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ON THE SINGULAR LOCI AND THE IMAGES OF PROPER HOLOMORPHIC MAPS FROM PSEUDOELLIPSOIDS

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Abstract

We prove a generalisation of Rudin's theorem on proper holomorphic maps from the unit ball to the case of proper holomorphic maps from pseudoellipsoids.

1. Introduction

In the beginning of the '80's, W. Rudin proved a theorem that gives an exhaustive description of proper holomorphic maps $F : B^n \to \Omega$, from the unit ball B^n onto a domain Ω of C^n, in terms of finite unitary reflection groups.

Such result can be stated as follows. Recall that for any finite group Γ of automorphisms of the unit ball there exists some $h \in \mathrm{Aut}(B_n)$ such that $\Gamma_o = h\Gamma h^{-1}$ is a finite subgroup of the unitary group U_n, i.e. of the group of automorphisms of B^n fixing the origin. Let us denote by $\Gamma_{o(\mathrm{ref})} \subset \Gamma_o$ the maximal subgroup of reflections in Γ_o and by $(P_1, \ldots, P_n)$ a fixed set of generators for the space of $\Gamma_{o(\mathrm{ref})}$-invariant polynomials in n-variables. One can check that the holomorphic map

$$P_\Gamma = P_{\Gamma_o} \circ h^{-1} : B^n \longrightarrow B^n_\Gamma := P_\Gamma(B^n),$$

with $P_{\Gamma_o} := (P_1, \ldots, P_n)$, is proper and is uniquely associated with Γ, up to composition with an element of a special group of polynomial biholomorphisms (see §2). Rudin's theorem is the following.

THEOREM 1.1 ([15]). *For any proper holomorphic map $F : B^n \to \Omega$ onto a domain $\Omega \subset \mathsf{C}^n$, of multiplicity $m > 1$ and $\mathscr{C}^1$ up to the boundary, there exists a finite subgroup $\Gamma \subset \mathrm{Aut}(B^n)$ and a biholomorphism $\Psi : \Omega \to P_\Gamma(B^n)$ such that $\Psi \circ F = P_\Gamma$.*

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This immediately implies that any domain, which is image of a proper holomorphic map from B^n that is $\mathscr{C}^1$ on $\overline{B^n}$, is necessarily biholomorphic to one of the domains $B^n_\Gamma = P_\Gamma(B^n)$, whose classification can be reduced to that of finite reflection subgroups of U_n.

A crucial element of Rudin's proof is the celebrated Alexander Theorem on global extendability of local automorphisms of B^n. One can therefore ask if a result, similar to Rudin's theorem, can be proved for the pseudoellipsoids of C^n, on which several analogues of properties of the unit ball have been obtained by appropriate applications of Alexander Theorem (see e.g. [13], [9], [5], [10]).

So, let us focus on the pseudoellipsoids of C^n, namely the domains $\mathscr{E}^n_{(p)}$, with $p = (p_1, \dots, p_k) \in \mathsf{N}^k$, $p_i \geq 2$, defined by

$$\mathscr{E}^n_{(p)} := \left\{ z \in \mathsf{C}^n : \sum_{j=1}^{n-k} |z_j|^2 + |z_{n-k+1}|^{2p_1} + \cdots + |z_n|^{2p_k} < 1 \right\}.$$

Let also denote by $\varphi^{(p)} : \mathsf{C}^n \to \mathsf{C}^n$ the holomorphic map

$$\varphi^{(p)}(z) = \big(z_1, \dots, z_{n-k}, (z_{n-k+1})^{p_1}, \dots, (z_n)^{p_k}\big), \tag{1.1}$$

whose restriction $\varphi^{(p)}|_{\mathscr{E}^n_{(p)}} : \mathscr{E}^n_{(p)} \to B^n$ is directly seen to be a proper map.

Some ideas of Rudin's theorem can be actually implemented to study proper maps from pseudoellipsoids and they bring to the following theorem.

THEOREM 1.2. *For any proper holomorphic map $F : \mathscr{E}^n_{(p)} \to \Omega$ onto a domain $\Omega \subset \mathsf{C}^n$, of multiplicity $m > 1$ and $\mathscr{C}^1$ up to the boundary, there exists a finite subgroup $\Gamma \subset \mathrm{Aut}(B^n)$ and a proper holomorphic map $\Psi : \Omega \to P_\Gamma(B^n)$ such that $\Psi \circ F = P_\Gamma \circ \varphi^{(p)}$.*

In other words, if we call *factoring of f* any expression of the form $f = g \circ h$, where f appears as composition of two *factors* g, h, our theorem says that any proper holomorphic map F, defined on a pseudoellipsoid and $\mathscr{C}^1$ up to the boundary, is always a factor of a map of the form $P_\Gamma \circ \varphi^{(p)}$. This reduces the analysis of the first to that of factorings of the second.

We would like to stress that our result is optimal, in the sense that one cannot expect that Ψ can be proved to be a biholomorphism, as in Rudin's theorem: just consider the case $F = \mathrm{Id}_{\mathscr{E}^n_{(p)}} : \mathscr{E}^n_{(p)} \to \mathscr{E}^n_{(p)}$.

It is also clear that there exist several proper maps F that are not equivalent to the trivial examples $\mathrm{Id}_{\mathscr{E}^n_{(p)}}$, $\varphi^{(p)}$ or $P_\Gamma \circ \varphi^{(p)}$. Consider for instance the pseudoellipsoid

$$\mathscr{E}^4_{(2,2)} = \{ |z_1|^2 + |z_2|^2 + |z_3|^4 + |z_4|^4 < 1 \}$$

and the map

$$F : \mathscr{E}^4_{(2,2)} \longrightarrow \Omega = F(\mathscr{E}^4_{(2,2)}), \qquad F(z) = (z_1 z_2, z_1 + z_2, (z_3)^2, z_4),$$

which is a non trivial factor of the map $P_\Gamma \circ \varphi^{(2,2)}$, given by

$$\varphi^{(2,2)}(z) = (z_1, z_2, (z_3)^2, (z_4)^2) \quad \text{and} \quad P_\Gamma(z) = (z_1 z_2, z_1 + z_2, z_3, z_4)$$

(here P_Γ is associated with the group $\Gamma = \{\mathrm{Id}_{B^4}, g(z) = (z_2, z_1, z_3, z_4)\}$).

Nonetheless, the fact that F is always a factor of $P_\Gamma \circ \varphi^{(p)}$ gives precise information on the singular locus $Z_F = \{\det J_F(z) = 0\}$. Indeed, it is necessarily an analytic subvariety of $\mathscr{E}^n_{(p)}$ mapped by $\varphi^{(p)}$ into a subvariety of B^n contained in the union of the hyperplanes $\{z_i = 0\}$ and the fixed point set of a finite reflection subgroup of $\mathrm{Aut}(B^n)$.

It also gives strong restrictions on the class of the images Ω of the proper holomorphic maps from pseudoellipsoids, since, in their turn, they are constrained to admit a proper holomorphic map onto a domain B^n_Γ. We believe that such information can bring to the classification of such domains at least in the most simple cases, as for instance when Γ is trivial and $B^n_\Gamma = B^n$ (see e.g. [3], [7], [8] for the case $n = 2$).

