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SOME REMARKS ON CLOSE-TO-CONVEX AND STRONGLY CONVEX FUNCTIONS

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Abstract

We consider questions of the following kind: When does boundedness of $|\arg\{1 + zp'(z)/p(z)\}|$, for a given analytic function p, imply boundedness of $|\arg\{p(z)\}|$? The paper determines the order of strong close-to-convexity in the class of strongly convex functions. Also, we consider conditions that are sufficient for a function to be a Bazilevič function.

1. Introduction

Let $\mathscr{H}$ be the class of analytic functions in the disc $\mathsf{U} = \{z : \ |z| < 1\}$ in the complex plane C. Let $\mathscr{A}$ be the subclass of $\mathscr{H}$ consisting of functions f of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Moreover, by $\mathscr{S}$, $\mathscr{S}^*$, $\mathscr{K}$ and $\mathscr{C}$ we denote the subclasses of $\mathscr{A}$ which consist of univalent, starlike, convex and close-to-convex functions, respectively.

Robertson introduced in [12] the classes $\mathscr{S}^*_\alpha$, $\mathscr{K}_\alpha$ of starlike and convex functions of order α which are defined by

$$\mathscr{S}^*_\alpha = \left\{ f \in \mathscr{A} : \Re\mathrm{e}\, \frac{zf'(z)}{f(z)} > \alpha, z \in \mathsf{U} \right\}, \qquad \alpha < 1,$$

$$\begin{aligned}\mathscr{K}_\alpha &= \left\{ f \in \mathscr{A} : \Re\mathrm{e}\left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, z \in \mathsf{U} \right\} \\ &= \left\{ f \in \mathscr{A} : zf'(z) \in \mathscr{S}^*_\alpha \right\}, \qquad \alpha < 1.\end{aligned}$$

If $\alpha \in [0, 1)$, then a function in either of these sets is univalent, if $\alpha < 0$ it may fail to be univalent. In particular, we have $\mathscr{S}^*_0 = \mathscr{S}^*$ and $\mathscr{K}_0 = \mathscr{K}$.

Let $\mathscr{SS}^*(\beta)$ denote the class of strongly starlike functions of order β

$$\mathscr{SS}^*(\beta) = \left\{ f \in \mathscr{S} : \left| \arg \frac{zf'(z)}{f(z)} \right| < \frac{\beta\pi}{2}, z \in \mathsf{U} \right\}, \qquad \beta \in (0, 1],$$

which was introduced in [13] and [3]. Furthermore,

$$\mathscr{SK}(\beta) = \left\{ f \in \mathscr{S} : \ zf'(z) \in \mathscr{SS}^*(\beta) \right\}, \qquad \beta \in (0, 1]$$

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denotes the class of strongly convex functions of order β. Recall also that an analytic function f is said to be a close-to-convex function of order β, $\beta \in [0, 1)$, if and only if there exists a number $\varphi \in \mathsf{R}$ and a function $g \in \mathscr{K}$, such that

$$\Re\mathrm{e}\left\{e^{i\varphi}\frac{f'(z)}{g'(z)}\right\} > \beta \qquad \text{for} \quad z \in \mathsf{U}. \tag{1}$$

Reade [11] introduced the class of strongly close-to-convex functions of order β, $\beta < 1$, which is defined by

$$\left|\arg\left\{e^{i\varphi}\frac{f'(z)}{g'(z)}\right\}\right| < \frac{\pi\beta}{2} \qquad \text{for} \quad z \in \mathsf{U}, \tag{2}$$

instead of (1). Kaplan [5] investigated the class of functions satisfying the condition (1) in which $g \in \mathscr{K}_\alpha$. He denoted this class by $\mathscr{C}_\alpha(\beta)$. Let $\mathscr{S}\mathscr{C}_\alpha(\beta)$ denote the class of strongly close-to-convex functions of order β with respect to a convex function of order α, i.e. the class of functions $f \in \mathscr{A}$ satisfying (2) for some $g \in \mathscr{K}_\alpha$ and $\varphi \in \mathsf{R}$. Functions defined by (1) with $\varphi = 0$ were discussed by Ozaki [10] (see also Umezawa [15], [16]). Moreover, Biernacki [2] defined the class of functions $f \in \mathscr{A}$ for which the complement of $f(\mathsf{U})$ with respect to the complex plane is a linearly accessible domain in a broad sense. Lewandowski [6], [7] observed that the class $\mathscr{C}_0(0)$ of close-to-convex functions is the same as the class of linearly accessible functions.

