MATH. SCAND. 118 (2016), 106–118

A DECOMPOSITION THEOREM FOR POSITIVE MAPS, AND THE PROJECTION ONTO A SPIN FACTOR

ERLING STØRMER

Abstract

It is shown that each positive map between matrix algebras is the sum of a maximal decomposable map and an atomic map which is both optimal and co-optimal. The result is studied in detail for the projection onto a spin factor.

Introduction

The structure of positive maps between C^*-algebras, even in the finite dimensional case, is still poorly understood. The only maps which are well understood are the decomposable ones, which are sums of completely positive and co-positive maps, hence in the finite dimensional case, are sums of maps of the form $\operatorname{Ad} v$ and $t \circ \operatorname{Ad} v$, where t is the transpose map, and $\operatorname{Ad} v$ the map $x \to v^*xv$. In the present paper we shall shed some light on the structure of positive maps by showing that they are the sum of a maximal decomposable map and an atomic map, which is bi-optimal, i.e. it majorizes neither a non-zero completely positive map nor a co-positive map.

In order to obtain a deeper understanding of this decomposition we study it in detail in Section 2 for the trace invariant positive projection of the full matrix algebra M_{2^n} onto a spin factor inside it. We shall obtain explicit formulas for the decomposable map and the bi-optimal map in the decomposition when the spin factor is irreducible and contained in the $2^{n-1} \times 2^{n-1}$ matrices over the quaternions.

For the reader's convenience we recall the main definitions concerning positive maps, see also [8]. We let A be a finite dimensional C^*-algebra and $B(H)$ the bounded operators on a finite dimensional Hilbert space H.

Let $\phi\colon A \to B(H)$ be a linear map. Then ϕ is *positive*, written $\phi \geq 0$ or $0 \leq \phi$ if it carries positive operators to positive operators. If ψ is another positive map, ψ *majorizes* ϕ, written $\psi \geq \phi$, if $\psi - \phi \geq 0$. The map ϕ is *k-positive* if $\iota_k \otimes \phi\colon M_k \otimes A \to M_k \otimes B(H)$ is positive, where ι_k is the identity

Received 13 August 2013.

map on the $k \times k$ matrices M_k. The map ϕ is *completely positive* if ϕ is k-positive for all k. Let t denote the transpose map on $B(H)$ with respect to some fixed orthonormal basis. Then ϕ is *k-co-positive*, (resp. *co-positive*) if $t \circ \phi$ is k-positive (resp. *completely positive*). The map ϕ is *k-decomposable* (resp. *decomposable*) if ϕ is the sum of a k-positive and a k-co-positive map (resp. sum of a completely positive and a co-positive map). The map ϕ is *atomic* if ϕ is not 2-decomposable. The map ϕ is *extremal*, or just extreme, if $\phi \geq \psi$ for a positive map ψ implies $\psi = \lambda\phi$ for some non-negative number λ. The map ϕ is *optimal* (resp. *co-optimal*) if $\phi \geq \psi$ for ψ completely positive (resp. co-positive) implies $\psi = 0$. Combining the last two concepts we introduce the following definition, which has also been introduced by Ha and Kye [3].

DEFINITION 1. ϕ is *bi-optimal* if ϕ is both optimal and co-optimal.

The author is grateful to E. Alfsen for many helpful discussions on spin factors.

1. The decomposition theorem

Let K and H be finite dimensional Hilbert spaces. In [5], Theorem 3.4, Marciniak showed the surprising result that if ϕ is a 2-positive map (resp. 2-co-positive) which is extremal, then ϕ is completely positive (resp. co-positive). His proof, see also [8], Theorem 3.3.7, contained more information, namely the following result.

LEMMA 2. *Let ϕ be a non-zero 2-positive map of $B(K)$ into $B(H)$. Then there exists a non-zero completely positive map $\psi: B(K) \to B(H)$ such that $\phi \geq \psi$.*

A slight extension of the above lemma yields the following.

PROPOSITION 3. *Let A be a finite dimensional C^*-algebra and $\phi: A \to B(H)$ a non-zero 2-decomposable map. Then there exists a non-zero decomposable map $\psi: A \to B(H)$ such that $\phi \geq \psi$.*

PROOF. We first consider the case when $A = B(K)$. Since ϕ is 2-decomposable there exist a 2-positive map ϕ_1 and a 2-co-positive map ϕ_2 such that $\phi = \phi_1 + \phi_2$. By Lemma 2 there is a completely positive map ψ_1, non-zero if ϕ_1 is non-zero, such the $\phi_1 \geq \psi_1$. Applying Lemma 2 to $t \circ \phi_2$ we find a co-positive map $\psi_2 \leq \phi_2$. Thus $\phi \geq \psi_1 + \psi_2$, proving the proposition when $A = B(K)$.