We finally note that, when $\mathscr{E}^n_{(p)} = B^n$, by Rudin's theorem the map $\Psi : \Omega \to B^n_\Gamma$, given in Theorem 1.2, is necessarily invertible and the holomorphic correspondence

$$\Psi^{-1} = F \circ \varphi^{(p)-1} \circ P_\Gamma^{-1} : B^n_\Gamma \multimap \Omega$$

splits. Therefore a question worth of further investigations could be under which conditions on Γ or on Z_F one can infer that Ψ^{-1} necessarily splits or, equivalently, that Ψ is actually a biholomorphism.

After a section of preliminaries, in §3 we prove a crucial property of the proper holomorphic maps $F : \mathscr{E}^n_{(p)} \to \Omega$ that are $\mathscr{C}^1$ up to the boundary, namely we show that the subsets of B^n of the form $\varphi^{(p)}(F^{-1}(w))$, $w \in \Omega$, coincide with the orbits of a finite group Γ of automorphisms of B^n. With the help of this fact, we prove Theorem 1.2 in §4.

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2. Preliminaries

2.1. Finite subgroups $\Gamma \subset \mathrm{Aut}(B^n)$ *and the proper maps* P_Γ

Let us call *geodesic hyperplane of* B^n any $(n-1)$-dimensional subvariety of B^n of the form $g(\{z \in B^n : z_n = 0\})$ for some $g \in \mathrm{Aut}(B^n)$. Note that the geodesic hyperplanes $g(\{z_n = 0\})$, determined by elements $g \in \mathrm{U}_n = \mathrm{Aut}_0(B^n)$, are the usual affine hyperplanes through the origin.

DEFINITION 2.1. An element $h \in \mathrm{Aut}(B^n)$ is called *reflection* if it has finite period and its fixed point set is a geodesic hyperplane.

For a given finite subgroup $\Gamma \subset \mathrm{Aut}(B^n)$, we denote by $\Gamma_{(\mathrm{ref})} \subset \Gamma$ the subgroup generated by all reflections in Γ. Note that a non trivial element $g \in \Gamma$ is a reflection if and only if its fixed point set is $(n-1)$-dimensional (indeed, up to conjugation in $\mathrm{Aut}(B^n)$, any such element is in U_n). This implies that $\Gamma_{(\mathrm{ref})}$ is normal in Γ.

Consider a finite reflection group $\Gamma_o = \Gamma_{o(\mathrm{ref})}$ in U_n. By a classical result of Chevalley ([4], [16], [6]), there are n homogeneous, Γ_o-invariant polynomials $P_1, \ldots, P_n$ that constitute a basis for the invariants of Γ_o (i.e, the Γ_o-invariant polynomials $f \in \mathbb{C}[z_1, \ldots, z_n]$ are exactly those of the form $f = q(P_1, \ldots, P_n)$ for some $q \in \mathbb{C}[z_1, \ldots, z_n]$). The map

$$P_{\Gamma_o} = (P_1, \ldots, P_n) : B^n \longrightarrow \mathbb{C}^n$$

is uniquely determined by Γ_o, up to composition with the polynomial maps that interchange the bases of homogeneous polynomials for the invariants Γ_o. The group of such basis changes is the same for all groups Γ'_o of the conjugacy class of $\Gamma_o = \Gamma_{o(\mathrm{ref})}$ in U_n.

Consider now an arbitrary finite group of automorphisms $\Gamma \subset \mathrm{Aut}(B^n)$, with reflections subgroup $\Gamma_{(\mathrm{ref})}$. It is known that the elements of Γ have a common fixed point x_o (see e.g. [15], Thm. 3.1), so that for any $h \in \mathrm{Aut}(B^n)$ with $h(x_o) = 0$, the conjugate group $\Gamma_o = h\Gamma h^{-1}$ is in U_n and has $\Gamma_{o(\mathrm{ref})} = h\Gamma_{(\mathrm{ref})}h^{-1}$ as reflection subgroup. We may therefore consider the map

$$P_\Gamma : B^n \longrightarrow \mathbb{C}^n, \qquad P_\Gamma = P_{\Gamma_{o(\mathrm{ref})}} \circ h,$$

whose components are $\Gamma_{(\mathrm{ref})}$-invariant rational functions. Up to compositions with the basis changes described above, P_Γ is uniquely determined by $\Gamma_{(\mathrm{ref})}$. By [15], Thm. 2.5, the image $B^n_\Gamma = P_\Gamma(B^n)$ is a domain of $\mathbb{C}^n$, which is uniquely determined by the subgroup $\Gamma_{(\mathrm{ref})} \subset \Gamma$ up to biholomorphisms, and $P_\Gamma : B^n \to B^n_\Gamma$ is a proper holomorphic map.

We conclude recalling the statement of Rudin's generalisation of Alexander Theorem. Let us call *local automorphism of* B^n any biholomorphism $f : \mathcal{U}_1 \subset B^n \to \mathcal{U}_2 \subset B^n$ between connected open subsets of B^n such that:

a) each of the intersections $\partial \mathcal{U}_i \cap \partial B^n$, $i = 1, 2$, contains a boundary open set $\Gamma_i \subset \partial B^n$;

b) there exists a sequence $\{x_k\} \subset \mathcal{U}_1$ which converges to a point $x_o \in \Gamma_1$, which is not a limit point of $\partial \mathcal{U}_1 \cap B^n$, and so that $\{f(x_k)\}$ converges to a point $\hat{x}_o \in \Gamma_2$, which is not a limit point of $\partial \mathcal{U}_2 \cap B^n$.

Let also say that f *extends to a global automorphism* if there exists $F \in \mathrm{Aut}(B^n)$ such that $F|_{\mathcal{U}_1} = f$.

THEOREM 2.2 ([1], [13]). *Any local automorphism of B^n extends to a global one.*

2.2. *Correspondences*

Let D, $D' \subset \mathbb{C}^n$ be two bounded domains and denote by $\pi : D \times D' \to D$ and $\pi' : D \times D' \to D'$ the two natural projections. We recall that a *holomorphic correspondence between D and D'* is a subvariety $V \subset D \times D'$. It is called *proper* if the restricted projections $\pi|_V : V \to D$ and $\pi'|_V : V \to D'$ are proper maps. A holomorphic correspondence is called *irreducible* if it is an irreducible subvariety.

A holomorphic correspondence V is uniquely determined by the associated multivalued map

$$f : D \longrightarrow\!\!\circ\; D', \qquad f(z) = \pi'\big(\pi|_V^{-1}(z)\big),$$

which is a (single-valued) holomorphic map if and only if $\pi|_V$ is injective. We often denote a holomorphic correspondence V by the corresponding multivalued map f, so that the subvariety V coincides with the graph

$$V = \Gamma_f := \{(z, w) \in D \times D' : w \in f(z)\}.$$

If $f : D \longrightarrow\!\!\circ\; D'$ is a holomorphic correspondence, we denote by $f^{-1} : D' \longrightarrow\!\!\circ\; D$ the holomorphic correspondence with

$$\Gamma_{f^{-1}} = \{(w, z) \in D' \times D : (z, w) \in \Gamma_f\}.$$

If $f : D \longrightarrow\!\!\circ\; D'$ and $f' : D' \longrightarrow\!\!\circ\; D''$ are two (proper) holomorphic correspondences, it is known that the multivalued map $f' \circ f : D \longrightarrow\!\!\circ\; D''$ with

$$\Gamma_{f' \circ f} = \{(z, v) \in D \times D'' : (z, w) \in \Gamma_f, (w, v) \in \Gamma_{f'} \text{ for some } w \in D'\}$$

is a (proper) holomorphic correspondence as well ([17]).