Many classes can be defined using the notion of subordination. Recall that for $f, g \in \mathscr{H}$, we write $f \prec g$ and say that f is subordinate to g in U, if and only if there exists an analytic function $w \in \mathscr{H}$ satisfying $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$ for $z \in \mathsf{U}$. Therefore, $f \prec g$ implies $f(\mathsf{U}) \subset g(\mathsf{U})$. In particular, if g is univalent in U, then

$$f \prec g \iff \big[f(0) = g(0) \text{ and } f(\mathsf{U}) \subset g(\mathsf{U})\big].$$

The class $\mathscr{S}^*[A, B]$

$$\mathscr{S}^*[A, B] = \left\{f \in \mathscr{A} : \frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz}, z \in \mathsf{U}\right\}, \quad -1 \le B < A \le 1,$$

was investigated in [4]. For $-1 \le B < A \le 1$ the function $w(z) = (1 + Az)/(1 + Bz)$ maps the unit disc onto a disc in the right half plane, therefore the class $\mathscr{S}^*[A, B]$ is a subclass of $\mathscr{S}^*$ so if $f \in \mathscr{S}^*[A, B]$, then f is univalent in the unit disc.

2. Preliminaries

To prove the main results, we need the following generalization of the Nunokawa Lemmas from [8].

LEMMA 2.1 ([8]). *Let $p(z) = 1+\sum_{n=m}^{\infty} c_n z^n$, $c_m \neq 0$ be an analytic function in U with $p(z) \neq 0$. If there exists a point z_0, $|z_0| < 1$, such that*

$$|\arg\{p(z)\}| < \frac{\pi\beta}{2} \qquad \textit{for} \quad |z| < |z_0|$$

and

$$|\arg\{p(z_0)\}| = \frac{\pi\beta}{2}$$

for some $\beta > 0$, then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = \frac{2ik \arg\{p(z_0)\}}{\pi},$$

for some $k \geq m(a + a^{-1})/2 > m$, where

$$\{p(z_0)\}^{1/\beta} = \pm ia, \qquad \textit{and} \quad a > 0.$$

LEMMA 2.2. *[9]Let $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ be an analytic function in U. If there exists a point z_0, $z_0 \in \mathsf{U}$, such that*

$$\Re\mathrm{e}\{p(z)\} > c, \qquad \textit{for} \quad |z| < |z_0|$$

and

$$\Re\mathrm{e}\{p(z_0)\} = c, \qquad p(z_0) \neq c$$

for some $c \in (0, 1)$, then we have

$$\Re\mathrm{e}\, \frac{z_0 p'(z_0)}{p(z_0)} \leq \gamma(c),$$

where

$$\gamma(c) = \begin{cases} c/(2c-2) & \textit{when } c \in (0, 1/2], \\ (c-1)/(2c) & \textit{when } c \in (1/2, 1). \end{cases} \tag{3}$$

3. Main result

THEOREM 3.1. *Suppose that a function $f \in \mathscr{A}$ of the form*

$$f(z) = z + a_m z^m + a_{m+1} z^{m+1} + \cdots, \qquad a_m \neq 0$$

satisfies the conditions $f'(z) \neq 0$ *in* U *and*

$$\left|\arg\left\{1 + \frac{zf''(z)}{f'(z)}\right\}\right| < \tan^{-1}\lambda \qquad \textit{for} \quad z \in \mathsf{U}, \tag{4}$$

where $\lambda > 0$*. Then we have*

$$|\arg\{f'(z)\}| < \frac{\pi\lambda}{2(m-1)} \qquad \textit{for} \quad z \in \mathsf{U}. \tag{5}$$

PROOF. First, we note that from (4) it follows that $\Re\mathfrak{e}\{1+zf''(z)/f'(z)\} > 0$ and f is convex univalent in the unit disc, since $f'(z) \neq 0$ and $\arg\{f'(z)\}$ is well defined. If $f'(z) = p(z)$, then

$$p(z) = 1 + ma_m z^{m-1} + \cdots, \quad p(z) \neq 0, \qquad \text{for} \quad z \in \mathsf{U}. \tag{6}$$