In the general case let $e_1, \ldots, e_m$ be the minimal central projections in A, so $A = \bigoplus_{i=1}^{m} Ae_i$. Then each Ae_i is isomorphic to some $B(K)$, and $\phi_{|Ae_i}$ is 2-decomposable. By the first part $\phi_{|Ae_i} \geq \alpha_i + \beta_i$ with α_i completely positive and β_i co-positive. Let $\alpha = \sum \alpha_i$ and $\beta = \sum \beta_i$. Then α is completely positive and

β co-positive, hence $\alpha+\beta$ is a decomposable map majorized by ϕ, completing the proof of the proposition.

COROLLARY 4. *Each bi-optimal map of a finite dimensional C^*-algebra into $B(H)$ is atomic.*

PROOF. By definition a map ϕ is atomic if it is not 2-decomposable. By the definition of being bi-optimal such a map ϕ cannot majorize a decomposable map, hence by Proposition 3, ϕ cannot be 2-decomposable, completing the proof.

Since completely positive maps are sums of maps of the form $\operatorname{Ad} v$, and each co-positive map a sum of maps $t \circ \operatorname{Ad} v$, our next result reduces much of the study of positive maps to that of bi-optimal maps. If $\phi\colon A \to B(H)$ is positive, A a C^*-algebra, we say a decomposable map $\alpha\colon A \to B(H), \alpha \leq \phi$ is a *maximal decomposable map majorized by* ϕ if there is no decomposable map $\psi\colon A \to B(H)$ such that $\psi \neq \alpha$ and $\alpha \leq \psi \leq \phi$.

THEOREM 5. *Let A be a finite dimensional C^*-algebra and H a finite dimensional Hilbert space. Let $\phi\colon A \to B(H)$ be a positive map. Then there are a maximal decomposable map $\alpha\colon A \to B(H)$ majorized by ϕ and a bi-optimal, hence atomic, map $\beta\colon A \to B(H)$ such that $\phi = \alpha + \beta$.*

PROOF. We first assume $A = B(K)$ for a finite dimensional Hilbert space K. Let $C = \{\psi\colon B(K) \to B(H) : \psi \text{ decomposable}, \psi \leq \phi\}$. Then C is bounded and norm closed, hence is compact in the norm topology, as K and H are finite dimensional. Furthermore C is an ordered set with the usual ordering on positive maps. We show C has a maximal element. For this let $X = \{\phi_v \in C : v \in F\}$ be a totally ordered set with $\phi_v \leq \phi_{v'}$ if $v \leq v'$ in F. For each $v \in F$ let $X_v = \{\phi_{v'} \in X : v \leq v'\}$. Then X_v is closed, and $X_v \supset X_{v'}$ if $v \leq v'$. Since X is totally ordered it follows that the sets X_v with $v \in F$ have the finite intersection property. Thus the intersection $\bigcap_{v\in F} X_v \neq \emptyset$, hence a map $\psi \in \bigcap X_v$ is an upper bound for X. By Zorn's lemma, C has a maximal element α. Since C is closed, α is decomposable, $\alpha \leq \phi$, and there is no decomposable map $\psi\colon B(K) \to B(H)$ different from α such that $\alpha \leq \psi \leq \phi$. Thus α is maximal decomposable map majorized by ϕ.

Let $\beta = \phi - \alpha$. Then β is bi-optimal, for if $\gamma \leq \beta$, $\gamma \neq 0$ and decomposable, then $\alpha+\gamma$ is decomposable, and $\alpha+\gamma \leq \alpha+\beta = \phi$, contradicting maximality of α. Thus $\gamma = 0$, and β is bi-optimal.

In the general case we imitate the proof of Proposition 3 and write A as $A = \bigoplus Ae_i$ where the e_i are minimal central projections in A, so Ae_i is isomorphic to some $B(K)$, and we apply the first part of the proof to each Ae_i in the same way as we did in the proof of Proposition 3. The proof is complete.

If we do not require α in the theorem to be maximal decomposable we can have different decompositions. For example, if ϕ is a bi-optimal map, and Tr is the trace on $B(K)$, then the map $\psi(x) = \phi(1)\,\mathrm{Tr}(x) + \phi(x)$ is super-positive, hence in particular completely positive, see [8], Theorem 7.5.4. But ψ has a decomposition $\psi = \alpha + \beta$, where $\alpha = \phi(1)\,\mathrm{Tr}$ is completely positive, and $\beta = \phi$ is bi-optimal.

COROLLARY 6. *With assumptions as in Theorem 5, if ϕ is extreme, then ϕ is either of the form* Ad v, $t \circ$ Ad v *or ϕ is bi-optimal, so atomic.*

If we in the proof of Theorem 5 replace decomposable map by completely positive map and bi-optimal by optimal and define maximal completely positive map majorized by ϕ in analogy with the definition for decomposable maps, we obtain the following result.