Finally, given two (proper) holomorphic correspondences $f_1, f_2 : D \longrightarrow\!\!\circ\; D'$, we denote by $f = f_1 \cup f_2$ the (proper) holomorphic correspondence with $\Gamma_f = \Gamma_{f_1} \cup \Gamma_{f_2} \subset D \times D'$.

If $f : D \multimap D'$ is a proper holomorphic correspondence, there exist a positive integer p and a subvariety $W \subset D$ such that, for every $z_o \in D \setminus W$, there are an open neighbourhood $\mathcal{U} \subset D \setminus W$ of z_o and p holomorphic maps $f_i : \mathcal{U} \to D'$ such that the sets $f(z)$, $z \in \mathcal{U}$, have cardinality p and are equal to

$$f(z) = \{f_1(z), \ldots, f_p(z)\}.$$

We shortly say that "f is a p-valued map" . For a given $z_o \in D$, we say that f *splits at* z_o if there exists a neighbourhood $\mathcal{U} \subset \mathbb{C}^n$ of z_o such that

$$\Gamma_{f|_{\mathcal{U}}} = \Gamma_f \cap \pi^{-1}(\mathcal{U} \cap D) = \Gamma_{f_1} \cup \ldots \cup \Gamma_{f_q}$$

for some single-valued holomorphic maps $f_i : D \cap \mathcal{U} \to D'$. If f is p-valued, the number of such single-valued maps has to coincide with p.

We say that f *splits* if it splits at all points. If D is simply connected, f splits if and only if there are p holomorphic maps $f_i : D \to D'$ such that $f = f_1 \cup \ldots \cup f_p$. The f_i's are called *single-valued components of* f. The following is a direct consequence of [3], Lemma 3.1.

LEMMA 2.3. *If $f : D \multimap D'$ is a holomorphic correspondence, either it splits or there exists an analytic subvariety $S_f \subset D$ of dimension* $n - 1$ *such that f does not split at z for any $z \in S_f$.*

2.3. A technical fact concerning proper holomorphic maps

Let $F : D \to D'$ be a proper holomorphic map with multiplicity m and denote $Z_F = \{x \in D : \det J_F = 0\}$. If F extends to a $\mathscr{C}^1$-map $F : \mathcal{U} \to \mathbb{C}^n = \mathbb{R}^{2n}$ on a neighbourhood $\mathcal{U}$ of $\overline{D}$, we denote by $\mathsf{J}_F(x)$, $x \in \mathcal{U}$, the (real) Jacobian of F at x, where F is considered as a map between open subsets of $\mathbb{R}^{2n}$. If such (real) map is expressed in terms of the complex coordinates $(z_i, \overline{z}_i)$ and F is holomorphic at x, then

$$\mathsf{J}_F(x) = \begin{pmatrix} \frac{\partial F_i}{\partial z_j} & 0 \\ 0 & \overline{\frac{\partial F_i}{\partial z_j}} \end{pmatrix} \quad \text{and hence} \quad \operatorname{rank} \mathsf{J}_F(x) = 2 \operatorname{rank}\left(\frac{\partial F_i}{\partial z_j}\right). \tag{2.1}$$

By continuity, (2.1) holds in $\overline{D}$, so that $\operatorname{rank}(\mathsf{J}_F(x))$ is even for all $x \in \overline{D}$.

LEMMA 2.4. *Let $D \subset \mathbb{C}^n$ be a bounded domain with smooth boundary and $F : D \to D'$ a proper holomorphic map admitting a $\mathscr{C}^1$ extension to $\overline{D}$. Let us also use the notation $Z_F \cap \partial D := \{x \in \partial D : \det \mathsf{J}_F = 0\}$.*

Then, the $(2n - 1)$*-dimensional Hausdorff measure of* $F(Z_F \cap \partial D)$ *is* 0.

PROOF. Let $x \in \partial D$ and consider a system of real coordinates $\xi = (x_1, \ldots, x_{2n})$ on a neighbourhood $\mathscr{V}$ of x such that $\partial D \cap \mathscr{V} = \{x_{2n} = 0\}$. In such coordinates, the restriction $\widetilde{F} = F|_{\partial D}$ is of the form $\widetilde{F}(x_1, \dots, x_{2n-1}) = F(x_1, \dots, x_{2n-1}, 0)$ and the Jacobian $\mathsf{J}_F(x)$ of F is of the form

$$\mathsf{J}_F(x) = \left(\begin{array}{c|c} J_{\widetilde{F}}(x) & \begin{array}{c} \frac{\partial F_1}{\partial x_{2n}}(x) \\ \vdots \\ \frac{\partial F_{2n}}{\partial x_{2n}}(x) \end{array} \end{array} \right).$$

This means that $\operatorname{rank} \widetilde{F}|_x \leq \operatorname{rank} F|_x \leq \operatorname{rank} \widetilde{F}|_x + 1$. If $x \in Z_F \cap \partial D$, previous remarks imply that $\operatorname{rank} \widetilde{F}|_x \leq \operatorname{rank} F|_x \leq 2n-2$ and, conversely, if $\operatorname{rank} \widetilde{F}|_x \leq 2n-2$, one has that $\operatorname{rank} F|_x \leq 2n-1$ and hence $x \in Z_F \cap \partial D$. This means that $Z_F \cap \partial D = \{x \in \partial D : \operatorname{rank} \widetilde{F}|_x \leq 2n-2\}$ and the claim follows from generalised Morse-Sard Theorem (see e.g. [11]).

3. F-related points in $\mathscr{E}^n_{(p)}$ and B^n

In all the following, $F : \mathscr{E}^n_{(p)} \to \Omega \subset \mathsf{C}^n$ is a proper holomorphic map of multiplicity m and $\varphi^{(p)} : \mathscr{E}^n_{(p)} \to B^n$ is the proper holomorphic map defined in (1.1). We also set

$$\pi = \{z \in \mathsf{C}^n : z_{n-k+1} \cdot z_{n-k+2} \cdot \ldots \cdot z_n = 0\}. \tag{3.1}$$

DEFINITION 3.1. A subset $J \subset \mathscr{E}^n_{(p)}$ is called *complete F-set in $\mathscr{E}^n_{(p)}$* if $J = F^{-1}(w_o)$ for some $w_o \in \Omega$. It is called *good* if it is the pre-image of a point

$$w_o \in \Omega \setminus F(Z_F \cup \pi).$$

Similarly, a subset $\widetilde{J} \subset B^n$ is called *complete F-set in B^n* if it is of the form $\widetilde{J} = \varphi^{(p)}(J)$ for a complete F-set in $\mathscr{E}^n_{(p)}$. If J is good, also $\widetilde{J}$ is called *good*.

Two points of a complete F-set in $\mathscr{E}^n_{(p)}$ (resp. in B^n) are called *F-related*. Similarly, two sequences $\{x_k\}$, $\{x'_k\}$ in $\mathscr{E}^n_{(p)}$ (resp. in B^n) are called *F-related* if x_k and x'_k are F-related for all k's.

