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A STEIN CRITERION VIA DIVISORS FOR DOMAINS OVER STEIN MANIFOLDS

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Abstract

It is shown that a domain X over a Stein manifold is Stein if the following two conditions are fulfilled: a) the cohomology group $H^i(X, \mathcal{O})$ vanishes for $i \geq 2$ and b) every topologically trivial holomorphic line bundle over X admits a non-trivial meromorphic section.

As a consequence we recover, with a different proof, a known result due to Siu stating that a domain X over a Stein manifold Y is Stein provided that $H^i(X, \mathcal{O}) = 0$ for $i \geq 1$.

1. Introduction

Let Y be a complex manifold of pure dimension n. By a *branched domain* (resp. *domain*) over Y we mean a couple (X, π) (or simply X) consisting of a connected complex manifold X of dimension n and a holomorphic map $\pi : X \longrightarrow Y$ which has discrete fibers (resp. π is locally biholomorphic). (Note that in this setting π is an open map.)

If π is injective, we say that X is a *schlicht domain over* Y; in that case we view X as an open subset of Y.

The (branched) domains over $\mathbf{C}^n$ are also called (*branched*) *Riemann domains*. Note that Riemann domains over $\mathbf{C}^n$ appear naturally as domains of existence of families of holomorphic functions defined on open subsets of $\mathbf{C}^n$.

In this paper we prove the following result:

Theorem 1. *Let Y be a Stein manifold of dimension n. Then a domain X over Y is Stein provided the following two conditions are fulfilled:*

a) *The cohomology groups $H^2(X, \mathcal{O}), \dots, H^{n-1}(X, \mathcal{O})$ vanish.*

b) *Every holomorphic line bundle over X that is topologically trivial admits a non trivial meromorphic section.*

Remark 1. Note that for a holomorphic line bundle L over a connected complex manifold M the following statements are equivalent:

- L is associated to a Cartier divisor.
- L admits a non-trivial meromorphic section.

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(This follows easily because every stalk $\mathcal{O}_{M,\zeta}$ is a factorial ring!)

Furthermore L is the line bundle of an effective divisor if and only if L has a nontrivial global holomorphic section.

Now a few comments on the statement of the theorem are in order here. We use the hypothesis b) as follows. We cover X with two open sets X_1 and X_2 and let $\xi_{12} \in \mathcal{O}(X_1 \cap X_2)$. With $\exp(\xi_{12})$ as transition function one gets a holomorphic line bundle L over X which is topologically trivial. The hypothesis reads: for every $\xi_{12} \in \mathcal{O}(X_1 \cap X_2)$ there are meromorphic functions m_1 on X_1 and m_2 on X_2 such that

$$\exp(\xi_{12}) = m_1/m_2.$$

This furnishes a way to produce examples where condition b) fails. See Lemma 2 in §2.

On the other hand condition a) holds in each of the subsequent settings:

i) either X admits a Stein morphism into a Stein space S, meaning that there is a holomorphic map $f : X \longrightarrow S$ together with an open covering of S by open sets V such that $f^{-1}(V)$ are Stein (see [16]);

ii) or X is the union of two Stein open sets, *a posteriori* if X is 2-complete.

Note. A complex manifold Z is called *q-complete* (the normalization is such that "1-complete $\equiv$ Stein") if there is a smooth function $\varphi \in C^\infty(Z, \mathbb{R})$ which is exhaustive and its Levi form $L(\varphi, \cdot)$ has at any point of X at most $q-1$ non-positive eigenvalues. It is known from [3] that a q-complete manifold has trivial cohomology for coefficients in coherent analytic sheaves in dimension from q on.

On the other hand, in the surface case the condition a) is superfluous, so one has:

PROPOSITION 1. *A domain (X, π) over a smooth Stein surface is Stein if every topologically trivial holomorphic line bundle on X is associated to some Cartier divisor.*

Here we mention that an important point in the proof of this proposition is the generalization of the notion of boundary map from [8]; for more details see §4 and Proposition 3.

REMARK 2. Although this proposition might be seen as a particular case of Theorem 1, it is, in fact, the starting induction step in the proof of our theorem.

Also, from this proposition we recover a result due to Abe [1] when X is a *schlicht* domain.