THEOREM 7. *Let A be a finite dimensional C^*-algebra and H a finite dimensional Hilbert space. Let $\phi\colon A \to B(H)$ be a positive map. Then there are a maximal completely positive map $\alpha\colon A \to B(H)$ majorized by ϕ and an optimal map $\beta\colon A \to B(H)$ such that $\phi = \alpha + \beta$.*

2. Spin factors

In the present section we illustrate the decomposition theorems, Theorem 5 and Theorem 7, by the projection of $B(H)$ onto a spin factor. Following [2] we recall that a *spin system* in $B(H)$ is a set of symmetries, i.e. self-adjoint unitaries $s_1, \dots, s_m$ satisfying the anti-commutation relations $s_i s_j + s_j s_i = 0$ for $i \neq j$. Let

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

denote the Pauli matrices in M_2. Then we can construct a spin system $\{s_1, \dots, s_{2n}\}$ in $M_{2^n} = \bigotimes_1^n M_2$ as follows, where $1 \le k < n-1$:

$$\begin{aligned}
s_1 &= \sigma_1 \otimes 1^{\otimes n-1} \\
s_2 &= \sigma_2 \otimes 1^{\otimes n-1} \\
&\dots\dots\dots\dots\dots\dots \\
s_{2k+1} &= \sigma_3^{\otimes k} \otimes \sigma_1 \otimes 1^{\otimes n-k-1} \\
s_{2k+2} &= \sigma_3^{\otimes k} \otimes \sigma_2 \otimes 1^{\otimes n-k-1} \\
&\dots\dots\dots\dots\dots\dots \\
s_{2n-1} &= \sigma_3^{\otimes n-1} \otimes \sigma_1 \\
s_{2n} &= \sigma_3^{\otimes n-1} \otimes \sigma_2
\end{aligned}$$

where for $a \in M_2$, $a^{\otimes k}$ denotes the k-fold tensor product of a with itself.

Let V_m denote the linear span of $s_0 = 1, s_1, \dots, s_m$. Then V_m is a spin factor of dimension $m+1$ in M_{2^n}. For $m = 2n$ the C^*-algebra $C^*(V_m)$ generated by V_{2n} equals M_{2^n}, so in that case V_m is irreducible, see [2], Theorem 6.2.2. If $m = 2n-1$ then $C^*(V_m) = M_{2^{n-1}} \oplus M_{2^{n-1}} \subset M_{2^n}$.

By [1] or [8], Proposition 2.2.10, if Tr denotes the usual trace on M_{2^n} then there exists a positive idempotent map $P\colon M_{2^n} \to V_m + iV_m$ given by $\mathrm{Tr}(P(a)b) = \mathrm{Tr}(ab)$ for all $a \in M_{2^n}$, $b \in V_m + iV_m$, $m \leq 2n$. Then P restricted to the self-adjoint part of M_{2^n} is a projection map onto V_m. With the Hilbert-Schmidt structure the set $\{1, s_1, \dots, s_m\}$ is an orthonormal basis for V_m with respect to the normalized trace $2^{-n}\,\mathrm{Tr}$ on M_{2^n}. Thus P has the form

$$P(a) = 2^{-n} \sum_{i=0}^{m} \mathrm{Tr}(s_i a) s_i.$$

By [7] or [8], Theorem 2.3.4, P is atomic if $n \neq 2, 3, 5$. By [2], Theorem 6.2.3, V_m is a JW-factor of type I_2, i.e. for each minimal projection $e \in V_m$, the operator $1 - e$ is also a minimal projection. Thus $\mathrm{Tr}(e) = 2^{n-1}$. Note that for each $i \geq 1$, the maps $e_+ = \frac{1}{2}(1 + s_i)$ and $e_- = \frac{1}{2}(1 - s_i)$ are such projections.

Let t denote the transpose on M_2. Then

$$\sigma_1^t = \sigma_1, \qquad \sigma_2^t = \sigma_2, \qquad \sigma_3^t = -\sigma_3.$$

Since the transpose on M_{2^n} is the tensor product $t^{\otimes n}$, it follows from the defining equations for s_k that

$$s_{2k+1}^t = (-1)^k s_{2k+1}, \qquad s_{2k+2}^t = (-1)^k s_{2k+2}.$$

It follows in particular that $P \circ t = t \circ P$.

LEMMA 8. *Define a symmetry $W \in M_{2^n}$ as follows:*

(i) *If n is odd, $n = 2m+1$, $W = (1 \otimes \sigma_3)^{\otimes m} \otimes 1$.*

(ii) *If n is even, $n = 2m$, $W = (1 \otimes \sigma_3)^{\otimes m}$.*

Then $\mathrm{Ad}\, W(s_k) = s_k^t$ *for all* $1 \leq k \leq 2n$. *Hence* $\mathrm{Ad}\, W(a) = a^t$ *for all* $a \in V_n$. *Furthermore, if n is of the form $n = 4m + i$, $i = 0, 1$, then $W \in C^*(V_n)$.*

PROOF. If $k = 1, 2$, then $\mathrm{Ad}\, W(s_k) = s_k = {s_k}^t$, so we may assume $k \geq 3$. We first consider the case when $k = 2j + 1$ with j odd. Then

$$s_k = \sigma_3^{\otimes j} \otimes \sigma_1 \otimes 1^{\otimes n-j-1}.$$

Thus by definition of W, since $\mathrm{Ad}\, \sigma_3(\sigma_1) = -\sigma_1$, we have

$$\mathrm{Ad}\, W(s_k) = \sigma_3^{\otimes j} \otimes (-\sigma_1) \otimes 1^{\otimes n-j-1} = -s_k = (-1)^j s_k = s_k^t.$$