LEMMA 3.2. *Let $F : \mathscr{E}^n_{(p)} \to \Omega \subset \mathsf{C}^n$ be a proper holomorphic map of multiplicity $m > 1$, admitting a $\mathscr{C}^1$ extension to $\overline{\mathscr{E}^n_{(p)}}$. Then there exist m pairwise F-related sequences $\{x^{(1)}_k\}, \dots, \{x^{(m)}_k\}$ in $\mathscr{E}^n_{(p)}$ with the following properties:*

i) *they converge to m distinct points $x^{(1)}_o, \dots, x^{(m)}_o \in \partial\mathscr{E}^n_{(p)}$;*

ii) *there are disjoint connected open sets $\mathscr{U}^{(i)} \subset \mathsf{C}^n$, $1 \leq i \leq m$, such that:*
   - *$x^{(i)}_o \in \mathscr{U}^{(i)}$;*
   - *the restrictions $F|_{\mathscr{U}^{(i)} \cap \mathscr{E}^n_{(p)}} : \mathscr{U}^{(i)} \cap \mathscr{E}^n_{(p)} \to F(\mathscr{U}^{(i)} \cap \mathscr{E}^n_{(p)})$ are biholomorphisms onto the same open set $\mathscr{W} = F(\mathscr{U}^{(i)} \cap \mathscr{E}^n_{(p)}) \subset \Omega$;*

– *the restrictions* $\varphi^{(p)}|_{\mathscr{U}^{(i)}\cap\mathscr{E}^n_{(p)}} : \mathscr{U}^{(i)}\cap\mathscr{E}^n_{(p)} \to \mathscr{V}^{(i)} = \varphi^{(p)}(\mathscr{U}^{(i)}\cap\mathscr{E}^n_{(p)})$ *are biholomorphisms.*

Proof. Let $\widetilde{Z}_F = Z_F \cap \partial\mathscr{E}^n_{(p)}$ and $\widetilde{\pi} = \pi \cap \partial\mathscr{E}^n_{(p)}$. Since F is Lipschitz in $\overline{\mathscr{E}^n_{(p)}}$ and the $(2n-1)$-dimensional Hausdorff measure $H_{2n-1}(\widetilde{\pi})$ is zero, we have $H_{2n-1}(F(\widetilde{\pi})) = 0$. Hence, by Lemma 2.4,

$$H_{2n-1}(F(\widetilde{Z}_F \cup \widetilde{\pi})) \leq H_{2n-1}(F(\widetilde{Z}_F)) + H_{2n-1}(F(\widetilde{\pi})) = 0.$$

Since $\partial\Omega$ surely includes pieces of smooth hypersurfaces, $H_{2n-1}(\partial\Omega) > 0$ and consequently

$$\partial\Omega \setminus F(\widetilde{Z}_F \cup \widetilde{\pi}) \neq \emptyset.$$

Pick a point $w_o \in \partial\Omega\setminus F(\widetilde{Z}_F\cup\widetilde{\pi})$, a pre-image $x_o^{(1)} \in F^{-1}(w_o)$ and a small arc $\gamma_t^{(1)} \subset \mathscr{E}^n_{(p)} \setminus F^{-1}(F(Z_F))$, $t \in [0, 1)$, ending at $x_o^{(1)} = \lim_{t\to 1}\gamma_t^{(1)}$. Since the restriction of F to $\mathscr{E}^n_{(p)} \setminus F^{-1}(F(Z_F))$ is a proper, unbranched cover (see e.g. [2]), there are exactly $m-1$ disjoint arcs $\gamma_t^{(2)}, \ldots, \gamma_t^{(m)}$, $t \in [0, 1)$, determined by the points that are F-related to the points $\gamma_t^{(1)}$.

By construction, $x_o^{(i)} = \lim_{t\to 1}\gamma_t^{(i)} \in F^{-1}(w_o)$ for all $2 \leq i \leq m$. Since $\det \mathsf{J}_F(x_o^{(i)}) \neq 0$, any such point admits a connected neighbourhood, on which F is an homeomorphism, implying that $x_o^{(1)}, \ldots, x_o^{(m)}$ are all distinct.

We may consider disjoint connected neighbourhoods $\widehat{\mathscr{U}}^{(i)}$ of the $x_o^{(i)}$ that do not intersect $Z_F \cup \pi$, so that $F|_{\widehat{\mathscr{U}}^{(i)}}$ and $\varphi^{(p)}|_{\widehat{\mathscr{U}}^{(i)}}$ are homeomorphisms onto their images in $\mathbb{C}^n$ and the restrictions $F|_{\widehat{\mathscr{U}}^{(i)}\cap\mathscr{E}^n_{(p)}}$ and $\varphi^{(p)}|_{\widehat{\mathscr{U}}^{(i)}\cap\mathscr{E}^n_{(p)}}$ are biholomorphisms onto their images in Ω and B^n, respectively. Setting $\widehat{\mathscr{W}} = \bigcap_{i=1}^m F(\widehat{\mathscr{U}}^{(i)})$ and $\mathscr{U}^{(i)} = F^{-1}(\widehat{\mathscr{W}}) \cap \widehat{\mathscr{U}}^{(i)}$, any choice of F-related sequences $\{x_k^{(i)} = \gamma_{t_k}^{(i)}\}$ with $\lim_{k\to\infty} t_k = 1$ satisfies the claim.

From now on, we consider a fixed choice of m sequences $\{x_k^{(j)}\}$, $1 \leq j \leq m$, converging to $x_o^{(j)} \in \partial\mathscr{E}^n_{(p)}$, and open sets

$$\mathscr{U}^{(i)} \quad \text{and} \quad \mathscr{V}^{(i)} = \varphi^{(p)}(\mathscr{U}^{(i)} \cap \mathscr{E}^n_{(p)}) \subset B^n \tag{3.2}$$

satisfying the statement of Lemma 3.2. We also denote by $g^{(i,j)} : \mathscr{V}^{(i)} \to \mathscr{V}^{(j)}$ the biholomorphisms

$$g^{(i,j)} = \big(\varphi^{(p)}|_{\mathscr{U}^{(j)}\cap\mathscr{E}^n_{(p)}}\big)\circ\big(F|_{\mathscr{U}^{(j)}\cap\mathscr{E}^n_{(p)}}\big)^{-1}\circ\big(F|_{\mathscr{U}^{(i)}\cap\mathscr{E}^n_{(p)}}\big)\circ\big(\varphi^{(p)}|_{\mathscr{U}^{(i)}\cap\mathscr{E}^n_{(p)}}\big)^{-1}. \tag{3.3}$$

Notice that *the $g^{(i,j)}$'s are local automorphisms of B^n and, by Theorem 2.2, they all extend to global automorphisms of B^n*. We finally set

$$\Gamma = \{g^{(i,j)}, 1 \leq i, j \leq m\} \subset \mathrm{Aut}(B^n). \tag{3.4}$$

PROPOSITION 3.3. *Two points y, $y' \in B^n$ are F-related in B^n if and only if $y' = g(y)$ for some $g \in \Gamma$. In particular, Γ is a finite subgroup of* $\mathrm{Aut}(B^n)$.

PROOF. We first prove the necessity. Let y, $y' \in B^n$ be F-related points, i.e. $y = \varphi^{(p)}(x)$, $y' = \varphi^{(p)}(x')$ for two points x, x' of a complete F-set $J = \{x_1, \dots, x_m\} = F^{-1}(w)$ in $\mathscr{E}^n_{(p)}$. Let also $Z_{F,\varphi^{(p)}}$ be the analytic subvariety of $\mathscr{E}^n_{(p)}$ defined by

$$Z_{F,\varphi^{(p)}} = F^{-1}(F(Z_F \cup \pi))$$

where, as usual, $\pi = \{z_{n-k+1} \cdot \dots \cdot z_n = 0\}$. We consider two cases.

