}{\alpha^{(n-1)k}}, \qquad k, n \geq 1.$$

From the usual Cauchy estimates $|a_{nk}| \leq \|f\| \gamma^{nk}$, and from $\gamma = \alpha\sigma$, we obtain

$$|\lambda^{n-1} a_{1k}| \leq \|f\| \frac{\gamma^{nk}}{\alpha^{(n-1)k}} = \|f\| \gamma^k \sigma^{(n-1)k}, \qquad n \geq 1, k \geq 1.$$

Dividing by λ^n and sending n to ∞, we see that $a_{1k} = 0$ for $\sigma^k < |\lambda|$.

THEOREM 3.4. *Let ℓ be the largest integer such that $\alpha^\ell > \sigma$. The eigenvalues λ of C_φ satisfying $|\lambda| > \sigma$ are the numbers $\lambda = \alpha^j$, $0 \leq j \leq \ell$. Each such eigenvalue is simple, with corresponding eigenfunction z^j.*

PROOF. Suppose $C_\varphi f = \lambda f$, where $|\lambda| > \sigma$. The lemma, with $m = 0$, shows that $P_1 f = 0$. Then also $P_n f = 0$ for $n \geq 2$, by (3.2), so $f \in H^\infty(\mathsf{D})$. Now apply Lemma 3.2.

LEMMA 3.5. *Let $m \geq 1$, and suppose $|\lambda| \leq \sigma^m$. If f_1 is a linear combination of the functions $1/(z - c_1)^k$, $1 \leq k \leq m$, and $(f_n)(\alpha^{n-1} z) = \lambda^{n-1}(f_1)(z)$ for $n \geq 2$, then $f = \sum f_n$ is bounded on U, that is, $f \in H^\infty(U)$. Further, f satisfies equation (3.2).*

PROOF. We may assume that $f_1(z) = 1/(z - c_1)^k$, where k is fixed and $1 \leq k \leq m$. For $n \geq 2$, set

$$(f_n)(z) = \lambda^{n-1}(f_1)(\alpha^{1-n} z) = \lambda^{n-1} \frac{1}{(\alpha^{1-n} z - c_1)^k} = \lambda^{n-1} \frac{1}{(\alpha^{1-n}(z - c_n))^k}.$$

It suffices to show that the partial sums of $\sum_n |f_n(z)|$ are uniformly bounded on $\cup\partial D_j$. Then the partial sums are uniformly bounded on U and this guarantees that the series $\sum_n f_n(z)$ converges normally to a function $f \in H^\infty(U)$ that satisfies (3.2).

Fix a point z in the boundary of the qth disk D_q, so that $|z - c_q| = \gamma^q$. Since $k \le m$, we have $|\lambda| \le \sigma^k$, and

$$|f_q(z)| = \frac{|\lambda|^{q-1}\alpha^{(q-1)k}}{|z - c_q|^k} = \frac{|\lambda|^{q-1}\alpha^{(q-1)k}}{\gamma^{kq}}$$
$$\le \frac{\sigma^{(q-1)k}\alpha^{(q-1)k}}{\gamma^{kq}} = \frac{\gamma^{(q-1)k}}{\gamma^{kq}} = \gamma^{-k}.$$

For $n < q$, we have $|\alpha^{1-n}z - c_1| = \alpha^{1-n}|z - c_n| \ge \alpha^{1-n}(c_n - c_q) - \alpha^{1-n}|z - c_q| \ge \alpha^{1-n}(c_n - c_{n+1}) - \alpha^{1-n}\gamma^q \ge c_1 - c_2 - \gamma > 0$. Consequently

$$\sum_{n=1}^{q-1} |f_n(z)| \le \sum_{n=1}^{q-1} \frac{|\lambda^{n-1}|}{|\alpha^{1-n}z - c_1|^k} \le (c_1 - c_2 - \gamma)^{-k} \sum_{n=1}^{\infty} |\sigma|^{m(n-1)}.$$

Similarly, for $n > q$, we have $|\alpha^{1-n}z - c_1| = \alpha^{1-n}|z - c_n| \ge \alpha^{1-n}(c_q - c_n) - \alpha^{1-n}|z - c_q| \ge \alpha^{1-n}(c_{n+1} - c_n) - \alpha^{1-n}\gamma^q \ge \alpha(c_1 - c_2 - \gamma) > 0$, and the sum over the terms for which $n > q$ is also bounded by a constant independent of q.

THEOREM 3.6. *If $m \ge 1$ and $\sigma^{m+1} < |\lambda| \le \sigma^m$, then the dimension of the null space of $\lambda I - C_\varphi$ is m.*

PROOF. Let g_1 be a linear combination of $1/(z - c_1)^k$, $1 \le k \le m$. For $n \ge 2$, define g_n as in Lemma 3.5, and $G = \sum_{n \ge 1} g_n$. By Lemma 3.5, $G \in H^\infty(U)$, and $g_n = P_n G$ satisfies (3.2).