Similarly if $k = 2j+2$ with j odd, then $\mathrm{Ad}\, W(s_k) = {s_k}^t$. Now let $k = 2j+1$ with j even. Then

$$\mathrm{Ad}\, W(s_k) = \sigma_3^{\otimes j} \otimes \sigma_1 \otimes 1^{\otimes n-j-1} = s_k = (-1)^j s_k = s_k^t.$$

Similarly for $k = 2j + 2$ with j even. Thus in every case $\mathrm{Ad}\, W(s_k) = s_k^t$. Since V_n is the real linear span of s_k, $k = 0, 1, \ldots, n$, we have $\mathrm{Ad}\, W(a) = a^t$ for all $a \in V_n$.

If $n = 4m+i$, $i = 0, 1$, then, since $\sigma_3^t = -\sigma_3$, and there are $2m$ factors of σ_3 in W, we have $W^t = W$. If $i = 0$ then by [2], Theorem 6.2.2, $C^*(V_n) = M_{2^n}$, so clearly $W \in C^*(V_n)$. If $n = 4m + 1$ then again by [2], Theorem 6 2.2,

$$C^*(V_{4m+1}) = M_{2^{4m}} \oplus M_{2^{4m}} \subset M_{2^{4m+1}}.$$

Since in this case $W = (1 \otimes \sigma_3)^{\otimes 2m} \otimes 1$, it follows that $W \in M_{4^m} \otimes \mathbf{C} \subset C^*(V_{4m+1}) = C^*(V_n)$, completing the proof of the lemma.

LEMMA 9. *Let $m \leq 2n$ and $P\colon M_{2^n} \to V_m$ be the trace invariant projection. Let W be as in Lemma 8. Then*

$$P = P \circ t \circ \mathrm{Ad}\, W.$$

PROOF. By Lemma 8 if $a \in V_m$ then $t \circ \mathrm{Ad}\, W(a) = a$. Thus if $x \in M_{2^n}$ then

$$(P \circ t \circ \mathrm{Ad}\, W) \circ (P \circ t \circ \mathrm{Ad}\, W)(x) = P \circ (P \circ t \circ \mathrm{Ad}\, W)(x) = P \circ t \circ \mathrm{Ad}\, W(x).$$

Thus $P \circ t \circ \mathrm{Ad}\, W$ is idempotent with range V_m and being the identity on V_m. Since P is trace invariant, if $x \in M_{2^n}$, $y \in V_m$ we have

$$\begin{aligned}\mathrm{Tr}(P \circ t \circ \mathrm{Ad}\, W(x)y) &= \mathrm{Tr}(t \circ \mathrm{Ad}\, W(x)y) = \mathrm{Tr}(\mathrm{Ad}\, W(x)y^t)\\ &= \mathrm{Tr}(x\, \mathrm{Ad}\, W \circ t(y)) = \mathrm{Tr}(xy) = \mathrm{Tr}(P(x)y),\end{aligned}$$

using that $\mathrm{Ad}\, W \circ t = t \circ \mathrm{Ad}\, W = \iota$ on V_m, where ι is the identity map on V_m. The lemma follows.

The following lemma is probably well known, but is included for completeness.

LEMMA 10. *Let $a \in B(H)$ be positive and e, f projections in $B(H)$ with sum* 1. *Then*

$$2(eae + faf) \geq a.$$

PROOF. We have

$$a = (e + f)a(e + f) = eae + eaf + fae + faf.$$

Let

$$b = (e - f)a(e - f) = eae - eaf - fae + faf \geq 0.$$

Thus

$$a \leq a + b = 2(eae + faf),$$

as asserted.

We shall need the following slight extension of a result of Robertson [6]. For simplicity we show it in the finite-dimensional case. Recall that M' denotes the commutant for a set $M \subset B(H)$ and that B_{sa} denotes the set of self-adjoint operators in M.

LEMMA 11. *Let H be a finite-dimensional Hilbert space, let $B \subset B(H)$ be a C^*-algebra and let $A \subset B_{sa}$ be a Jordan algebra with $1 \in A$. Suppose $P: B_{sa} \to A$ is a positive projection map. Suppose $\phi \leq P$ is a completely positive map, $\phi: B \to B$. Then $\phi(1) \in C^*(A)'$, and $\phi(x) = \phi(1)x$ for $x \in C^*(A)$.*