Theorem 1 has several consequences. First it gives another proof of a theorem due to Siu ([21], Theorem B):

COROLLARY 1. *Let X be a domain over a Stein manifold of dimension n such that $H^k(X, \mathcal{O}) = 0$ for $k = 1, 2, \ldots, n-1$. Then X is Stein.*

PROOF. This is because under the hypothesis of the corollary every topologically trivial holomorphic line bundles on X is holomorphically trivial and since each meromorphic function on Y lifts via π to a meromorphic function on X, whence condition b) is trivially fulfilled.

Second, it extends Ballico's result (see [5], Theorem 1, p. 23) as well as Abe's main result in [1]. But, before quoting them, let us recall that a real-valued smooth function φ of class C^∞ on a complex manifold X is called *weakly q-convex* if the Levi form of φ, $L(\varphi, \cdot)$, has at any point of X at most $q-1$ strictly negative eigenvalues. An open set Ω of X is said to be *weakly q-pseudoconvex* if locally its boundary is defined by a weakly q-convex function. It is known from [3] that a weakly q-pseudoconvex domain in a Stein manifold is q-complete; *a fortiori* it has trivial cohomology for coefficients in coherent analytic sheaves in dimension from q on.

THEOREM. *Let Y be a Stein manifold and $X \subset Y$ a weakly 2-pseudoconvex open subset. Then the following conditions are equivalent:*

a) *X is Stein;*

b) *Every holomorphic line bundle on X is associated to an effective Cartier divisor on X.*

This in turn has been extended in [2] to:

THEOREM. *Let X be an open set in a Stein manifold Y of dimension n such that $H^k(X, \mathcal{O}) = 0$ for $2 \le k < n$. Then X is Stein provided that every holomorphic line bundle on X is associated to an effective Cartier divisor on X.*

Finally, we mention that, besides Proposition 1, another key point in the proof of Theorem 1 is a Lelong type characterization theorem for domain over Stein manifolds (see also Theorem 3 and Proposition 2 in §3), namely:

THEOREM 2. *Let Y be a connected Stein manifold of dimension $n \ge 3$. Let (X, π) be a domain over Y. Then X is Stein if $\pi^{-1}(Z_f)$ is Stein, for any holomorphic function f on Y such that its zero set Z_f is smooth and f has multiplicity one on every connected component of Z_f.*

2. Preliminaries

Let X be a reduced complex space. Let $\mathscr{D}_X$ denote the sheaf of Cartier divisors on X, that is $\mathscr{D}_X = \mathscr{M}_X^\star/\mathscr{O}_X^\star$, so that one has a natural short exact sequence on X

$$(\star) \qquad 1 \longrightarrow \mathscr{O}_X^\star \longrightarrow \mathscr{M}_X^\star \longrightarrow \mathscr{D}_X \longrightarrow 0.$$

We denote the group $H^0(X, \mathscr{D}_X)$ by $\mathrm{Div}(X)$. The elements of $\mathrm{Div}(X)$ are called Cartier divisors on X.

A Cartier divisor D is called effective if it is in the image of the canonical map $H^0(X, \mathscr{O}_X \cap \mathscr{M}_X^\star) \longrightarrow H^0(X, \mathscr{D}_X)$.

To every divisor D on X one associates in a canonical way an invertible sheaf $\mathscr{O}_X(D)$, which is a subsheaf of $\mathscr{M}_X$ and determines canonically an equivalence class of holomorphic line bundles; in other words an element of the Picard group of X, denoted by $\mathrm{Pic}(X)$. It is known that $\mathrm{Pic}(X)$ is isomorphic to $H^1(X, \mathscr{O}_X^\star)$. As a matter of fact, the short exact sequence in $(\star)$ induces a canonical map

$$\delta_X : \mathrm{Div}(X) \longrightarrow \mathrm{Pic}(X),$$

sending Cartier divisors into their canonically associated class of holomorphic line bundles and δ_X is a homomorphism of groups.

The kernel of δ_X is clearly understood as the set of principal divisors, *i.e.* those divisors defined by globally meromorphic functions on X that are invertible, that is $\mathscr{M}(X)^\star$.

On the other hand there are a couple of natural hypotheses to guarantee that every holomorphic line bundle arises from a divisor, meaning that δ_X is surjective, namely if:

- either X is a projective algebraic manifold ([14], p. 161) or
- X is a Stein space ([12], p. 149).

Therefore it is an interesting question to study the geometry of X under the assumption that δ_X is surjective.