For this function p, we suppose that there exists a point $z_0 \in \mathsf{U}$ such that

$$|\arg\{p(z)\}| < \frac{\pi\lambda}{2(m-1)} \qquad \text{for} \quad |z| < |z_0|$$

and

$$|\arg\{p(z_0)\}| = \frac{\pi\lambda}{2(m-1)}.$$

By Nunokawa's Lemma 2.1 and by (6), for all $\beta \in (0, 1)$ there exists a real $k \geq (m-1)(a + a^{-1})/2 > (m-1)$ such that

$$\frac{z_0 p'(z_0)}{p(z_0)} = \frac{2ik \arg\{p(z_0)\}}{\pi},$$

where

$$\{p(z_0)\}^{(m-1)/\lambda} = \pm ia, \qquad \text{and} \quad a > 0.$$

From (6) we get

$$\frac{f''(z)}{f'(z)} = \frac{p'(z)}{p(z)}.$$

If $\arg\{p(z_0)\} = \pi\lambda/(2m-2) > 0$, then we have

$$\begin{aligned}\arg\left\{1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right\} &= \arg\left\{1 + \frac{z_0 p'(z_0)}{p(z_0)}\right\} = \arg\left\{1 + \frac{2ik \arg\{p(z_0)\}}{\pi}\right\} \\ &= \arg\left\{1 + \frac{i\lambda k}{m-1}\right\} \geq \arg\{1 + i\lambda\} \geq \tan^{-1}\lambda.\end{aligned}$$

This contradicts assumption (4). If $\arg\{p(z_0)\} = -\pi\lambda/(2m-2)$, then applying the same method we get

$$\arg\left\{1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right\} \leq -\tan^{-1}\lambda,$$

which also contradicts assumption (4). Thus, there is no $z_0 \in \mathsf{U}$ such that

$$|\arg\{p(z)\}| < \frac{\pi\lambda}{2(m-1)} \qquad \text{for} \quad |z| < |z_0|$$

and

$$|\arg\{p(z_0)\}| = \frac{\pi\lambda}{2(m-1)}.$$

Because $\arg\{p(0)\} = \arg\{1\} = 0$ this implies that

$$|\arg\{p(z)\}| < \frac{\pi\lambda}{2(m-1)} \qquad \text{for all} \quad z \in \mathsf{U}.$$

COROLLARY 3.2. *Suppose that a function $p \in \mathscr{H}$ of the form*

$$p(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \cdots, \qquad c_n \neq 0$$

satisfies the conditions $p(z) \neq 0$ and

$$\left|\arg\left\{1 + \frac{zp'(z)}{p(z)}\right\}\right| < \tan^{-1}\lambda \qquad \textit{for} \quad z \in \mathsf{U}, \tag{7}$$

where $\lambda > 0$. Then we have

$$|\arg\{p(z)\}| < \frac{\pi\lambda}{2n} \qquad \textit{for} \quad z \in \mathsf{U}. \tag{8}$$

PROOF. Consider a function f, $f(z) = z + \cdots$ such that $p(z) = f'(z)$. Then we have

$$f(z) = z + \frac{c_n}{n+1} z^{n+1} + \cdots, \qquad c_n \neq 0.$$

Moreover, (7) becomes (4). By Theorem 3.1, we then have (8).

THEOREM 3.3. *Suppose that a function f of the form*

$$f(z) = z + a_m z^m + a_{m+1} z^{m+1} + \cdots, \qquad a_m \neq 0 \tag{9}$$

is in the class $\mathcal{SK}(\gamma)$*, where* $\gamma = \gamma(\alpha, \beta) = \frac{2}{\pi} \tan^{-1} \frac{\beta(m-1)}{1-\alpha}$, $\alpha, \beta \in (0, 1)$*. Then there exists a function* $g \in \mathcal{K}_{1-\alpha} \cap \mathcal{SK}(\gamma)$ *such that*

$$\left|\arg \frac{f'(z)}{g'(z)}\right| < \frac{\pi\beta}{2} \qquad \textit{for} \quad z \in \mathsf{U}, \tag{10}$$

or $f \in \mathcal{SC}_{1-\alpha}(\beta)$.