PROOF. By [8], Lemma 2.3.5, since $P(x) = x$ for $x \in A$, we have $\phi(1) \in A$ and $\phi(x) = \phi(1)x = x\phi(1)$, for $x \in A$. Since $C^*(A)$ is the C^*-algebra generated by A, $\phi(1) \in C^*(A)'$. Since H is finite dimensional, if e is the range projection of $\phi(1)$, then $\phi(1)$ has a bounded inverse $\phi(1)^{-1}$ on eH. Thus

$$\psi = \phi(1)^{-1}e\phi$$

is a unital map of B into eBe such that for $x \in A$,

$$\psi(x) = \phi(1)^{-1}e\phi(x) = \phi(1)^{-1}\phi(1)x = ex.$$

Thus $\psi_{|A}$ is a Jordan homomorphism, so $A \subset D = \{x \in B_{sa} : \psi(x^2) = \psi(x)^2\}$, the definite set for ψ. Since ψ is completely positive, by [6] or [8], Proposition 2.1.8, D is the self-adjoint part of a C^*-algebra, hence ψ is a homomorphism on $C^*(A)$. Since by the above $\psi(x) = ex$ for $x \in A$, $\psi(x) = ex$ for $x \in C^*(A)$. If $x \in C^*(A)$, $0 \leq x \leq 1$, then $\phi(x) \leq \phi(1) = e\phi(1)$. Thus $\phi(x) = e\phi(x)$, so that for all $x \in C^*(A)$, we have

$$\phi(x) = e\phi(x) = \phi(1)\psi(x) = \phi(1)ex = \phi(1)x,$$

proving the lemma.

LEMMA 12. *Let $P: M_{2^n} \to V_m$, $m \leq 2n$ be the trace invariant projection. Then $P \geq 2^{-n}\iota$, and $P \geq 2^{-n}t \circ \mathrm{Ad}\, W$, with W as in Lemma 8. Furthermore there exists a 1-dimensional projection $q \in M_{2^n}$ such that $P(q) = 2^{-n}1$, hence*

$$2^{-n} = \max\{\lambda \geq 0 : P \geq \lambda\iota\}.$$

PROOF. Let p be a 1-dimensional projection in M_{2^n}. Since V_m is a JW-factor of type I_2, [2], Theorem 6.1.8, there are two minimal projections e and f in V_m with sum 1 and $a, b \geq 0$ such that

$$P(p) = ae + bf.$$

By [8], Proposition 2.1.7, $P(epe) = eP(p)e = ae$, so that

$$a2^{n-1} = \mathrm{Tr}(ae) = \mathrm{Tr}(P(epe)) = \mathrm{Tr}(epe).$$

Hence

$$a = 2^{-n+1}\,\mathrm{Tr}(epe), \quad b = 2^{-n+1}\,\mathrm{Tr}(fpf).$$

Since epe is positive of rank 1, $\mathrm{Tr}(epe) \geq epe$. Thus, using Lemma 10 we get

$$\begin{aligned} P(p) &= 2^{-n+1}(\mathrm{Tr}(epe)e + \mathrm{Tr}(fpf)f) \\ &\geq 2^{-n+1}(epe + fpf) \\ &\geq 2^{-n+1}\tfrac{1}{2}(epe + epf + fpe + fpf) \\ &= 2^{-n}p. \end{aligned}$$

Since this holds for all 1-dimensional projections p, $P \geq 2^{-n}\iota$. By Lemma 9 it thus follows that

$$P = P \circ t \circ \mathrm{Ad}\,W \geq 2^{-n} t \circ \mathrm{Ad}\,W,$$

proving the first part of the lemma.

To show the second part we exhibit a 1-dimensional projection q such that $P(q) = 2^{-n}1$. The Pauli matrix σ_3 is of the form $\sigma_3 = e_0 - f_0 \in M_2$ with e_0, f_0 1-dimensional projections in M_2. Let Tr_2 denote the usual trace on M_2. Then for $j = 1, 2$, we have

$$\begin{aligned} 0 = \mathrm{Tr}_2(\sigma_3\sigma_j) &= \mathrm{Tr}_2(e_0\sigma_j) - \mathrm{Tr}_2(f_0\sigma_j) \\ &= \mathrm{Tr}_2(e_0\sigma_j - (1 - e_0)\sigma_j) \\ &= 2\,\mathrm{Tr}_2(e_0\sigma_j) - \mathrm{Tr}_2(\sigma_j) \\ &= 2\,\mathrm{Tr}_2(e_0\sigma_j). \end{aligned}$$

Furthermore, $\mathrm{Tr}_2(e_0\sigma_3) = \mathrm{Tr}_2(e_0(e_0 - f_0)) = \mathrm{Tr}_2(e_0) = 1$. Let $q = e_0^{\otimes n} \in M_{2^n}$. If $j = 2k - i$, $i = 0, 1$, then $s_j = \sigma_3^{\otimes k-1} \otimes \sigma_j \otimes 1^{\otimes n-k}$. From the above we thus have

$$\mathrm{Tr}(qs_j) = \mathrm{Tr}_2(e_0\sigma_j) = 0.$$

Thus, since $s_0 = 1$, we have

$$P(q) = 2^{-n}\left(\sum_{j=0}^{m} \mathrm{Tr}(qs_j)s_j\right) = 2^{-n}\,\mathrm{Tr}(qs_0)s_0 = 2^{-n}1,$$

completing the proof.