This is done when X is an open set of a Stein manifold Y of dimension two as shown by Abe [1], namely the surjectivity of δ_X implies that X is Stein (see §1). (We do not know whether or not if the surjectivity of δ_X implies that X is Stein if we allow singularities for Y. However, one can prove that X is locally Stein at boundary points of X which are non-singular points for Y. See the subsequent Theorem 4.)

This result does not extends in this form to higher dimensions; for instance in the case of the non-Stein open set $X = \mathbb{C}^3 \setminus \{0\}$ in $\mathbb{C}^3$ the map δ_X is trivially surjective.

Let $\mathrm{Pic}^0(X)$ denote the subgroup of $\mathrm{Pic}(X)$ consisting of topologically trivial holomorphic line bundles. Granting the exponential sequence,

$$0 \longrightarrow \mathsf{Z} \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_X^\star \longrightarrow 1,$$

the subgroup $\mathrm{Pic}^0(X)$ of $\mathrm{Pic}(X)$ is the image of the canonical map from $H^1(X, \mathscr{O}_X)$ into $\mathrm{Pic}(X)$. Therefore $\mathrm{Pic}^0(X)$ is trivial if X satisfies the so called Cousin condition $H^1(X, \mathscr{O}_X) = 0$, which guarantees the universal solvability of the additive Cousin problem.

LEMMA 1. *Let Ω be a smooth Stein surface such that $H^2(\Omega, \mathsf{Z}) = 0$. Let $A \subset \Omega$ be a discrete subset. Then a holomorphic line bundle L over $\Omega \setminus A$ admits a non-trivial meromorphic section if, and only if, L is analytically trivial. In particular, $\Omega \setminus A$ is a Thullen type domain* (*i.e. the multiplicative Cousin problem is universally solvable*).

PROOF. We prove the "only if" assertion because the reverse implication is trivial. So let σ be a meromorphic section of L. If σ has no pole or zero, then σ is a nowhere vanishing holomorphic section of L so that L is holomorphically trivial. Now, as $\mathrm{div}(\sigma)$ is non empty, its support has pure dimension one; thus its closure defines a divisor on Ω where every multiplicative Cousin problem is universally solvable (thanks to the hypothesis). Therefore there is a meromorphic function m on Ω whose canonically associated divisor restricted to $\Omega \setminus A$ is $\mathrm{div}(\sigma)$. It follows that σ/m is a nowhere vanishing holomorphic section of L, whence the conclusion.

To show the additional statement, let $\{(U_i, m_i)\}_i$ be multiplicative Cousin data, that is $\{U_i\}_i$ is an open covering of $\Omega \setminus A$ and $m_i \in \mathscr{M}^\star(U_i)$ such that $m_i/m_j \in \mathscr{O}^\star(U_i \cap U_j)$. One gets a holomorphic line bundle over $\Omega \setminus A$ that has a non-trivial meromorphic section so that this holomorphic line bundle is trivial, which means that there is $m \in \mathscr{M}^\star(\Omega \setminus A)$ such that $m/m_i \in \mathscr{O}^\star(U_i)$, concluding the proof.

Here we show a simple

LEMMA 2. *Let D be a Stein open set in C^2. Let $(a, b) \in D$ and set $X := D \setminus \{(a, b)\}$. Then the* (*topologically trivial*) *holomorphic line bundle on X defined by* $\exp(1/(z_1 - a)(z_2 - b))$ *is not associated to a Cartier divisor on X.*

PROOF. Assume, in order to reach a contradiction, that the corresponding line bundle L comes from a Cartier divisor. So L is analytically trivial from Lemma 1. There is no loss in generality to assume that $a = b = 0$ and after restriction and scaling to deal with the case $D = \Delta \times \Delta$. Hence there are $h_1 \in \mathscr{M}^\star(\Delta^\star \times \Delta)$ and $h_2 \in \mathscr{M}^\star(\Delta \times \Delta^\star)$ such that $h_1/h_2 = \exp(1/z_1z_2)$.

Using the exponential sequence we deduce that there are integers m and n and holomorphic functions $f \in \mathscr{O}^\star(\Delta^\star \times \Delta)$ and $g \in \mathscr{O}^\star(\Delta \times \Delta^\star)$ such that

$$z_1^m \exp(f) = z_2^n \exp(g) \exp 1/z_1 z_2$$

so that we obtain $z_1^m z_2^{-n} = \exp(-f + g + 1/z_1 z_2)$. If either m or n is not zero, restricting this equation to $\{z_2 = 1/2\}$ or $\{z_1 = 1/2\}$ accordingly, we get a continuous branch of the logarithm on $\Delta^\star$, which is not possible. Thus $m = n = 0$ which implies that $1/z_1 z_2 = f - g + c$ on $\Delta^\star \times \Delta^\star$ for some $c \in \mathbb{C}$ which, again is not possible. The lemma follows.