Case 1: J is good, i.e. $J \cap Z_{F,\varphi^{(p)}} = \emptyset$.

Since $Z_{F,\varphi^{(p)}}$ is analytic subvariety of $\mathscr{E}^n_{(p)}$, the set $\mathscr{E}^n_{(p)} \setminus Z_{F,\varphi^{(p)}}$ is connected (see e.g. [12], Ch. 4, Prop. 1). We may therefore consider a $\mathscr{C}^0$ curve $\eta : [0, 1] \to \overline{\mathscr{E}^n_{(p)}}$ such that

- $\eta_0 = x$ and $\eta_1 = x^{(1)}_o$;
- $\eta_t \in \mathscr{E}^n_{(p)} \setminus Z_{F,\varphi^{(p)}}$ for any $0 \leq t < 1$.

The corresponding curve $\gamma = \varphi^{(p)} \circ \eta : [0, 1] \to \overline{B^n}$ is such that

- $\gamma_0 = y$ and $\gamma_1 = y^{(1)}_o = \varphi^{(p)}(x^{(1)}_o)$;
- $\gamma_t \in B^n \setminus \varphi^{(p)}(Z_{F,\varphi^{(p)}})$ for any $0 \leq t < 1$.

Consider now a $\mathscr{C}^0$-curve $\eta' : [0, 1] \to \overline{\mathscr{E}^n_{(p)}}$ such that $\eta'_0 = x'$ and η'_t is F-related to η_t for any $0 \leq t < 1$. By the properties of proper holomorphic maps and the fact that $\eta_t \notin F^{-1}(F(Z_F))$, such curve exists and it is unique. In particular, $\eta'_t \in \mathscr{E}^n_{(p)} \setminus Z_{F,\varphi^{(p)}}$ for any $t < 1$ and $\eta'_1 \in \partial\mathscr{E}^n_{(p)}$.

Finally, let $\gamma' : [0, 1] \to \overline{B^n}$ be the curve $\gamma' = \varphi^{(p)} \circ \eta'$. By construction,

$$\gamma'_0 = y', \qquad \gamma'_t \in B^n \setminus \varphi^{(p)}(Z_{F,\varphi^{(p)}}) \text{ if } t < 1, \qquad \gamma'_1 = \varphi^{(p)}(\eta'_1) \in \partial B^n.$$

Notice that, being η_t and η'_t distinct and F-related, the end-point η'_1 must be one of the points $x^{(2)}_o, x^{(3)}_o, \dots, x^{(m)}_o$. For simplicity, we assume $\eta'_1 = x^{(2)}_o$.

Now, we observe that, for any $t \in [0, 1)$, there exist neighbourhoods $\mathscr{U}_t$, $\mathscr{U}'_t \subset \mathscr{E}^n_{(p)}$ of η_t and η'_t, respectively, and neighbourhoods $\mathscr{V}_t$, $\mathscr{V}'_t \subset B^n$ of γ_t and γ'_t, such that the restrictions

$$F|_{\mathscr{U}_t} : \mathscr{U}_t \longrightarrow F(\mathscr{U}_t), \qquad \varphi^{(p)}|_{\mathscr{U}_t} : \mathscr{U}_t \longrightarrow \mathscr{V}_t,$$
$$F|_{\mathscr{U}'_t} : \mathscr{U}'_t \longrightarrow F(\mathscr{U}'_t) = F(\mathscr{U}_t), \qquad \varphi^{(p)}|_{\mathscr{U}'_t} : \mathscr{U}'_t \to \mathscr{V}'_t$$

are biholomorphisms, so that also

$$h_t = \varphi^{(p)}|_{\mathscr{U}'_t} \circ (F|_{\mathscr{U}'_t})^{-1} \circ F|_{\mathscr{U}_t} \circ (\varphi^{(p)}|_{\mathscr{U}_t})^{-1} : \mathscr{V}_t \to \mathscr{V}'_t \tag{3.5}$$

is a biholomorphism. For $t = 1$, we set $\mathscr{V}_1 = \mathscr{V}^{(1)}$, $\mathscr{V}'_1 = \mathscr{V}^{(2)}$ and

$$h_1 = g^{(1,2)}|_{\mathscr{V}_1} : \mathscr{V}_1 \to \mathscr{V}'_1. \tag{3.6}$$

We claim that, for any $t, s \in [0, 1]$, with $\mathscr{V}_t \cap \mathscr{V}_s \neq \emptyset$,

$$h_t|_{\mathscr{V}_t \cap \mathscr{V}_s} = h_s|_{\mathscr{V}_t \cap \mathscr{V}_s}. \tag{3.7}$$

Indeed, if $\mathscr{V}_t \cap \mathscr{V}_s \neq \emptyset$ (hence, it contains a subarc of γ), then $\mathscr{U}_t \cap \mathscr{U}_s \neq \emptyset$ (it contains a subarc of η) and $\varphi^{(p)}|_{\mathscr{U}_t \cap \mathscr{U}_s}$ is a biholomorphism onto $\varphi^{(p)}(\mathscr{U}_t \cap \mathscr{U}_s)$ with inverse

$$\left(\varphi^{(p)}|_{\mathscr{U}_t \cap \mathscr{U}_s}\right)^{-1} = \left(\varphi^{(p)}|_{\mathscr{U}_t}\right)^{-1}\Big|_{\varphi^{(p)}(\mathscr{U}_t \cap \mathscr{U}_s)} = \left(\varphi^{(p)}|_{\mathscr{U}_s}\right)^{-1}\Big|_{\varphi^{(p)}(\mathscr{U}_t \cap \mathscr{U}_s)}.$$

By a similar argument

$$\left(F|_{\mathscr{U}'_t \cap \mathscr{U}'_s}\right)^{-1} = \left(F|_{\mathscr{U}'_t}\right)^{-1}\Big|_{F(\mathscr{U}'_t \cap \mathscr{U}'_s)} = \left(F|_{\mathscr{U}'_s}\right)^{-1}\Big|_{F(\mathscr{U}'_t \cap \mathscr{U}'_s)}$$

and (3.7) follows directly from the definitions of the h_t's.

By compactness, there are $t_1, \ldots, t_{N-1}, t_N = 1 \in [0,1]$ such that $\gamma([0,1]) \subset \bigcup_{k=1}^N \mathscr{V}_{t_k}$ and, by (3.7), the maps h_{t_i} can be glued together to determine a holomorphic map

$$h : \mathscr{V} = \bigcup_{k=1}^{N} \mathscr{V}_{t_k} \longrightarrow \mathscr{V}' = \bigcup_{k=1}^{N} \mathscr{V}'_{t_k}.$$

Since $h|_{\mathscr{V}_1} = h_1 = g^{(1,2)}|_{\mathscr{V}_1}$, by the Identity Principle, $h = g^{(1,2)}|_{\mathscr{V}}$ and $y' = h(y) = g^{(1,2)}(y)$, proving the claim.

Case 2: J is not good, i.e. $J \cap Z_{F,\varphi^{(p)}} \neq \emptyset$.