The projection q above is not symmetric because $\sigma_3^t = -\sigma_3 = f_0 - e_0$, so that $e_0^t = f_0$. Furthermore $\mathrm{Ad}\,W(q) = \mathrm{Ad}\,W(e_0^{\otimes n}) = q$, hence $t \circ \mathrm{Ad}\,W(q) = q^t \perp q$. These properties of q will limit our choice of V_m in our study of P.

In the case $m = 2^n$ there are four classes of non-isomorphic irreducible Jordan subalgebras of $(M_m)_{sa}$, namely $(M_m)_{sa}$ itself, V_{2n}, S_m, the real symmetric matrices in M_m, and $M_{2^{n-1}}(\mathsf{H})_{sa}$, the self-adjoint $2^{n-1} \times 2^{n-1}$ matrices over the quaternions H represented as 2×2 matrices, see [2], Ch. 6. Presently we shall specialize to the case when $V_{2n} \subset (M_{2^{n-1}})_{sa}$. We refer the reader to [4] for further information on this case.

With our previous notation with W defined as in Lemma 8 let

$$Q(X) = \tfrac{1}{2}(x + t \circ \mathrm{Ad}\,W(x)).$$

Then Q is the projection of M_{2^n} onto the fixed point set of the anti-automorphism $t \circ \mathrm{Ad}\,W$, hence by Lemma 8 is the projection onto the reversible Jordan algebra A_{2n} containing V_{2n}. Thus, if $V_{2n} \subset M_{2^{n-1}}(\mathsf{H})_{sa}$ then $Q\colon M_{2^n} \to M_{2^{n-1}}(\mathsf{H})_{sa}$.

Lemma 13. *With the above notation, if $V_{2n} \subset A_{2n} = M_{2^{n-1}}(\mathsf{H})_{sa}$ and P the projection $P\colon M_{2^n} \to V_{2n}$, then*

$$P = P|_{A_{2n}} \circ Q \geq 2^{-n+1}Q.$$

Proof. It suffices to show $P(p) \geq 2^{-n+1}p$ for all minimal projections p in A_{2n}. For such a projection $\mathrm{Tr}(p) = 2$. We have $P(p) = ae + bf$, $a, b \geq 0$, as in the proof of Lemma 12. Then $a = 2^{-n+1}\,\mathrm{Tr}(epe)$, $b = 2^{-n+1}\,\mathrm{Tr}(fpf)$. Since p is a minimal projection in A_{2n}, $pep = \lambda p$, $pfp = \mu p$ with $\lambda, \mu \geq 0$. Then

$$(epe)^2 = epepe = \lambda epe.$$

Since $\mathrm{rank}\,epe = \mathrm{rank}\,pep = 2$, $epe = \lambda_0 e_0$ with e_0 a projection in A_{2n} of dimension 2. Thus

$$\lambda_0 2 = \mathrm{Tr}(\lambda_0 e_0) = \mathrm{Tr}(epe) = \mathrm{Tr}(pep) = \mathrm{Tr}(\lambda p) = \lambda 2.$$

Therefore $\lambda_0 = \lambda$. Thus $epe = \lambda e_0$, and similarly $fpf = \mu f_0$. We then have, since $e \geq e_0$ and $f \geq f_0$,

$$\begin{aligned} P(p) &= 2^{-n+1}(\mathrm{Tr}(epe)e + \mathrm{Tr}(fpf)f) \\ &= 2^{-n+1}(\mathrm{Tr}(\lambda e_0)e + \mathrm{Tr}(\mu f_0)f) \\ &\geq 2^{-n+1}(2\lambda e_0 + 2\mu f_0) \\ &= 2^{-n+1}(2epe + 2fpf) \\ &\geq 2^{-n+1}(epe + epf + fpe + fpf) \\ &= 2^{-n+1}p, \end{aligned}$$

where we used Lemma 10. The proof is complete.

LEMMA 14. *Given V_{2n} and A_{2n} as above, and assume $A_{2n} \cong M_{2^{n-1}}(\mathsf{H})_{sa}$. Then there exists a 1-dimensional projection q in M_{2^n} such that $Q(q) = \frac{1}{2}(q + q^t)$ with $q \perp q^t$, $P(q) = 2^{-n}1$, and $\beta = P - 2^{-n+1}Q$ is bi-optimal.*

PROOF. By Lemma 13 $P|_{A_{2n}} \geq 2^{-n+1}\iota$. Since $P = P \circ Q$ we therefore have $\beta = P \circ Q - 2^{-n+1}Q \geq 0$. V_{2n} is irreducible by [2], Theorem 6.2.2, so $C^*(V_{2n}) = M_{2^n}$, so by Lemma 12 there is a 1-dimensional $q \in C^*(V_{2n})$ such that $2^{-n}1 = P(q) = P(Q(q))$. By the comments after Lemma 12, $q^t = t \circ \mathrm{Ad}\, W(q) \perp q$, so in particular

$$Q(q) = \tfrac{1}{2}(q + t \circ \mathrm{Ad}\, W(q)) = \tfrac{1}{2}(q + q^t).$$

Furthermore

$$\beta(Q(q)) = P(Q(q)) - 2^{-n+1}Q(q) = 2^{-n}(1 - (q + q^t)).$$

To show β is bi-optimal, let $\phi \leq \beta$ be completely positive. Then by Lemma 11, $\phi(x) = \phi(1)x = \lambda x$, $\lambda \geq 0$, since $\phi(1) \in C^*(V_{2n})' = \mathsf{C}$. Thus

$$\lambda(q + q^t) = \phi(q + q^t) = 2\phi(Q(q)) \leq 2\beta(Q(q)) = 2^{-n}(1 - (q + q^t)).$$