In particular, the holomorphic line bundle on $\mathbb{C}^2 \setminus \{0\}$ defined by $\exp(1/zw)$ does not arises from a Cartier divisor.

3. Proof of Theorem 2

Below we first extend a well-known result due to Lelong [19] stating that *an open set D in $\mathbb{C}^n$, $n \geq 3$, is Stein iff for every affine hyperplane H of $\mathbb{C}^n$ its trace $H \cap D$ is Stein*, to the following:

THEOREM 3. *Let (D, π) be a Riemann domain over $\mathbb{C}^n$ with $n \geq 3$. Assume that for every point $z \in D$ there is a dense subset $\mathscr{H}_z \subset \mathrm{Gr}(n-1, n)$ such that for any hyperplane $\Sigma \in \mathscr{H}_z$, $\pi^{-1}(\Sigma)$ is Stein. Then D is Stein.*

PROOF (SKETCH). Here $\mathrm{Gr}(n-1, n)$ is the Grassmann complex manifold of all complex hyperplanes of $\mathbb{C}^n$ passing through the origin.

Denote by S the unit sphere in $\mathbb{C}^n$, *i.e.* $S = \{w \in \mathbb{C}^n : \|w\| = 1\}$. For each $w \in S$ define the *Hartogs radius of* (D, π) *in direction* w as a function

$$R_w : D \longrightarrow (0, \infty],$$

where for $\xi \in D$ we set $R_w(\xi) :=$ the supremum of all $r > 0$ such that there is a neighborhood U of ξ in $\pi^{-1}(L_w)$ which is mapped biholomorphically via π onto a disc in L_w centered at $\pi(\xi)$ and of radius r, where L_w is the complex line $L_w = \{\pi(\xi) + tw : t \in \mathbb{C}\}$.

Then R_w is lower semi-continuous and if δ denotes the boundary distance function for the domain (D, π) over $\mathbb{C}^n$, then

$$\delta = \inf_{w \in S} R_w.$$

In general note that $-\log R_w$ is subharmonic on $\pi^{-1}(\zeta + \mathbb{C}w)$ for all $\zeta \in \mathbb{C}^n$. Moreover, if D is Stein, then each $-\log R_w$ is plurisubharmonic.

Then our proof reduces, via standard arguments to the following (see [23], Prop. 4, p. 511):

LEMMA 3. *Let Ω be an open set in $\mathbb{C}^n$ and φ be an upper semi-continuous function on Ω. Then φ is plurisubharmonic if, for every point $a \in \Omega$ there is a dense subset $T_a \subset S$ such that the restriction of φ to $(\{a\} + \mathbb{C} \cdot w) \cap \Omega$ is subharmonic for all $w \in T_a$.*

In the same vein we have:

PROPOSITION 2. *Let (D, π) be a Riemann domain over $\mathbb{C}^n$. If, for every point $z \in D$ there is a dense subset $\Gamma_z \subset \mathrm{Gr}(2, n)$ such that for any $\Sigma \in \Gamma_z$, $\pi^{-1}(\Sigma)$ is Stein, then D is Stein.*

REMARK 3. A similar statement to Theorem 3 for branched Riemann domains does not hold. More precisely, there is a non Stein complex manifold D of dimension three and a holomorphic map $\pi : D \longrightarrow \mathbb{C}^3$ making D a branched Riemann domain over $\mathbb{C}^3$ and, however, for every hypersurface Σ of $\mathbb{C}^3$, $\pi^{-1}(\Sigma)$ is Stein.

This can be done using the counter-example to the hypersection problem (see [7], Theorem 0.1, p. 176) and a theorem of Grauert [13] asserting that for a reduced complex space X of dimension k which is holomorphically spreadable at any point (by this we mean that for any $x_0 \in X$ there is a holomorphic mapping $F : X \longrightarrow \mathbb{C}^N$, with N that might depend on x_0, such that x_0 is isolated in its fibre $F^{-1}(F(x_0))$), there is a holomorphic map $\tau : X \longrightarrow \mathbb{C}^k$ with discrete fibers.

