SOME REMARKS ON CLOSE-TO-CONVEX AND STRONGLY CONVEX FUNCTIONS

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Abstract
We consider questions of the following kind: When does boundedness of $|\arg\left\{1 + zp'(z)/p(z)\right\}|$, for a given analytic function $p$, imply boundedness of $|\arg(p(z))|$? The paper determines the order of strong close-to-convexity in the class of strongly convex functions. Also, we consider conditions that are sufficient for a function to be a Bazilevič function.

1. Introduction
Let $H$ be the class of analytic functions in the disc $U = \{z : |z| < 1\}$ in the complex plane $C$. Let $A$ be the subclass of $H$ consisting of functions $f$ of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Moreover, by $S^*$, $K^*$, $K$ and $C$ we denote the subclasses of $A$ which consist of univalent, starlike, convex and close-to-convex functions, respectively.

Robertson introduced in [12] the classes $S^*_\alpha$, $K^*_\alpha$ of starlike and convex functions of order $\alpha$ which are defined by

\[
S^*_\alpha = \left\{ f \in A : \Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha \right\}, \quad \alpha < 1,
\]

\[
K^*_\alpha = \left\{ f \in A : \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \right\}
\]

\[
= \left\{ f \in A : zf'(z) \in S^*_\alpha \right\}, \quad \alpha < 1.
\]

If $\alpha \in [0, 1)$, then a function in either of these sets is univalent, if $\alpha < 0$ it may fail to be univalent. In particular, we have $S^*_0 = S^*$ and $K^*_0 = K$.

Let $S^*_\beta$ denote the class of strongly starlike functions of order $\beta$

\[
S^*_\beta = \left\{ f \in S : \left| \frac{zf'(z)}{f(z)} \right| < \frac{\beta \pi}{2}, z \in U \right\}, \quad \beta \in (0, 1],
\]

which was introduced in [13] and [3]. Furthermore,

\[
\mathcal{K}^*_\beta = \left\{ f \in S : zf''(z) \in S^*_\beta \right\}, \quad \beta \in (0, 1]
\]

Received February 1 2013, in final form October 20 2014.
denotes the class of strongly convex functions of order \( \beta \). Recall also that an analytic function \( f \) is said to be a close-to-convex function of order \( \beta \), \( \beta \in [0, 1) \), if and only if there exists a number \( \varphi \in \mathbb{R} \) and a function \( g \in \mathcal{H} \), such that

\[
\Re \left\{ e^{i\varphi} \frac{f'(z)}{g'(z)} \right\} > \beta \quad \text{for } z \in \mathbb{U}.
\]

Reade [11] introduced the class of strongly close-to-convex functions of order \( \beta \), \( \beta < 1 \), which is defined by

\[
\left| \arg \left\{ e^{i\varphi} \frac{f'(z)}{g'(z)} \right\} \right| < \frac{\pi \beta}{2} \quad \text{for } z \in \mathbb{U},
\]

instead of (1). Kaplan [5] investigated the class of functions satisfying the condition (1) in which \( g \in \mathcal{H}_\alpha \). He denoted this class by \( \mathcal{C}_\alpha(\beta) \). Let \( \mathcal{S}_\alpha(\beta) \) denote the class of strongly close-to-convex functions of order \( \beta \) with respect to a convex function of order \( \alpha \), i.e. the class of functions \( f \in \mathcal{A} \) satisfying (2) for some \( g \in \mathcal{H}_\alpha \) and \( \varphi \in \mathbb{R} \). Functions defined by (1) with \( \varphi = 0 \) were discussed by Ozaki [10] (see also Umezawa [15], [16]). Moreover, Biernacki [2] defined the class of functions \( f \in \mathcal{A} \) for which the complement of \( f(U) \) with respect to the complex plane is a linearly accessible domain in a broad sense. Lewandowski [6], [7] observed that the class \( \mathcal{C}_0(0) \) of close-to-convex functions is the same as the class of linearly accessible functions.