If λ is not an eigenvalue of $T = C_\varphi | H^\infty(\mathsf{D})$, we set $g_0 = (\lambda I - T)^{-1} C_\varphi g_1 \in H^\infty(\mathsf{D})$. Then $F = g_0 + G$ satisfies (3.1), so $(\lambda I - C_\varphi)F = 0$. By Lemmas 3.1 and 3.3, all functions in the null space of $\lambda I - C_\varphi$ arise in this manner. Since the g_1's form a space of dimension m, the dimension of the null space of $\lambda I - C_\varphi$ is m.

Suppose λ is an eigenvalue of T, say $\lambda = \alpha^j$. The equation $(\lambda I - T)g_0 = g_1$ is solvable if only if $g_1^{(j)}(0) = 0$. Since the subspace $\{g_1 : g_1^{(j)}(0) = 0\}$ is $(m-1)$-dimensional, we may select a linearly independent set of $m - 1$ functions g_1 for which (3.1) is solvable, and then every solution of (3.1) and (3.2) is a linear combination of these and the function z^j. Again the dimension of the null space of $\lambda I - C_\varphi$ is m.

4. The Spectrum of C_φ

Recall the definition of M_r, and the characterization of functions in M_r given in Lemma 2.2. To determine the range of $\lambda I - C_\varphi$, we first solve $(\lambda I - C_\varphi) f = G$ for $G \in M_r$.

LEMMA 4.1. *Fix $\lambda \neq 0$, and suppose $r \geq 1$ satisfies $\sigma^r < |\lambda|$. Then $(\lambda I - C_\varphi)(M_r)$ is a subspace of M_r of codimension at most one. If additionally λ is not an eigenvalue of the restriction T of C_φ to $H^\infty(\mathsf{D})$, then $(\lambda I - C_\varphi) M_r = M_r$.*

PROOF. Let $G \in M_r$, and express $G = g_0 + \sum_n \gamma^{nr}(z - c_n)^{-r} g_n$ as in Lemma 2.2. Set

$$h = \sum_{k=1}^{\infty} \left[\frac{\gamma^{kr}}{(z - c_k)^r} \frac{1}{\lambda} \sum_{n=0}^{\infty} \left(\frac{\sigma^r}{\lambda} \right)^n C_\varphi^n g_{k+n} \right].$$

For fixed k, the inside sum over n is analytic for $|z - c_k| > \gamma^k$ and bounded by $\sum_n (\sigma^r/|\lambda|)^n \sup \|g_n\| \leq (1 - \sigma^r/|\lambda|) \sup \|g_n\|$. Hence $h \in M_r$. Using

$$C_\varphi \left(\frac{\gamma^{kr}}{(z - c_k)^r} \right) = \sigma^r \frac{\gamma^{(k-1)r}}{(z - c_{k-1})^r},$$

we compute that

$$(\lambda I - C_\varphi) h = \sum_{n=0}^{\infty} \frac{\sigma^{r(n+1)}}{\lambda^{n+1}} \frac{C_\varphi^{n+1} g_{n+1}}{(z - c_0)^r} + \sum_{k=1}^{\infty} \frac{\gamma^{kr}}{(z - c_k)^r} g_k.$$

The first sum is a bounded analytic function on D since $c_0 > 1$, and the second sum coincides with $G - g_0$. Hence $(\lambda I - C_\varphi) h = G - f_0$, where $f_0 \in H^\infty(\mathsf{D})$. If λ is not an eigenvalue of T, we may solve $(\lambda I - T) h_0 = f_0$ for $h_0 \in H^\infty(\mathsf{D})$. Then $(\lambda I - C_\varphi)(h + h_0) = G$. This proves the second statement of the lemma.

Suppose λ is an eigenvalue of T, say $\lambda = \alpha^j$. Let h and f_0 be as above, and choose a constant β such that the qth derivative of $f_0 - \beta z^j$ vanishes at $z = 0$. Then we may solve $(\lambda I - T) h_0 = f_0 - \beta z^j$ for $h_0 \in H^\infty(\mathsf{D})$, to obtain $(\lambda I - C_\varphi)(h + h_0) = G - \beta z^j$. Thus $(\lambda I - C_\varphi) M_r$ and z^j span M_r.