More precisely, from ([7], see also [6]) there is a normal Stein space X of dimension three and an analytic subset $A \subset X$ of dimension two, containing the singular set of X, such that the for any hypersurface Σ of X (analytic subset of X of pure dimension two), $(X \setminus A) \cap \Sigma$ is Stein. Now, as X is Stein, *a fortiori* holomorphically spreadable at any point, there is a holomorphic map $\pi : X \longrightarrow \mathbb{C}^3$ with discrete fibers so that $D := X \setminus A$ is as desired.

Now, in order to complete the proof of Theorem 2 we proceed as follows. For a complex manifold Y we introduce a subset $\mathcal{O}^\natural(Y)$ of $\mathcal{O}(Y)$ which consists of all holomorphic functions f on Y such that the following two properties hold:

- its zero set $Z_f = \{y \in Y;\, f(y) = 0\}$ is non-singular and
- the multiplicity of f along each connected component of Z_f is one.

Regarding this class of functions we notice a straightforward functorial property, namely if W is a domain over Y, then $\pi^\star(\mathcal{O}^\natural(Y)) \subset \mathcal{O}^\natural(W)$.

A few remarks are in order here. First, for Y a Stein manifold, by using Bertini type arguments, one shows that there are "enough functions" in $\mathcal{O}^\natural(Y)$. As a matter of fact, let $Y \hookrightarrow \mathbb{C}^N$ be a holomorphic embedding. It is known by

Siu's theorem that Y has a Stein open neighborhood W in $\mathbb{C}^N$ such that there is a holomorphic retract $\rho : W \longrightarrow Y$.

For every $\lambda \in \mathbb{C}^{N-1}$, we let f_λ be the restriction to Y of the linear function $\lambda_1 z_1 + \cdots + \lambda_{N-1} z_{N-1}$, where $z_1, \ldots, z_{N-1}, z_N$ are the coordinate functions of $\mathbb{C}^N$. It is shown (see [23], p. 523) that the set

$$\{\lambda \in \mathbb{C}^{N-1}; \exists\, y \in Y, \text{such that } f_\lambda(y) = 0 \text{ and } df_\lambda(y) = 0\}$$

has zero Lebesgue measure in $\mathbb{C}^{N-1}$. Thus the restrictions of such f's to Y will be in $\mathscr{O}^\natural(Y)$.

As an immediate consequence of this fact we deduce the following: For $\alpha \in \mathbb{C}^N$, $\alpha = (\alpha_1, \ldots, \alpha_N)$, we let g_α be the restriction to Y of the linear function $\alpha_1 z_1 + \cdots + \alpha_N z_N$, where $z_1, \ldots, z_{N-1}, z_N$ are the coordinate functions of $\mathbb{C}^N$. Then the set

$$T := \{\alpha \in \mathbb{C}^N; \exists\, y \in Y, \text{such that } g_\alpha(y) = 0 \text{ and } dg_\alpha(y) = 0\}$$

has zero Lebesgue measure in $\mathbb{C}^N$. (For this one consider the above setting in $\mathbb{C}^{N+1}$ with $X \times \{0\}$ and $B \times \mathbb{C}$ instead of X and B respectively.)

Notice that for $\alpha \in \mathbb{C}^N \setminus T$, Z_{g_α} is a complex submanifold of Y and the multiplicity of g_α along each connected component of Z_{g_α} is one.

Now consider the following cartesian square of canonically induced holomorphic mappings

$$\begin{array}{ccc} \Omega & \xrightarrow{\tau} & X \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\pi} \\ W & \xrightarrow{\rho} & Y \end{array}$$

where $\Omega = \{(w, x) \in W \times X; \rho(w) = \pi(x)\}$ so that (Ω, σ) becomes a domain over $W \subset \mathbb{C}^N$. Note also that Ω is a closed complex submanifold of the product $W \times X$.

Let $\iota : Y \hookrightarrow B$ be the canonical inclusion. As ρ is a holomorphic retract, $\rho \circ \iota = \mathrm{id}_Y$. Therefore the mapping from X into Ω

$$X \ni x \mapsto ((\iota \circ \pi)(x), x) \in \Omega$$

is a holomorhic embedding. Hence to show that X is Stein reduces to prove that Ω is Stein.