Many classes can be defined using the notion of subordination. Recall that for \( f, g \in \mathcal{H} \), we write \( f \prec g \) and say that \( f \) is subordinate to \( g \) in \( \mathbb{U} \), if and only if there exists an analytic function \( w \in \mathcal{H} \) satisfying \( w(0) = 0 \) and \( |w(z)| < 1 \) such that \( f(z) = g(w(z)) \) for \( z \in \mathbb{U} \). Therefore, \( f \prec g \) implies \( f(U) \subset g(U) \). In particular, if \( g \) is univalent in \( \mathbb{U} \), then

\[
f \prec g \iff [f(0) = g(0) \text{ and } f(U) \subset g(U)].
\]

The class \( \mathcal{S}^*[A, B] \)

\[
\mathcal{S}^*[A, B] = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} < \frac{1 + Az}{1 + Bz}, z \in \mathbb{U} \right\}, \quad -1 \leq B < A \leq 1,
\]

was investigated in [4]. For \( -1 \leq B < A \leq 1 \) the function \( w(z) = (1 + Az)/(1 + Bz) \) maps the unit disc onto a disc in the right half plane, therefore the class \( \mathcal{S}^*[A, B] \) is a subclass of \( \mathcal{S}^* \) so if \( f \in \mathcal{S}^*[A, B] \), then \( f \) is univalent in the unit disc.
2. Preliminaries

To prove the main results, we need the following generalization of the Nunokawa Lemmas from [8].

**Lemma 2.1** ([8]). Let \( p(z) = 1 + \sum_{n=m}^{\infty} c_n z^n, c_m \neq 0 \) be an analytic function in \( U \) with \( p(z) \neq 0 \). If there exists a point \( z_0, |z_0| < 1 \), such that
\[
|\arg\{p(z)\}| < \frac{\pi \beta}{2} \quad \text{for} \quad |z| < |z_0|
\]
and
\[
|\arg\{p(z_0)\}| = \frac{\pi \beta}{2}
\]
for some \( \beta > 0 \), then we have
\[
\frac{z_0 p'(z_0)}{p(z_0)} = \frac{2i k \arg\{p(z_0)\}}{\pi},
\]
for some \( k \geq (a + a^{-1})/2 > m \), where
\[
\{p(z_0)\}^{1/\beta} = \pm ia, \quad \text{and} \quad a > 0.
\]

**Lemma 2.2.** ([9]) Let \( p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \) be an analytic function in \( U \). If there exists a point \( z_0, z_0 \in U \), such that
\[
\Re\{p(z)\} > c, \quad \text{for} \quad |z| < |z_0|
\]
and
\[
\Re\{p(z_0)\} = c, \quad p(z_0) \neq c
\]
for some \( c \in (0, 1) \), then we have
\[
\Re \frac{z_0 p'(z_0)}{p(z_0)} \leq \gamma(c),
\]
where
\[
\gamma(c) = \begin{cases} 
  c/(2c - 2) & \text{when } c \in (0, 1/2], \\
  (c - 1)/(2c) & \text{when } c \in (1/2, 1).
\end{cases}
\]

3. Main result

**Theorem 3.1.** Suppose that a function \( f \in \mathcal{A} \) of the form
\[
f(z) = z + a_m z^m + a_{m+1} z^{m+1} + \cdots, \quad a_m \neq 0
\]
satisfies the conditions \( f'(z) \neq 0 \) in \( U \) and

\[
\left| \arg \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \right| < \tan^{-1} \lambda \quad \text{for} \quad z \in U,
\]

where \( \lambda > 0 \). Then we have

\[
|\arg\{f'(z)\}| < \frac{\pi \lambda}{2(m - 1)} \quad \text{for} \quad z \in U.
\]

**Proof.** First, we note that from (4) it follows that \( \Re\{1+zf''(z)/f'(z)\} > 0 \) and \( f \) is convex univalent in the unit disc, since \( f'(z) \neq 0 \) and \( \arg\{f'(z)\} \) is well defined. If \( f'(z) = p(z) \), then

\[
p(z) = 1 + ma_m z^{m-1} + \cdots, \quad p(z) \neq 0, \quad \text{for} \quad z \in U.
\]