THEOREM 4.2. *The spectrum of C_φ consists of the disk $\{|\lambda| \leq \sigma\}$, together with the simple eigenvalues $\{1, \alpha, \alpha^2, \ldots, \alpha^\ell\}$, where ℓ is the largest integer such that $\alpha^\ell > \sigma$.*

PROOF. Theorem 3.6 shows that the spectrum includes the disk $\{|\lambda| \leq \sigma\}$. Lemma 4.1, applied in the case $r = 1$, shows that if $|\lambda| > \sigma$, the range of $\lambda I - C_\varphi$ on $M_1 = H^\infty(U)$ has codimension at most one, and moreover, $\lambda I - C_\varphi$

is onto unless λ is an eigenvalue of T. By Theorem 3.4, the eigenvalues λ of C_φ satisfying $|\lambda| > \sigma$ are eigenvalues of T. Thus the spectral points λ satisfying $|\lambda| > \sigma$ are the powers α^j of α satisfying $\alpha^j > \sigma$.

5. The Essential Spectrum and Fredholm Index

The work in the preceding section shows that $\lambda I - C_\varphi$ is a Fredholm operator if $|\lambda| > \sigma$. To complete the description of the Fredholm points, we need the following lemma.

LEMMA 5.1. *Fix $\lambda \neq 0$. Suppose $r \geq 1$ satisfies $\sigma^r > |\lambda|$. For any function $g \in H^\infty(U)$ of the form*

$$g = \sum_{n\geq 1} a_n \frac{\gamma^{nr}}{(z - c_n)^r},$$

there is a function $f \in H^\infty(U)$ of the form

$$f = \sum_{n\geq 2} b_n \frac{\gamma^{nr}}{(z - c_n)^r},$$

such that $(\lambda I - C_\varphi) f = g$.

PROOF. We compute that

$$(\lambda I - C_\varphi) f = -b_2 \sigma^r \frac{\gamma^r}{z - c_1} + \sum_{n=2}^{\infty} (\lambda b_n - \sigma^r b_{n+1}) \frac{\gamma^{rn}}{(z - c_n)^r}.$$

Equating coefficients, we have $(\lambda I - C_\varphi) f = g$ whenever $-b_2 \sigma^r = a_1$ and $\lambda b_n - \sigma^r b_{n+1} = a_n$ for $n \geq 2$. This occurs if $b_2 = -a_1/\sigma^{-r}$, and $b_n = \lambda \sigma^{-r} b_{n-1} - \sigma^{-r} a_{n-1}$ for $n \geq 3$. We check by induction that

$$|b_n| \leq \sigma^{-r} \bigl(1 + |\lambda| \sigma^{-r} + (|\lambda| \sigma^{-r})^2 + \cdots + (|\lambda| \sigma^{-r})^{n-2}\bigr) \sup |a_j|, \quad n \geq 2,$$

so the b_n's are bounded, and $f \in H^\infty(U)$.

THEOREM 5.2. *The essential spectrum of C_φ consists of the circles $\{|\lambda| = \sigma^r\}$ for $r \geq 1$, together with the point $\{0\}$. If $r \geq 1$ and $\sigma^{r+1} < |\lambda| \leq \sigma^r$, then $\lambda I - C_\varphi$ is onto, the dimension of the null space of $\lambda I - C_\varphi$ is r, and the Fredholm index of $\lambda I - C_\varphi$ is r. If $|\lambda| > \sigma$, then the Fredholm index of $\lambda I - C_\varphi$ is* 0.

PROOF. Suppose $r \geq 1$ and $\sigma^{r+1} < |\lambda| < \sigma^r$. By Lemma 4.1, functions in M_{r+1} belong to the range of $\lambda I - C_\varphi$. If $1 \leq s \leq r$, then by Lemma 5.1, each function $G_s \in H^\infty(U)$ of the form $G_s = \sum_{n\geq 1} a_n \gamma^{ns}/(z - c_n)^s$ belongs to the range of $\lambda I - C_\varphi$. Each $G \in H^\infty(U)$ can be represented as a sum of such functions G_s, $1 \le s \le r$, and a function in M_{r+1}, hence the range of $\lambda I - C_\varphi$ coincides with $H^\infty(U)$. By Theorem 3.6, the dimension of the null space of $\lambda I - C_\varphi$ is r. Consequently the points λ in the annulus $\{\sigma^{r+1} < |\lambda| < \sigma^r\}$ are Fredholm points with index r. Since the set of Fredholm points is open, and the Fredholm index is locally constant, the circles forming the boundaries of these annuli lie in the essential spectrum, as does $\lambda = 0$.

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DEPARTMENT OF MATHEMATICS AND STATISTICS
CALIFORNIA STATE UNIVERSITY, CHICO
CHICO, CA 95928

Current address:
DEPARTMENT OF MATHEMATICS
SALEM STATE UNIVERSITY
352 LAFAYETTE STREET
SALEM, MA 01970
E-mail: cyakes@salemstate.edu