For $\alpha \in \mathbb{C}^N \setminus T$, which is a dense subset of $\mathbb{C}^N$, we let H_α be the hyperplane in $\mathbb{C}^N$ given by the vanishing of $\alpha_1 z_1 + \cdots + \alpha_N z_N$. Clearly these $\{H_\alpha\}_\alpha$ induces a dense subset $\mathscr{H}$ of $\mathrm{Gr}(N-1, N)$ and, for every $H \in \mathscr{H}$, $\sigma^{-1}(H \cap W)$ is a closed analytic subset of $W \times \pi^{-1}(H \cap Y)$. But $\pi^{-1}(H \cap Y)$ is Stein thanks to the hypothesis. But W is Stein, hence $W \times \pi^{-1}(H \cap Y)$ is Stein, so that $\sigma^{-1}(H \cap W)$ is Stein being a closed complex submanifold of a Stein manifold.

Therefore from Theorem 3 we get readily the proof of Theorem 2.

4. A remark on pseudoconvex domains

The point we want to address in this section concerns a weakening of the notion of pseudoconvex domain over a complex euclidean space. In order to do this, let us recall some facts about boundary points of domains over complex spaces (see [10], p. 101). Let (X, π) be a domain over a complex manifold Y. Consider sequences $\{x_n\}_n$ of points in X with the following properties:

(1) $\{x_n\}$ has no cluster point in X;

(2) The sequence of images $\{\pi(x_n)\}$ has a limit point $a \in Y$;

(3) For every connected open neighborhood V of a in Y there is $n_0 \in \mathsf{N}$ such that for $n, m \geq n_0$ the points x_n and x_m can be joined by a continuous path $\gamma : [0, 1] \longrightarrow X$ with $\pi \circ \gamma([0, 1]) \subset V$.

Two such sequences $\{x_n\}, \{y_n\}$ are called equivalent if the sequence $\{z_n\}$ defined by $z_{2n+1} = x_n$ and $z_{2n} = y_n$ satisfies the above three properties, or equivalently that:

(1) $\lim \pi(x_n) = \lim \pi(y_n) = a$.

(2) For every connected open neighborhood V of a in Y there is $n_0 \in \mathsf{N}$ such that for $n, m \geq n_0$ the points x_n and y_m can be joined by a continuous path $\gamma : [0, 1] \longrightarrow X$ with $\pi \circ \gamma([0, 1]) \subset V$.

An *accessible boundary point* is an equivalence class of such sequences. Let bX be the set of accessible boundary points of X. (Even if X is schlicht, this set may be different from the topological boundary ∂X. There may be points in ∂X that are not accessible, and it may happen that an accessible boundary point is the limit of two inequivalent sequences.)

We define $\widehat{X} = X \cup bX$. If ξ is an accessible boundary point defined by a sequence $\{x_n\}$, we define a neighborhood of ξ in $\widehat{X}$ as follows: Take a connected open set U in X such that almost all x_n lie in U. Then add all accessible boundary points defined by sequences $\{y_n\}$ such that almost all y_n lie in U and $\lim \pi(y_n)$ is a cluster point of $\pi(U)$. For an ordinary point $x \in X$ its neighborhood system in $\widehat{X}$ is the same as in X.

With this neighborhood definition $\widehat{X}$ becomes a separated space and π extends to a continuous map $\widehat{\pi} : \widehat{X} \longrightarrow Y$, $\widehat{\pi}(\xi) = a = \lim \pi(x_\nu)$. Observe that $\pi(bX)$ is contained in $\partial\pi(X)$ (topological boundary of $\pi(X)$ with respect to Y) and for every point $\xi \in bX$ there is a continuous path $\alpha : [0, 1] \longrightarrow \widehat{X}$ such that $\alpha(1) = \xi$ and $\alpha(s) \in X$ for $s \in [0, 1)$.

The following lemma is Satz 4 in [8].

LEMMA 4. *Let T be a locally connected topological space and $N \subset T$ be a nowhere dense subset of T nowhere disconnecting T. Let (X, π) be a domain over a complex manifold M, $\tau : T \setminus N \longrightarrow X$ a continuous map such that $\pi \circ \tau$ extends to a continuous mapping from T to M. Then τ extends uniquely to a continuous mapping $\widehat{\tau} : T \longrightarrow \widehat{X}$.*

Now let (X, π) be a Riemann domain over $\mathbb{C}^n$. (Notice that if (X, π) is a domain over an open set $\Omega \subset \mathbb{C}^n$ with canonical injection $\iota : \Omega \longrightarrow \mathbb{C}^n$, then $(X, \iota \circ \pi)$ is a Riemann domain over $\mathbb{C}^n$.)