Proof. If $f \in \mathcal{SK}(\gamma)$, then f is univalent and $f'(z) \neq 0$ in the unit disc. Let a function $g \in \mathcal{A}$ be defined by

$$g'(z) = (f'(z))^{\alpha}. \tag{11}$$

This implies that

$$\frac{zg''(z)}{g'(z)} = \alpha \frac{zf''(z)}{f'(z)}.$$

Furthermore, $f \in \mathcal{SK}(\gamma)$ follows that $\Re\mathfrak{e}\{1 + zf''(z)/f'(z)\} > 0$. Therefore

$$\begin{aligned}\Re\mathfrak{e}\left\{1 + \frac{zg''(z)}{g'(z)}\right\} &= \Re\mathfrak{e}\left\{1 + \alpha\frac{zf''(z)}{f'(z)}\right\} \\ &= \Re\mathfrak{e}\left\{1 - \alpha + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > 1 - \alpha,\end{aligned}$$

which means that $g \in \mathcal{K}_{1-\alpha}$. Moreover,

$$\begin{aligned}\left|\arg\left\{1 + \frac{zg''(z)}{g'(z)}\right\}\right| &= \left|\arg\left\{1 + \alpha\frac{zf''(z)}{f'(z)}\right\}\right| \\ &= \left|\arg\left\{\frac{1-\alpha}{\alpha} + \left(1 + \frac{zf''(z)}{f'(z)}\right)\right\}\right| \\ &< \left|\arg\left\{1 + \frac{zf''(z)}{f'(z)}\right\}\right| < \frac{\gamma\pi}{2}.\end{aligned}$$

This means that $g \in \mathcal{SK}(\gamma)$, thus $g \in \mathcal{K}_{1-\alpha} \cap \mathcal{SK}(\gamma)$.

From assumption $f \in \mathcal{SK}(\gamma)$ we have

$$\left|\arg\left\{1 + \frac{zf''(z)}{f'(z)}\right\}\right| < \tan^{-1} \frac{\beta(m-1)}{1-\alpha} \qquad \text{for} \quad z \in \mathsf{U}, \tag{12}$$

thus by Theorem 3.1 we obtain

$$|\arg\{f'(z)\}| < \frac{\pi}{2(m-1)} \frac{\beta(m-1)}{1-\alpha} = \frac{\pi\beta}{2(1-\alpha)} \qquad \text{for} \quad z \in \mathsf{U}. \tag{13}$$

By (13) we have

$$\left|\arg\left\{\frac{f'(z)}{g'(z)}\right\}\right| = \left|\arg\left\{\frac{f'(z)}{(f'(z))^{\alpha}}\right\}\right| = (1-\alpha)|\arg\{f'(z)\}|$$
$$< (1-\alpha)\frac{\pi\beta}{2(1-\alpha)} = \frac{\pi\beta}{2},$$

which proves (10).

Condition (10) means that f is a strongly close-to-convex function of order β with respect to a function g which is convex of order $1-\alpha$. Moreover, $g \in \mathscr{K}_{1-\alpha} \cap \mathscr{SK}(\gamma)$. We can rewrite Theorem 3.3 in the following form.

COROLLARY 3.4. *Assume that* $\alpha, \beta \in (0,1)$ *and a function* $f(z) = z + a_m z^m + a_{m+1}z^{m+1} + \cdots$, $a_m \neq 0$ *satisfies the condition* $f'(z) \neq 0$ *in* U. *Then*

$$\left[\left|\arg\left\{1 + \frac{zf''(z)}{f'(z)}\right\}\right| < \tan^{-1}\frac{\beta(m-1)}{1-\alpha}\right] \Longrightarrow \left[\left|\arg\frac{f'(z)}{g'(z)}\right| < \frac{\pi\beta}{2}\right]$$

for $z \in \mathsf{U}$ *and for some* $g \in \mathscr{K}_{1-\alpha} \cap \mathscr{SK}(\gamma)$, *where* $\gamma = \gamma(\alpha,\beta) = \frac{2}{\pi}\tan^{-1}\frac{\beta(m-1)}{1-\alpha}$.