In this case $J = F^{-1}(w)$ for some $w \in F(Z_F \cup \pi)$. Let $\{w_k\} \subset \Omega \setminus F(Z_F \cup \pi)$ be a sequence with $\lim_{k\to\infty} w_k = w$ and denote by $\widetilde{J}_k = \varphi^{(p)}(F^{-1}(w_k)) = \{y_{k,1}, \ldots, y_{k,r_k}\}$ the corresponding sequence of good complete F-sets in B^n. Taking a suitable subsequence, we may assume that y, y' are limits of two sequences $\{y_k\}$, $\{y'_k\}$ with $y_k, y'_k \in \widetilde{J}_k$ for any k. By the previous part of the proof, there are $g_k \in \Gamma$ such that $g_k(y_k) = y'_k$. Since Γ is a finite set, we may consider a subsequence $\{y_{k_n}\}$ and $g \in \Gamma$ such that $g(y_{k_n}) = y'_{k_n}$ for any n. Therefore $g(y) = \lim_{n\to\infty} g(y_{n_k}) = \lim_{n\to\infty} y'_{n_k} = y'$ and the claim follows.

Let us now prove the sufficiency. Let $y, y' \in B^n$ be such that $y' = g(y)$ for some $g \in \Gamma$. If $y = y'$, there is nothing to prove. Therefore, we assume $y \neq y'$ and $g \neq \mathrm{Id}_{B^n}$. For simplicity, we assume that $g = g^{(1,2)}$. Consider the analytic subvariety of $\overline{B^n}$

$$Z' = \bigcup_{h\in\Gamma} h\big(\varphi^{(p)}(Z_{F,\varphi^{(p)}})\big). \tag{3.8}$$

We prove the claim in the mutually exclusive cases $y, y' \notin Z'$ and $y, y' \in Z'$, respectively.

Case 1: $y, y' \in B^n \setminus Z'$.

Pick a point $y_o \in \mathscr{V}^{(1)} \setminus Z' \subset B^n$ and observe that, being $B^n \setminus Z'$ complementary to an analytic subvariety, there exists a $\mathscr{C}^0$ curve $\gamma : [0, 1] \to B^n$ such that

- $\gamma_0 = y$ and $\gamma_1 = y_o$;
- $\gamma_t \in B^n \setminus Z'$ for any $t \in [0, 1]$;

Secondly, consider the $\mathscr{C}^0$ curve $\gamma' = g^{(1,2)} \circ \gamma : [0, 1] \to B^n$. By construction, $\gamma'_0 = y'$ and γ'_1 is equal to a point $y'_o \in \mathscr{V}^{(2)} = g^{(1,2)}(\mathscr{V}^{(1)})$.

Since $y_o \in \mathscr{V}^{(1)}$ and $y'_o \in \mathscr{V}^{(2)}$, there are exactly two points $x_o \in \mathscr{U}^{(1)} \cap \mathscr{E}^n_{(p)}$ and $x'_o \in \mathscr{U}^{(2)} \cap \mathscr{E}^n_{(p)}$ such that $\varphi^{(p)}(x_o) = y_o$ and $\varphi^{(p)}(x'_o) = y'_o$. We may therefore consider the unique $\mathscr{C}^0$ curves $\eta, \eta' : [0, 1] \to \mathscr{E}^n_{(p)} \setminus Z_{F,\varphi^{(p)}}$ such that

- $\varphi^{(p)} \circ \eta = \gamma$ and $\varphi^{(p)} \circ \eta' = \gamma'$,
- $\eta_1 = x_o$ and $\eta'_1 = x'_o$.

For any $t \in [0, 1]$, consider the F-complete set $\{\eta^{(1)}_t = \eta_t, \eta^{(2)}_t, \dots, \eta^{(m)}_t\}$ which contains η_t. Then there exist m neighbourhoods $\mathscr{U}^{(j)}_t \subset \mathscr{E}^n_{(p)}$, $1 \le j \le m$, of the points $\eta^{(j)}_t$ such that the restrictions

$$\varphi|_{\mathscr{U}^{(j)}_t} : \mathscr{U}^{(j)}_t \longrightarrow \mathscr{V}^{(j)}_t = \varphi^{(p)}(\mathscr{U}^{(j)}_t),$$
$$F|_{\mathscr{U}^{(j)}_t} : \mathscr{U}^{(j)}_t \longrightarrow \mathscr{W}_t, \mathscr{W}_t = F(\mathscr{U}^{(1)}_t),$$

are biholomorphisms. Hence, also the maps

$$k^{(1,j)}_t = \varphi^{(p)} \circ \left(F|_{\mathscr{U}^{(j)}_t}\right)^{-1} \circ F|_{\mathscr{U}^{(1)}_t} \circ \left(\varphi^{(p)}|_{\mathscr{U}^{(1)}_t}\right)^{-1} : \mathscr{V}^{(1)}_t \longrightarrow \mathscr{V}^{(j)}_t, \tag{3.9}$$

$2 \le j \le m$, are biholomorphisms. Reordering the elements in the F-complete sets, we may always assume that

$$\mathscr{U}^{(1)}_{t=1} = \mathscr{U}^{(1)} \cap \mathscr{E}^n_{(p)}, \qquad \mathscr{V}^{(1)}_{t=1} = \mathscr{V}^{(1)}, \qquad k^{(1,2)}_{t=1} = g^{(1,2)}|_{\mathscr{V}^{(1)}}.$$

By compactness and reorderings, there exist $t_1, \dots, t_N = 1 \in [0, 1]$ such that $\gamma([0, 1]) \subset \bigcup_{k=1}^N \mathscr{V}^{(1)}_{t_k}$ and $\mathscr{V}^{(1)}_{t_j} \cap \mathscr{V}^{(1)}_{t_{j-1}} \neq \emptyset$ for all $2 \le j \le N$. By the same arguments for (3.7), we have that $k^{(1,2)}_{t_j}|_{\mathscr{V}^{(1)}_{t_j} \cap \mathscr{V}^{(1)}_{t_{j-1}}} = k^{(1,2)}_{t_{j-1}}|_{\mathscr{V}^{(1)}_{t_j} \cap \mathscr{V}^{(1)}_{t_{j-1}}}$ for all $2 \le j \le N$, so that the map $k^{(1,2)}_1$ extends to a holomorphic map

$$k^{(1,2)} : \mathscr{V} = \bigcup_{j=1}^N \mathscr{V}^{(1)}_{t_j} \longrightarrow \mathscr{V}' = \bigcup_{j=1}^N \mathscr{V}^{(2)}_{t_j}$$

between a neighbourhood $\mathscr{V}$ of $\gamma([0,1])$ and a neighbourhood $\mathscr{V}'$ of $k^{(1,2)}(\gamma([0,1]))$. Notice that, by construction, if $\widetilde{y}$, $\widetilde{y}'$ are such that $\widetilde{y}' = k^{(1,2)}(\widetilde{y})$, they are F-related. Since $k^{(1,2)}|_{\mathscr{V}_1^{(1)}} = g^{(1,2)}|_{\mathscr{V}_1^{(1)}}$, by the Identity Principle, $k^{(1,2)} = g^{(1,2)}|_{\mathscr{V}}$ and $y' = g^{(1,2)}(y) = k^{(1,2)}(y)$. Therefore y, y' are F-related, as we needed to prove.

Case 2: y, $y' \in Z'$.