Let $G := \{(t_1, \dots, t_n) \in \mathbb{C}^n; |t_1| \leq 1, |t_2| < 1, \dots, |t_n| < 1\}$ be the *semi-closed* unit polydisc, and $bG := \{t \in G; |t_1| = 1\}$.

A *boundary map* for (X, π) is a continuous map $\Phi : \overline{G} \longrightarrow \widehat{X}$ which fulfils the following three conditions:

(1) $\Phi(bG)$ is relatively compact in X and $\Phi(\operatorname{int} G) \subset X$.

(2) $\Phi(\overline{G}) \cap bX \neq \emptyset$.

(3) The map $\widehat{\pi} \circ \Phi$ extends to a biholomorphic map from $\mathbb{C}^n$ onto itself,

$$\begin{tikzcd} \overline{G} \arrow[r, "\Phi"] \arrow[d] & \widehat{X} \arrow[d, "\widehat{\pi}"] \\ \mathbb{C}^n \arrow[r, "F"] & \mathbb{C}^n \end{tikzcd}$$

REMARK 4. Notice that in [8] it is required that $\widehat{\pi} \circ \Phi$ extends to a biholomorphic map only from an open neighborhood of $\overline{G}$ in $\mathbb{C}^n$ onto an open subset in $\mathbb{C}^n$.

The next proposition is a slightly generalization of ([8], Satz 7, p. 111).

PROPOSITION 3. *Let (X, π) be a Riemann domain over $\mathbb{C}^n$ and δ the boundary distance function. If (X, π) admits no boundary map, then $-\log \delta$ is plurisubharmonic.*

PROOF. For the commodity of the reader we supply some arguments of the proof. First, without any loss in generality we may assume that X is not biholomorphic to $\mathbb{C}^n$ via π so that δ is finite.

Now we proceed by contradiction, so assume that $-\log \delta$ is not plurisubharmonic. Thus there is a complex line E in $\mathbb{C}^n$ such that $-\log \delta|_{\pi^{-1}(E)}$ is not subharmonic. After an affine transformation we may consider $E = \mathbb{C} \times \{(0, \dots, 0)\} \subset \mathbb{C}^n$. By standard arguments we arrive at the following situation: there is a holomorphic map $f : V \longrightarrow X$, where V is a non-empty open subset of C, such that $\pi \circ f(z) = (\alpha z + \beta, 0, \ldots, 0)$ ($\alpha \in \mathsf{C}^\star$, $\beta \in \mathsf{C}$), a disk $K \subset V$ and a holomorphic polynomial g in one complex variable such that:

- $-\log \delta(f(z)) < \operatorname{Re} g(z)$ on ∂K and
- $-\log \delta(f(z_0)) > \operatorname{Re} g(z_0)$ for some $z_0 \in K$.

This gives that

- $\delta(f(z)) > |e^{-g(z)}|$ for every $z \in \partial K$
- there is $z_0 \in K$ such that $\delta(f(z_0)) < |e^{-g(z_0)}|$

Now define the open subset U of C^n by:

$$U := \{w \in \mathsf{C}^n; \exists\, z \in K, \|w - \pi(f(z))\| < \delta(f(z)\}.$$

It is easily seen that U is a connected neighborhood of $L := \pi(f(K))$ in C^n and that there is a holomorhic section σ of π, that is $\sigma : U \longrightarrow X$ is holomorphic such that $\pi \circ \sigma = \mathrm{id}_U$ and $\sigma(\pi(f(z))) = f(z)$ for all $z \in K$.

As a matter of fact, if we denote for $x \in X$ by $\Omega(x)$ the open subset of X containing x and biholomorphic via π to the euclidean ball $B(\pi(x), \delta(x))$, then

$$\sigma(U) = \bigcup_{z \in K} \Omega(f(z)).$$

(Note that no set $\Omega(x)$ is relatively compact in X!) Then for each $c > 0$ consider the holomorphic mapping

$$\Psi_c : \mathsf{C} \times \mathsf{C}^{n-1} \longrightarrow \mathsf{C}^n$$

defined by

$$\Psi_c(z, t) = (\alpha z + \beta, t_1 e^{-g(z)}, ct_2, \ldots, ct_{n-1}),\ (z \in \mathsf{C}, t \in \mathsf{C}^{n-1}).$$

Clearly Ψ_c is a biholomorphic map of C^n onto itself, $\Psi_c(K \times \{0\} = L \subset U$ and $\|\Psi_c(z_0, 1, 0, \ldots, 0) - \pi(f(z_0))\| > \delta(f(z_0))$.