THEOREM 3.5. *Assume that* $\alpha \in [1/2, 1)$, $\beta \geq 1$ *and* $c \in (0,1)$. *Furthermore, let* $f \in \mathscr{K}_{\alpha}$ *and let a function* $g \in \mathscr{A}$ *satisfy the conditions*

$$\Re\mathfrak{e}\,\frac{zg'(z)}{g(z)} \leq \frac{\alpha - \gamma(c) + (\beta-1)\delta(\alpha)}{\beta}, \qquad g(z) \neq 0, \tag{14}$$

for $z \in \mathsf{U}\setminus\{0\}$, *where* $\gamma(c)$ *is given by* (3) *and*

$$\delta(\alpha) = \begin{cases} (1-2\alpha)/(2^{2-2\alpha}-2) & \textit{for } \alpha \neq 1/2, \\ 1/(2\log 2) & \textit{for } \alpha = 1/2. \end{cases} \tag{15}$$

Then we have

$$\Re\mathfrak{e}\,\frac{zf'(z)}{f^{1-\beta}(z)g^{\beta}(z)} > c \qquad \textit{for} \quad z \in \mathsf{U}.$$

PROOF. From [17] it follows that if $f \in \mathscr{K}_{\alpha}$, then $f \in \mathscr{S}^*_{\delta(\alpha)}$. Because $\beta \geq 1$, so

$$\Re\mathfrak{e}\left\{(1-\beta)\frac{zf'(z)}{f(z)}\right\} \leq (1-\beta)\delta(\alpha). \tag{16}$$

If f, g satisfy (16) and (14), respectively, then f is univalent in U, $f(z) \neq 0$ and $g(z) \neq 0$ for $z \in \mathsf{U} \setminus \{0\}$. If we put

$$p(z) = f'(z)\left\{\frac{z}{f(z)}\right\}^{1-\beta}\left\{\frac{z}{g(z)}\right\}^{\beta} = \frac{zf'(z)}{f^{1-\beta}(z)g^{\beta}(z)}, \tag{17}$$

then p is an analytic function in U and $p(0) = 1$. From (17) we get

$$1 + \frac{zf''(z)}{f'(z)} = \frac{zp'(z)}{p(z)} + (1-\beta)\frac{zf'(z)}{f(z)} + \beta\frac{zg'(z)}{g(z)}. \tag{18}$$

For this function p, we suppose that there exists a point $z_0 \in \mathsf{U}$ such that

$$\Re\mathfrak{e}\{p(z)\} > c, \qquad \text{for} \quad |z| < |z_0|$$

and

$$\Re\mathfrak{e}\{p(z_0)\} = c, \qquad p(z_0) \neq c.$$

Hence, Lemma 2.2 gives us

$$\Re\mathfrak{e}\,\frac{z_0 p'(z_0)}{p(z_0)} \leq \gamma(c), \tag{19}$$

where $\gamma(c)$ is given by (3).

Taking into account (14), (16), (18) and (19), we get

$$\begin{aligned}\Re\mathfrak{e}\left\{1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right\} &= \Re\mathfrak{e}\left\{\frac{z_0 p'(z_0)}{p(z_0)} + (1-\beta)\frac{z_0 f'(z_0)}{f(z_0)} + \beta\frac{z_0 g'(z_0)}{g(z_0)}\right\}\\ &\leq \gamma(c) + (1-\beta)\delta(\alpha) + \beta\frac{\alpha - \gamma(c) + (\beta-1)\delta(\alpha)}{\beta}\\ &= \alpha.\end{aligned}$$

This contradicts the hypothesis that $f \in \mathscr{K}_\alpha$. Thus, there is no $z_0 \in \mathsf{U}$ such that

$$\Re\mathfrak{e}\{p(z)\} > c \qquad \text{for} \quad |z| < |z_0|$$

and

$$\Re\mathfrak{e}\{p(z_0)\} = c, \qquad p(z_0) \neq c.$$

Because $p(0) = 1 > c$, this implies that $\Re\mathfrak{e}\{p(z)\} > c$ in the unit disc, which completes the proof.