For this function \( p \), we suppose that there exists a point \( z_0 \in U \) such that

\[
|\arg\{p(z)\}| < \frac{\pi \lambda}{2(m - 1)} \quad \text{for} \quad |z| < |z_0|
\]

and

\[
|\arg\{p(z_0)\}| = \frac{\pi \lambda}{2(m - 1)}.
\]

By Nunokawa’s Lemma 2.1 and by (6), for all \( \beta \in (0, 1) \) there exists a real \( k \geq (m - 1)(a + a^{-1})/2 > (m - 1) \) such that

\[
\frac{z_0 p'(z_0)}{p(z_0)} = \frac{2ik \arg\{p(z_0)\}}{\pi},
\]

where

\[
\{p(z_0)\}^{(m-1)/\lambda} = \pm ia, \quad \text{and} \quad a > 0.
\]

From (6) we get

\[
\frac{f''(z)}{f'(z)} = \frac{p'(z)}{p(z)}.
\]

If \( \arg\{p(z_0)\} = \pi \lambda/(2m - 2) > 0 \), then we have

\[
\arg\left\{ 1 + \frac{z_0 f''(z_0)}{f'(z_0)} \right\} = \arg\left\{ 1 + \frac{z_0 p'(z_0)}{p(z_0)} \right\} = \arg\left\{ 1 + \frac{2ik \arg\{p(z_0)\}}{\pi} \right\} = \arg\left\{ 1 + \frac{i \lambda k}{m - 1} \right\} \geq \arg\{1 + i \lambda\} \geq \tan^{-1} \lambda.
\]
This contradicts assumption (4). If \( \arg\{p(z_0)\} = -\pi \lambda / (2m - 2) \), then applying the same method we get
\[
\arg\left\{ 1 + \frac{z_0 f''(z_0)}{f'(z_0)} \right\} \leq -\tan^{-1} \lambda,
\]
which also contradicts assumption (4). Thus, there is no \( z_0 \in U \) such that
\[
|\arg\{p(z)\}| < \frac{\pi \lambda}{2(m - 1)} \quad \text{for} \quad |z| < |z_0|
\]
and
\[
|\arg\{p(z_0)\}| = \frac{\pi \lambda}{2(m - 1)}.
\]
Because \( \arg\{p(0)\} = \arg\{1\} = 0 \) this implies that
\[
|\arg\{p(z)\}| < \frac{\pi \lambda}{2(m - 1)} \quad \text{for all} \quad z \in U.
\]

**Corollary 3.2.** Suppose that a function \( p \in \mathcal{H} \) of the form
\[
p(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \cdots, \quad c_n \neq 0
\]
satisfies the conditions \( p(z) \neq 0 \) and
\[
|\arg\{p(z)\}| < \frac{\pi \lambda}{2(m - 1)} \quad \text{for} \quad z \in U,
\]
where \( \lambda > 0 \). Then we have
\[
|\arg\{p(z)\}| < \frac{\pi \lambda}{2n} \quad \text{for} \quad z \in U.
\]

**Proof.** Consider a function \( f, f(z) = z + \cdots \) such that \( p(z) = f'(z) \). Then we have
\[
f(z) = z + \frac{c_n}{n+1} z^{n+1} + \cdots, \quad c_n \neq 0.
\]
Moreover, (7) becomes (4). By Theorem 3.1, we then have (8).

**Theorem 3.3.** Suppose that a function \( f \) of the form
\[
f(z) = z + a_m z^m + a_{m+1} z^{m+1} + \cdots, \quad a_m \neq 0
\]
is in the class $\mathcal{KH}(\gamma)$, where $\gamma = \gamma(\alpha, \beta) = \frac{2}{\pi} \tan^{-1} \frac{\beta(m-1)}{1-\alpha}$, $\alpha, \beta \in (0, 1)$. Then there exists a function $g \in \mathcal{K}_{1-\alpha} \cap \mathcal{KH}(\gamma)$ such that

\begin{equation}
\left| \arg \frac{f'(z)}{g'(z)} \right| < \frac{\pi \beta}{2} \quad \text{for } z \in U,
\end{equation}

or $f \in \mathcal{EC}_{1-\alpha}(\beta)$.