Since $q + q^t \perp 1 - (q + q^t)$, $\lambda = 0$, so $\phi = 0$. Thus β is optimal.

Next, if $\phi \leq \beta$ is co-positive, then $t \circ \phi$ is completely positive, and

$$t \circ \phi \leq t \circ P = P \circ t = P \circ \mathrm{Ad}\, W,$$

since $P = P \circ t \circ \mathrm{Ad}\, W$ by Lemma 9. Thus by Lemma 11, $t \circ \phi = \lambda\iota$ with $\lambda \geq 0$. Hence

$$\begin{aligned} \lambda(q + q^t) &= t \circ \phi(q + q^t) = 2t \circ \phi(Q(q)) \\ &\leq 2t \circ \beta(Q(q)) = 2^{-n}(1 - (q + q^t))^t \\ &= 2^{-n}(1 - (q + q^t)), \end{aligned}$$

so again $\lambda = 0$, and $\phi = 0$. Thus β is bi-optimal, completing the proof to the lemma.

From the above we see that if $\phi \le P$ is completely positive or co-positive, then $\phi \le \lambda Q$ for some $\lambda \ge 0$. Since $P \ge \alpha = 2^{-n+1}Q$, and $P(q) = 2^{-n}1$, it follows that α is a maximal decomposable map majorized by P.

Summarizing Lemma 14 and the above comments we obtain the following result.

THEOREM 15. *Assume the reversible Jordan algebra A_{2n} containing V_{2n} is isomorphic to $M_{2^{n-1}}(\mathsf{H})_{sa}$, and let $Q\colon M_{2^n} \to A_{2n}$ be the trace-invariant projection. Let $\alpha = 2^{-n+1}Q$ and $\beta = P - \alpha$. Then $P = \alpha + \beta$ is a decomposition as in Theorem 5.*

The following result describes Theorem 7 in detail for P.

THEOREM 16. *Let $P\colon M_{2^n} \to V_{2n}$ be the trace invariant projection. Let $\alpha = 2^{-n}\iota$, and $\beta = P - 2^{-n}\iota$, where ι is the identity map. Then α is a maximal completely positive map majorized by P, β is optimal, and $P = \alpha + \beta$.*

PROOF. By Lemma 12, $P \ge \alpha$, so $\beta \ge 0$, and there exists a 1-dimensional projection $q \in M_{2^n}$ such that $P(q) = 2^{-n}1$. Since V_{2n} is irreducible the argument in the proof of Lemma 14 shows that if $\phi \le \beta$ is completely positive, then $\phi = \lambda\iota$ with $\lambda \ge 0$. Thus

$$\lambda q = \phi(q) \le \beta(q) = 2^{-n}1 - 2^{-n}q = 2^{-n}(1 - q),$$

which implies $\lambda = 0$. Thus β is optimal. As remarked before the statement of Theorem 15 α is a maximal completely positive map majorized by P. The proof is complete.

It was crucial in the proof of Theorem 15 that $A_{2n} = M_{2^{n-1}}(\mathsf{H})_{sa}$, so $\dim q = 2$ for a minimal projection q in A_{2n}. In the case when $A_{2n} = S_{2^n}$, the real $2^n \times 2^n$ matrices, we have been unable to find a 1-dimensional projection $p \in A_{2n}$ such that $P(p) = 2^{-n}1$, so that for each minimal projection $e \in V_{2n}$ we have

$$\mathrm{Tr}(pe) = \mathrm{Tr}(epe) = \mathrm{Tr}(P(epe)) = \mathrm{Tr}(eP(p)e) = \mathrm{Tr}(e2^{-n}1) = \tfrac{1}{2},$$

so $\mathrm{Tr}(p.)$ is the trace on V_{2n}.

If $n = 1$, $V_2 = S_2 = A_1$, so $\mathrm{Tr}(p.)$ is never a trace on A_1. We next show this for V_4 too, showing in particular the well-known result that $A_2 = M_2(\mathsf{H})_{sa}$. We thus leave it as an open question whether there is an n such that $\mathrm{Tr}(p.)$ can be a trace on V_{2n} for a 1-dimensional projection $p \in A_{2n}$, or even for $p \in M_{2^n}$.

EXAMPLE 17. If $n = 2$ then there is no positive rank 1 operator $x \in M_4$ such that $t \circ \mathrm{Ad}\, W(x) = x$.