For $\beta = 1$, Theorem 3.5 gives us the following corollary.

COROLLARY 3.6. *Assume that* $\alpha \in [1/2, 1)$. *Moreover, let* $f \in \mathcal{K}_\alpha$ *and let a function* $g \in \mathcal{A}$ *satisfy the conditions*

$$\Re\mathfrak{e}\, \frac{zg'(z)}{g(z)} \leq \alpha - \gamma(c), \quad g(z) \neq 0, \qquad \text{for} \quad z \in \mathsf{U} \setminus \{0\},$$

where $\gamma(c)$ *is given by (3) and* $c \in (0, 1)$ *is such that* $\alpha - \gamma(c) > 1$. *Then we have*

$$\Re\mathfrak{e}\, \frac{zf'(z)}{g(z)} > c \qquad \text{for} \quad z \in \mathsf{U}.$$

REMARK 3.7. If $\beta > 1$, α and f satisfy the conditions of Theorem 3.5, then f is a Bazilevič function of order c, $c \in (0, 1)$, see [14, p. 353].

If $g \in \mathcal{S}^*[A, B]$, then

$$\frac{1+A}{1+B} \leq \Re\mathfrak{e}\, \frac{zg'(z)}{g(z)} \leq \frac{1-A}{1-B}$$

Therefore, applying the same method as in the proof of Theorem 3.5, we obtain the following theorem.

THEOREM 3.8. *Suppose that* $\alpha \in [1/2, 1)$, $\beta > 1$ *and* $c \in (0, 1)$. *Assume also that* $f \in \mathcal{K}_\alpha$ *and that* $g \in \mathcal{S}^*[A, B]$ *with*

$$\frac{1-A}{1-B} \leq \frac{\alpha - \gamma(c) + (\beta - 1)\delta(\alpha)}{\beta},$$

where $\gamma(c)$ *and* $\delta(\alpha)$ *are given by* (3) *and* (15), *respectively. Then we have*

$$\Re\mathfrak{e}\, \frac{zf'(z)}{f^{1-\beta}(z) g^{\beta}(z)} > c \qquad \text{for} \quad z \in \mathsf{U}.$$

REMARK 3.9. If f satisfies the conditions of Theorem 3.8, then f is a Bazilevič function.

If we take that $\alpha = 3/4$, $\beta = 5/4$ and $c = 1/2$, then $\gamma(1/2) = -1/2$, $\delta(3/4) = (2+\sqrt{2})/4$, therefore Theorem 3.5 becomes the following corollary.

COROLLARY 3.10. *Suppose that* $f \in \mathcal{K}_{3/4}$ *and that for* $g \in \mathcal{A}$ *we have*

$$\Re\mathfrak{e}\, \frac{zg'(z)}{g(z)} \leq \frac{22+\sqrt{2}}{20} = 1.17\ldots, \quad g(z) \neq 0, \qquad \text{for} \quad z \in \mathsf{U} \setminus \{0\}.$$

Then we get

$$\Re\mathfrak{e}\, \frac{zf'(z)\sqrt[4]{f(z)}}{g(z)\sqrt[4]{g(z)}} > \frac{1}{2} \qquad \text{for} \quad z \in \mathsf{U}.$$

If $g \in \mathcal{S}^*(q_c)$, $c \in (0, 1]$, where the class

$$\mathcal{S}^*(q_c) = \left\{ g \in \mathcal{A} : \frac{zg'(z)}{g(z)} \prec q_c(z),\, g(z) \neq 0,\, z \in \mathsf{U} \setminus \{0\} \right\},$$

$q_c(z) = \sqrt{1 + cz}$, was introduced in [1], then $\Re\mathfrak{e}\{zg'(z)/g(z)\} < \sqrt{1+c}$. Therefore, if

$$c < \frac{43 + 22\sqrt{2}}{200} = 0.37\ldots,$$

then Corollary 3.10 becomes

$$\left[f \in \mathcal{K}_{3/4} \text{ and } g \in \mathcal{S}^*(q_c) \right] \implies \left[\Re\mathfrak{e}\, \frac{zf'(z)\sqrt[4]{f(z)}}{g(z)\sqrt[4]{g(z)}} > \frac{1}{2} \right].$$

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