**Proof.** If $f \in \mathcal{KH}(\gamma)$, then $f$ is univalent and $f'(z) \neq 0$ in the unit disc. Let a function $g \in \mathcal{A}$ be defined by

\begin{equation}
g'(z) = (f'(z))^{\alpha}.
\end{equation}

This implies that

\[ \frac{zg''(z)}{g'(z)} = \alpha \frac{zf''(z)}{f'(z)}. \]

Furthermore, $f \in \mathcal{KH}(\gamma)$ follows that $\Re\{1 + zf''(z)/f'(z)\} > 0$. Therefore

\[ \Re\left\{1 + \frac{zg''(z)}{g'(z)}\right\} = \Re\left\{1 + \alpha \frac{zf''(z)}{f'(z)}\right\} = \Re\left\{1 - \alpha + \alpha \left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > 1 - \alpha, \]

which means that $g \in \mathcal{K}_{1-\alpha}$. Moreover,

\[ \left| \arg \left\{1 + \frac{zg''(z)}{g'(z)}\right\}\right| = \left| \arg \left\{1 + \alpha \frac{zf''(z)}{f'(z)}\right\}\right| = \left| \arg \left\{\frac{1 - \alpha}{\alpha} + \left(1 + \frac{zf''(z)}{f'(z)}\right)\right\}\right| < \left| \arg \left\{1 + \frac{zf''(z)}{f'(z)}\right\}\right| < \frac{\gamma \pi}{2}. \]

This means that $g \in \mathcal{KH}(\gamma)$, thus $g \in \mathcal{K}_{1-\alpha} \cap \mathcal{KH}(\gamma)$.

From assumption $f \in \mathcal{KH}(\gamma)$ we have

\begin{equation}
\left| \arg \left\{1 + \frac{zf''(z)}{f'(z)}\right\}\right| < \tan^{-1} \frac{\beta(m-1)}{1-\alpha} \quad \text{for } z \in U,
\end{equation}

thus by Theorem 3.1 we obtain

\begin{equation}
\left| \arg \left\{f'(z)\right\}\right| < \frac{\pi}{2(m-1)} \frac{\beta(m-1)}{1-\alpha} = \frac{\pi \beta}{2(1-\alpha)} \quad \text{for } z \in U.
\end{equation}
By (13) we have
\[
\left| \arg \left\{ \frac{f'(z)}{g'(z)} \right\} \right| = \left| \arg \left\{ \frac{f'(z)}{(f'(z))^\alpha} \right\} \right| = (1 - \alpha) \left| \arg \left\{ f'(z) \right\} \right| \\
< (1 - \alpha) \frac{\pi \beta}{2(1 - \alpha)} = \frac{\pi \beta}{2},
\]
which proves (10).

Condition (10) means that \( f \) is a strongly close-to-convex function of order \( \beta \) with respect to a function \( g \) which is convex of order \( 1 - \alpha \). Moreover, \( g \in H_{1-\alpha} \cap H(\gamma) \). We can rewrite Theorem 3.3 in the following form.

**Corollary 3.4.** Assume that \( \alpha, \beta \in (0, 1) \) and a function \( f(z) = z + a_m z^m + a_{m+1} z^{m+1} + \cdots, a_m \neq 0 \) satisfies the condition \( f'(z) \neq 0 \) in \( U \). Then
\[
\left[ \left| \arg \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \right| < \tan^{-1} \frac{\beta(m - 1)}{1 - \alpha} \right] \Rightarrow \left[ \left| \arg \frac{f'(z)}{g'(z)} \right| < \frac{\pi \beta}{2} \right]
\]
for \( z \in U \) and for some \( g \in H_{1-\alpha} \cap H(\gamma) \), where \( \gamma = \gamma(\alpha, \beta) = \frac{2}{\pi} \tan^{-1} \frac{\beta(m-1)}{1-\alpha} \).