Let $\{y_k\} \subset B^n \setminus Z'$ be a sequence with $\lim_{k\to\infty} y_k = y$. By continuity, the sequence $y'_k = g(y_k)$ converges to $y' = g(y)$. By the result in the previous case, y_k and y'_k are F-related for any k and there exists a sequence $\{w_k\} \subset \Omega$ such that $y_k, y'_k \in \varphi^{(p)}(F^{-1}(w_k))$. Since $\varphi^{(p)}$ and F are proper, up to a subsequence, we may assume that $\{w_k\}$ converges to a point $w_o \in \Omega$. Using continuity, one can check that this implies that y, $y' \in \varphi^{(p)}(F^{-1}(w_o))$ and are therefore F-related.

Finally, the property that Γ is a subgroup follows from the fact that the composition of two elements $g^{(i,j)}, g^{(k,\ell)} \in \Gamma$ maps the connected open set $\mathscr{V}^{(1)}$ into one of the F-related sets $\mathscr{V}^{(r)}$. This can occur only if $g^{(i,j)} \circ g^{(k,\ell)}|_{\mathscr{V}^{(1)}} = g^{(1,r)}|_{\mathscr{V}^{(1)}}$ for some r, meaning that $g^{(i,j)} \circ g^{(k,\ell)} = g^{(1,r)} \in \Gamma$.

4. The Main Theorem

Consider now the proper holomorphic correspondence

$$\Psi = P_\Gamma \circ \varphi^{(p)} \circ F^{-1} : \Omega \multimap B^n_\Gamma, \tag{4.1}$$

where P_Γ and B^n_Γ are as defined in §2.1. Theorem 1.2 is direct consequence of the following:

PROPOSITION 4.1. *The correspondence* (4.1) *splits and each of its single-valued components $\Psi_i : \Omega \to B^n_\Gamma$, $1 \le i \le k$, is a proper holomorphic map such that*

$$\Psi_i \circ F = P_\Gamma \circ \varphi^{(p)}. \tag{4.2}$$

PROOF. By Lemma 2.3, it suffices to show that the subset $S_\Psi \subset \Omega$ of the points z, at which Ψ does not split, is included in an analytic subvariety of dimension less than or equal to $n-2$. Let $\Gamma_{(\mathrm{ref})} \subset \Gamma$ be the normal subgroup generated by the reflections in Γ and fix some elements $h_1, \ldots, h_k$ in $\Gamma \setminus \Gamma_{(\mathrm{ref})}$ such that Γ can be expressed as a disjoint union

$$\Gamma = \Gamma_{(\mathrm{ref})} \cup \Gamma_{(\mathrm{ref})} h_1 \cup \ldots \cup \Gamma_{(\mathrm{ref})} h_k.$$

For convenience of notation, we set $h_0 = \mathrm{Id}_{B^n}$ so that $\Gamma = \bigcup_{i=0}^k \Gamma_{(\mathrm{ref})} h_i$.

We first observe that for any $g, g' \in \Gamma_{(\mathrm{ref})}$ and $0 \leq i \neq j \leq k$, the element $(g'h_j)^{-1}(gh_i)$ is not in $\Gamma_{(\mathrm{ref})}$. Indeed, since $\Gamma_{(\mathrm{ref})}$ is normal, if $\widetilde{g} = h_j^{-1}g'^{-1}gh_i$ is in $\Gamma_{(\mathrm{ref})}$, then

$$g'^{-1}gh_i = h_j\widetilde{g} = \widetilde{g}'h_j \text{ for some } \widetilde{g}' \in \Gamma_{(\mathrm{ref})} \Longrightarrow \Gamma_{(\mathrm{ref})}h_i \cap \Gamma_{(\mathrm{ref})}h_j \neq \emptyset,$$

contradicting the choice of the h_m's. Due to this, any fixed point set $\mathrm{Fix}((g'h_j)^{-1}(gh_i))$ is an analytic variety of dimension less than or equal to $n-2$.

Let X be the union of such fixed point sets, that is

$$X = \bigcup_{\substack{g, g' \in \Gamma_{(\mathrm{ref})} \\ 0 \leq i \neq j \leq k}} \mathrm{Fix}((g'h_j)^{-1}(gh_i))$$

and note that $W = F(\varphi^{(p)-1}(X))$ is an analytic subvariety of Ω of dimension $\dim W \leq n-2$. Indeed, $\widetilde{X} = \varphi^{(p)-1}(X) \subset \mathscr{E}^n_{(p)}$ is an analytic variety, which is mapped onto X and W by the proper holomorphic maps $\varphi^{(p)}$ and F, respectively. By the Proper Mapping Theorem,

$$\dim W = \dim \widetilde{X} = \dim X \leq n-2.$$

Let $w_o \in \Omega \setminus W$ and $z_o \in \varphi^{(p)}(F^{-1}(w_o))$. By construction, $z_o \notin X$. We claim that there exists a ball $B_\varepsilon(z_o) \subset B^n$, centred at z_o and of radius ε, such that

$$gh_i(B_\varepsilon(z_o)) \cap g'h_j(B_\varepsilon(z_o)) = \emptyset \tag{4.3}$$

for any $g, g' \in \Gamma_{(\mathrm{ref})}$ and $0 \leq i \neq j \leq k$. Suppose not. Since Γ is finite, there exist $i \neq j$, $g, g' \in \Gamma_{(\mathrm{ref})}$ and two sequences z_n, z'_n such that

$$z_o = \lim_{n\to\infty} z_n = \lim_{n\to\infty} z'_n \qquad \text{and} \qquad (gh_i)(z_n) = (g'h_j)(z'_n) \quad \text{for any } n.$$

By continuity, $z_o = ((gh_i)^{-1}(g'h_j))(z_o)$, i.e. $z_o \in X$: contradiction.

In the following, we denote $\mathscr{V}_j = \bigcup_{g \in \Gamma_{(\mathrm{ref})}} gh_j(B_\varepsilon(z_o))$. By (4.3), we have that $\mathscr{V}_i \cap \mathscr{V}_j = \emptyset$ for any $0 \leq i \neq j \leq k$.

We now consider an open ball $B_\delta(w_o) \subset \Omega$ with the following property: for any $w \in B_\delta(w_o)$ there exists $z \in \varphi^{(p)}(F^{-1}(w))$ such that $z \in B_\varepsilon(z_o)$. The existence of such a ball can be checked as follows. Consider the F-complete set $F^{-1}(w_o) = \{x_o^1, \ldots, x_o^N\}$ in $\mathscr{E}^n_{(p)}$ and the corresponding F-complete set $\varphi^{(p)}(F^{-1}(w_o)) = \{z_o^1, \ldots, z_o^{N'}\}$ in B^n. Let r sufficiently small so that $F^{-1}(B_r(w_o))$ has exactly N connected components $U_1, \ldots, U_N$. Let $V_i = \varphi^{(p)}(U_i)$ and assume that $z_o = z_o^1 \in V_1$. If there is no $B_\delta(w_o)$ with the required property, there exists a sequence $\{w_\ell\} \subset \Omega$, converging to w_o such that

$$\varphi^{(p)}(F^{-1}(w_\ell)) \cap B_\varepsilon(z_o) = \emptyset \qquad \text{for any} \quad \ell.$$

Taking a suitable subsequence, we may assume that there exists a sequence $x_\ell \in U_1$ with $F(x_\ell) = w_\ell$ and x_ℓ converging to x^1_o. By construction, the sequence $\{\varphi^{(p)}(x_\ell) = z_\ell\}$ is in V_1 and tends to $\varphi^{(p)}(x^1_o) = z_o$. But this means that $z_\ell \in B_\varepsilon(z_o) \cap \varphi^{(p)}(F^{-1}(w_\ell))$ for all ℓ's sufficiently large and it contradicts our hypothesis.