PROOF. Let $\bar{\phi}: M_2 \to M_2$ be defined by

$$\bar{\phi}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

Then $\bar{\phi} = \operatorname{Ad}\sigma_3$ as is easily seen. Let $\phi = t \circ \bar{\phi}$. Then ϕ is an anti-automorphism of order 2, and

$$\phi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

is such that $\mathscr{R} = \{A \in M_2 : \phi(A^*) = A\}$ is the quaternions. Also $\phi = \operatorname{Ad} t \circ \sigma_3$. For simplicity of notation let $\rho = \operatorname{Ad}\sigma_3$. Let T denote the 4×4 matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with $A, B, C, D \in M_2$. Then

$$\iota \otimes \rho(T^*) = \begin{pmatrix} \rho(A)^* & \rho(C)^* \\ \rho(B)^* & \rho(D)^* \end{pmatrix}$$

Therefore

$$t \circ (\iota \otimes \rho)(T^*) = \begin{pmatrix} t \circ \rho(A)^* & t \circ \rho(B)^* \\ t \circ \rho(C)^* & t \circ \rho(D)^* \end{pmatrix}$$

Thus $t \circ (\iota \otimes \rho)(T^*) = T$ if and only if

$$A = \phi(A^*), \qquad B = \phi(B^*), \qquad C = \rho(C^*), \qquad D = \phi(D^*)$$

if and only if $A, B, C, D \in \mathsf{H}$, and so $T \in M_2(\mathsf{H})$. But $M_2(\mathsf{H})$ contains no positive rank 1 operators, so there is no positive rank 1 $x \in M_4$ such that $t \circ \operatorname{Ad} W(x) = x$, completing the proof of the example.

If $\mathscr{P} = \{s_i : i \in \mathsf{N}\}$ is an infinite spin system then the norm closed linear span V_∞ of 1 and $\mathscr{P}$ is the infinite spin factor. The C^*-algebra $C^*(V_\infty)$ generated by V_∞ is the CAR-algebra A which is isomorphic to the infinite tensor product of M_2 with itself, see e.g. [2], Theorem 6.2.2. By [1], Lemma 2.3, there exists a unique trace-invariant positive projection P of $C^*(V_\infty)_{sa}$ onto V_∞. If $M_{2^n} = \otimes_1^n M_2$ is imbedded in $C^*(V_\infty)$ by $x \to x \otimes 1 \in M_{2^n} \otimes \otimes_{n+1}^\infty M_2$, it is clear that $P|_{M_{2^n}} = P_n$, the trace invariant projection onto V_{2n}. Thus if $\phi \le P$ is decomposable then $\phi|_{M_{2^n}} \le P|_{M_{2^n}} = P_n$ for n even. Thus by Lemmas 11 and 12, $\phi|_{M_{2^n}} \le 2^{-n}\iota|_{M_{2^n}}$. But if $m \ge n$ is even then

$$\phi|_{M_{2^n}} = (\phi|_{M_{2^m}})|_{M_{2^n}} \le 2^{-m}(\iota|_{M_{2^m}})|_{M_{2^n}}.$$

Thus

$$\phi|_{M_{2^n}} \leq 2^{-m}\iota|_{M_{2^n}}$$

for all even $m \geq n$. Thus $\phi = 0$. Similarly if $\phi \leq t \leq P$. We have thus shown

COROLLARY 18. *Let P be the projection of the self-adjoint part of the CAR-algebra onto the spin factor V_∞. Then P is bi-optimal.*

REFERENCES

1. Effros, E. G., and Størmer, E., *Positive projections and Jordan structure in operator algebras*, Math. Scand. 45 (1979), no. 1, 127–138.
2. Hanche-Olsen, H., and Størmer, E., *Jordan operator algebras*, Monographs and Studies in Mathematics, 21. Pitman (Advanced Publishing Program), Boston, 1984.
3. Ha, K.-C., and Kye, S.-H., *Optimality for indecomposable entanglement witnesses*, Phys. Rev. A 86 (2012) no. 3, 034301, 4 pp.
4. Lawson, H. B., and Michelson, M.-L., *Spin geometry*, Princeton Mathematical Series, 38. Princeton University Press, 1989.
5. Marciniak, M., *On extremal positive maps acting between type I factors*, Noncommutative Harmonic Analysis with Applications to probability II, 201–221. Banach Center Publ., 89, Polish Acad. Sci. Inst. Math., Warsaw, 2010.
6. Robertson, A. G., *Positive extensions of automorphisms of spin factors*, Proc. Roy. Soc. Edinburgh Sect. A 94 (1983), no. 1-2, 71–77.
7. Robertson, A. G., *Positive projections on C^*-algebras and an extremal map*, J. London Math. Soc. (2) 32 (1985), no. 1, 133–140.
8. Størmer, E., *Positive linear maps of operator algebras*, Springer Monographs in Mathematics, Springer, Heidelberg, 2013.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF OSLO
0316 OSLO
NORWAY
E-mail: erlings@math.uio.no