**Theorem 3.5.** Assume that \( \alpha \in [1/2, 1), \beta \geq 1 \) and \( c \in (0, 1) \). Furthermore, let \( f \in H_\alpha \) and let a function \( g \in A \) satisfy the conditions
\[
\Re \left\{ \frac{g'(z)}{g(z)} \right\} \leq \frac{\alpha - \gamma(c) + (\beta - 1)\delta(\alpha)}{\beta}, \quad g(z) \neq 0, \quad (14)
\]
for \( z \in U \setminus \{0\} \), where \( \gamma(c) \) is given by (3) and
\[
\delta(\alpha) = \begin{cases} 
(1 - 2\alpha)/(2^{2-2\alpha} - 2) & \text{for } \alpha \neq 1/2, \\
1/(2 \log 2) & \text{for } \alpha = 1/2. 
\end{cases} \quad (15)
\]
Then we have
\[
\Re \left\{ \frac{zf'(z)}{f^{1-\beta}(z)g^\beta(z)} \right\} > c \quad \text{for } z \in U. \quad (16)
\]

**Proof.** From [17] it follows that if \( f \in H_\alpha \), then \( f \in F^*_\delta(\alpha) \). Because \( \beta \geq 1 \), so
\[
\Re \left\{ (1 - \beta) \frac{zf'(z)}{f(z)} \right\} \leq (1 - \beta)\delta(\alpha). \quad (16)
\]
If \( f, g \) satisfy (16) and (14), respectively, then \( f \) is univalent in \( U \), \( f(z) \neq 0 \) and \( g(z) \neq 0 \) for \( z \in U \setminus \{0\} \). If we put

\[
p(z) = f'(z) \left\{ \frac{z}{f(z)} \right\}^{1-\beta} \left\{ \frac{z}{g(z)} \right\}^{\beta} = \frac{zf'(z)}{f^{1-\beta}(z)g^\beta(z)},
\]

then \( p \) is an analytic function in \( U \) and \( p(0) = 1 \). From (17) we get

\[
1 + \frac{zf''(z)}{f'(z)} = \frac{zp'(z)}{p(z)} + (1 - \beta) \frac{zf'(z)}{f(z)} + \beta \frac{zg'(z)}{g(z)}.
\]

For this function \( p \), we suppose that there exists a point \( z_0 \in U \) such that

\[
\Re\{p(z)\} > c, \quad \text{for } |z| < |z_0|
\]

and

\[
\Re\{p(z_0)\} = c, \quad p(z_0) \neq c.
\]

Hence, Lemma 2.2 gives us

\[
\Re \frac{z_0p'(z_0)}{p(z_0)} \leq \gamma(c),
\]

where \( \gamma(c) \) is given by (3).

Taking into account (14), (16), (18) and (19), we get

\[
\Re \left\{ 1 + \frac{z_0f''(z_0)}{f'(z_0)} \right\} = \Re \left\{ \frac{zp'(z_0)}{p(z_0)} + (1 - \beta) \frac{zf'(z_0)}{f(z_0)} + \beta \frac{zg'(z_0)}{g(z_0)} \right\}
\]

\[
\leq \gamma(c) + (1 - \beta)\delta(\alpha) + \beta \frac{\alpha - \gamma(c) + (\beta - 1)\delta(\alpha)}{\beta}
\]

\[
= \alpha.
\]

This contradicts the hypothesis that \( f \in \mathcal{K}_\alpha \). Thus, there is no \( z_0 \in U \) such that

\[
\Re\{p(z)\} > c, \quad \text{for } |z| < |z_0|
\]

and

\[
\Re\{p(z_0)\} = c, \quad p(z_0) \neq c.
\]

Because \( p(0) = 1 > c \), this implies that \( \Re\{p(z)\} > c \) in the unit disc, which completes the proof.