We now consider the maps

$$\psi_j : B_\delta(w_o) \to B^n_\Gamma,\ \psi_j(w) = P_\Gamma(h_j(z)) \ \text{ for some } z \in \varphi^{(p)}(F^{-1}(w)) \cap \mathscr{V}_0$$

with $0 \le j \le k$. We claim that such maps are well defined and single valued. Indeed, if $z, z' \in \varphi^{(p)}(F^{-1}(w)) \cap \mathscr{V}_0$, then, by definition of $\mathscr{V}_0$,

$$z = g(\widetilde{z}), \quad z' = g'(\widetilde{z}') \qquad \text{for some} \quad \widetilde{z}, \widetilde{z}' \in B_\varepsilon(z_o),\ g, g' \in \Gamma_{(\mathrm{ref})}$$

and, by Proposition 3.3, $z' = h(z)$ for some $h \in \Gamma$ and hence of the form

$$h = g'' h_{i_o} \in \Gamma_{(\mathrm{ref})} h_{i_o} \quad \text{for some} \quad 0 \le i_o \le k.$$

These two facts and the normality of $\Gamma_{(\mathrm{ref})}$ imply that

$$g'(\widetilde{z}') = (g'' h_{i_o} g)(\widetilde{z}) = (g''' h_{i_o})(\widetilde{z}) \qquad \text{for some} \quad g''' \in \Gamma_{(\mathrm{ref})}$$

and hence that

$$\widetilde{z}' = (\widehat{g} h_{i_o})(\widetilde{z}) \in \mathscr{V}_{i_o} \qquad \text{with} \quad \widehat{g} = g'^{-1} g''' \in \Gamma_{(\mathrm{ref})}.$$

Since $\mathscr{V}_0 \cap \mathscr{V}_{i_o} = \emptyset$ for $i_o \ne 0$, we conclude that $h_{i_o} = h_0 = \mathrm{Id}_{B^n}$ and that $z' = g''(z)$. By normality of $\Gamma_{(\mathrm{ref})}$ and the properties of P_Γ, it follows that

$$P_\Gamma(h_j(z')) = P_\Gamma(h_j(g''(z))) = P_\Gamma(g'''(h_j(z))) \overset{g''' \in \Gamma_{(\mathrm{ref})}}{=} P_\Gamma(h_j(z)),$$

proving that ψ_j is well defined and single valued. Moreover, we have that

LEMMA 4.2. *Each map ψ_j is holomorphic.*

PROOF. Let us first show that the ψ_j's are continuous, i.e., that if $w_\ell \in B_\delta(w_o)$ is a sequence converging to $w \in B_\delta(w_o)$, then $\lim_{\ell \to \infty} \psi_j(w_\ell) = \psi_j(w)$. Consider the J-complete set $F^{-1}(w) = \{x^1, \ldots, x^N\} \subset \mathscr{E}^n_{(p)}$. By construction of $B_\delta(w_o)$, we may assume that $z = \varphi^{(p)}(x^1)$ belongs to $B_\varepsilon(z_o)$, so that $\psi_j(w) = P_\Gamma(h_j(z))$.

Let $\overline{B_{r_i}(x^i)} \subset \mathscr{E}^n_{(p)}$ be N disjoint closed balls such that

$$F^{-1}(w) \cap \overline{B_{r_i}(x^i)} = \{x^i\} \tag{4.4}$$

and denote by $S \subset \Omega$ the compact set $S = \bigcup_{i=1}^N F(\partial B_{r_i}(x^{(i)}))$. Since $w \notin S$, there exists $B_{\delta'}(w) \subset B_\delta(w_o)$ such that $B_{\delta'}(w) \cap S = \emptyset$. If we set

$$R^i = F^{-1}(B_{\delta'}(w)) \cap B_{r_i}(x^i) \tag{4.5}$$

the arguments of Prop. 15.1.6 in [13] imply that the maps $F|_{R^i} : R^i \to B_{\delta'}(w)$ are proper and hence surjective. With no loss of generality, we may assume that $\{w_\ell\} \subset B_{\delta'}(w)$ and we may consider a sequence $\{x_\ell\} \subset R^1$ such that $F(x_\ell) = w_\ell$. Up to a subsequence, $\{x_\ell\}$ converges to some $\widetilde{x} \in \overline{R^1}$. By (4.4), (4.5) and continuity, $F(\widetilde{x}) = w$ and $\widetilde{x} = x^1$.

Since $\{z_\ell = \varphi^{(p)}(x_\ell)\} \subset B^n$ converges to $z = \varphi^{(p)}(x^1)$, for all ℓ's sufficiently large z_ℓ is in $B_\varepsilon(z_o)$, so that $\lim_{\ell\to\infty} \psi_j(w_\ell) = \lim_{\ell\to\infty} P_\Gamma(h_j(z_\ell)) = P_\Gamma(h_j(z)) = \psi_j(w)$, as claimed.

We now prove that ψ_j's are holomorphic. Indeed, for any $w \in B_\delta(w_o) \setminus F(Z_F)$, there exist a neighbourhood $\mathscr{W}$ of w and neighbourhoods $\mathscr{U}^1, \dots, \mathscr{U}^m$ of the pre-images $x^1, \dots, x^m$ of w, such that $F|_{\mathscr{U}^i} : \mathscr{U}^i \to F(\mathscr{U}^i) = \mathscr{W}$ are biholomorphisms. For any $z \in \varphi^{(p)}(F^{-1}(w)) \cap B_\varepsilon(z_o)$, there exists $1 \leq j_o \leq m$ such that

$$z = \varphi^{(p)}(F|^{-1}_{\mathscr{U}^{j_o}}(w)).$$

Taking $\mathscr{W}$ sufficiently small, we may suppose that for any $w' \in \mathscr{W}$

$$z' = \varphi^{(p)}(F|^{-1}_{\mathscr{U}^{j_o}}(w')) \in B_\varepsilon(z_o) \Longrightarrow \psi_j(w') = P_\Gamma \circ h_j \circ \varphi^{(p)} \circ F|^{-1}_{\mathscr{U}^{j_o}}(w'),$$

proving that $\psi_j|_{\mathscr{W}}$ is holomorphic. This implies that ψ_j is holomorphic in $B_\delta(w_o) \setminus F(Z_F)$ and, by continuity and known facts on holomorphic extensions ([13], Cor. of Thm. 4.4.7), it is holomorphic on $B_\delta(w_o)$.

By construction, for any $w \in B_\delta(w_o)$ we have that $\Psi(w) = (\psi_0(w), \psi_1(w), \dots, \psi_k(w))$. By Lemma 4.2, the ψ_j's are holomorphic, meaning that Ψ splits at w_o. Since w_o is an arbitrary point of $\Omega \setminus W$ and $\dim W \leq n-2$, by Lemma 2.3 we have that Ψ splits. The equality (4.2) is a direct consequence of the definition of Ψ.

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