For \( \beta = 1 \), Theorem 3.5 gives us the following corollary.
Corollary 3.6. Assume that $\alpha \in [1/2, 1)$. Moreover, let $f \in \mathcal{K}_\alpha$ and let a function $g \in \mathcal{A}$ satisfy the conditions

$$\Re \frac{zg'(z)}{g(z)} \leq \alpha - \gamma(c), \quad g(z) \neq 0, \quad \text{for } z \in U \setminus \{0\},$$

where $\gamma(c)$ is given by (3) and $c \in (0, 1)$ is such that $\alpha - \gamma(c) > 1$. Then we have

$$\Re \frac{zf'(z)}{f(z)} > c \quad \text{for } z \in U.$$

Remark 3.7. If $\beta > 1$, $\alpha$ and $f$ satisfy the conditions of Theorem 3.5, then $f$ is a Bazilević function of order $c$, $c \in (0, 1)$, see [14, p. 353].

If $g \in \mathcal{S}^*[A, B]$, then

$$\frac{1 + A}{1 + B} \leq \Re \frac{zg'(z)}{g(z)} \leq \frac{1 - A}{1 - B}.$$

Therefore, applying the same method as in the proof of Theorem 3.5, we obtain the following theorem.

Theorem 3.8. Suppose that $\alpha \in [1/2, 1)$, $\beta > 1$ and $c \in (0, 1)$. Assume also that $f \in \mathcal{K}_\alpha$ and that $g \in \mathcal{S}^*[A, B]$ with

$$\frac{1 - A}{1 - B} \leq \frac{\alpha - \gamma(c) + (\beta - 1)\delta(\alpha)}{\beta},$$

where $\gamma(c)$ and $\delta(\alpha)$ are given by (3) and (15), respectively. Then we have

$$\Re \frac{zf'(z)}{f^{1-\beta}(z)g^\beta(z)} > c \quad \text{for } z \in U.$$

Remark 3.9. If $f$ satisfies the conditions of Theorem 3.8, then $f$ is a Bazilević function.

If we take that $\alpha = 3/4$, $\beta = 5/4$ and $c = 1/2$, then $\gamma(1/2) = -1/2$, $\delta(3/4) = (2 + \sqrt{2})/4$, therefore Theorem 3.5 becomes the following corollary.

Corollary 3.10. Suppose that $f \in \mathcal{K}_{3/4}$ and that for $g \in \mathcal{A}$ we have

$$\Re \frac{zg'(z)}{g(z)} \leq \frac{22 + \sqrt{2}}{20} = 1.17\ldots, \quad g(z) \neq 0, \quad \text{for } z \in U \setminus \{0\}.$$

Then we get

$$\Re \frac{zf'(z)\sqrt{f(z)}}{g(z)\sqrt{g(z)}} > \frac{1}{2} \quad \text{for } z \in U.$$
If \( g \in \mathcal{S}^*(q_c) \), \( c \in (0, 1] \), where the class

\[
\mathcal{S}^*(q_c) = \left\{ g \in \mathcal{A} : \frac{zg'(z)}{g(z)} < q_c(z), g(z) \neq 0, z \in U \setminus \{0\} \right\},
\]

\( q_c(z) = \sqrt{1 + cz} \), was introduced in [1], then \( \Re\{zg'(z)/g(z)\} < \sqrt{1 + c} \). Therefore, if

\[
c < \frac{43 + 22\sqrt{2}}{200} = 0.37 \ldots,
\]

then Corollary 3.10 becomes

\[
\left[ f \in \mathcal{H}_{3/4} \text{ and } g \in \mathcal{S}^*(q_c) \right] \implies \left[ \Re\left( \frac{zf'(z)}{\sqrt{f(z)}} \frac{\sqrt{f(z)}}{g(z)} \right) > \frac{1}{2} \right].
\]

ACKNOWLEDGEMENT. The authors wish to sincerely thank the referees for their suggestions for improvement to an earlier draft of this paper.

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