Now select $c > 0$ small enough such that $\Psi_c(\partial K \times \overline{\Delta^{n-1}}) \subset U$ and let $r \in (0, 1]$ be the supremum of all $s \in (0, 1]$ such that $\Psi_c(K \times \Delta^{n-1}(s)) \subset U$. (Here $\Delta^{n-1}(s)$ is the polydisc of center 0 in C^{n-1} and polyradius $(s, \ldots, s)$.)

Setting $\Phi := \sigma \circ (\Psi_c|_{K \times \Delta^{n-1}(r)})$ one has that $\Phi(\partial K \times \Delta^{n-1}(r))$ is relatively compact in X and that $\Phi(K \times \Delta^{n-1}(r))$ lies in X but it is not relatively compact in X.

Then, thanks to Lemma 4 we can extend Φ to a boundary map from $K \times \overline{\Delta^{n-1}(r)}$ to $\widehat{X}$ contradicting the hypothesis, whence the proof.

We give:

DEFINITION 1. A domain (X, π) over a connected complex manifold Y of pure dimension n is said to be *weakly pseudoconvex* if, for any point $y_0 \in Y$ there is a local chart V around y_0 in Y, where V is viewed as an open set in C^n, such that the canonically induced Riemann domain $(\pi^{-1}(V), \pi|_{\pi^{-1}(V)})$ over C^n admits no boundary map.

LEMMA 5. *A domain (X, π) over a connected Stein manifold Y is Stein if, and only if, (X, π) is* weakly pseudoconvex *over Y.*

PROOF. Let $x_0 \in X$. There is an open neighborhood V of $\pi(x_0)$ in C^n such that $U := \pi^{-1}(V)$ admits no boundary map; hence $-\log \delta_U$ is plurisubharmonic. (Here δ_U is the boundary distance function of the Riemann domain $\pi : U \longrightarrow \mathsf{C}^n$.) But $\delta = \delta_U$ on a neighborhood of x_0. Therefore $-\log \delta$ is plurisubharmonic near x_0. As x_0 is arbitrarily chosen in X, we deduce that $-\log \delta$ is plurisubharmonic on X. Then, thanks to Oka's classical theorem (see [20]), X follows Stein, as desired.

5. Proof of Proposition 1

We follow ideas from [18]. Assume, in order to reach a contradiction, that X is not Stein. Thus there is a point $y_0 \in Y$ such that, for every Stein open neighborhood V of y_0, $\pi^{-1}(V)$ fails to be Stein. Since Y is Stein and smooth, there is a holomorphic map $f = (f_1, f_2) : Y \longrightarrow \mathsf{C}^2$ such that $f(y_0) = 0$, $f^{-1}(0) = \{y_0\}$ and f maps biholomorphically an open neighborhood W of y_0 in Y onto $\Delta(2) \times \Delta(2)$.

Granting Lemma 5 and Proposition 3, after scaling and standard arguments, there is $\epsilon \in (0, 1)$ such that setting $W(\epsilon) := \{y \in W; f(y) \in \Delta(1+\epsilon) \times \Delta(1+\epsilon)\}$, one has: there is $F \in \mathrm{Aut}(\mathsf{C}^2)$ and a holomorphic mapping $\sigma : H(\epsilon) \longrightarrow X$ such that F extends $f \circ (\pi \circ \sigma)$, and there is a point (α, β), $|\alpha| \le 1-\epsilon$, $\beta = 1$, such that $(\alpha, \beta) \notin f(\pi(X) \cap W(\epsilon))$, where

$$H(\epsilon) = \{(z_1, z_2) \in \mathsf{C}^2; 1-\epsilon < |z_1| < 1+\epsilon, |z_2| < 1+\epsilon\} \cup (\Delta \times \Delta).$$

Clearly, $\widehat{H}(\epsilon)$, the envelope of holomorphy of $H(\epsilon)$ is $\Delta(1+\epsilon) \times \Delta(1+\epsilon)$. Set $Y^\sharp := \{y \in Y; |f_1(y)| < 1+\epsilon\}$, $Y_1 := \{y \in Y^\sharp; |f_2(y)| < 1+\epsilon\}$ and $Y_2 := \{y \in Y^\sharp; |f_2(y)| > 1+\epsilon/2\}$.

Then, as $\{Y_1, Y_2\}$ is an open covering of the Stein manifold $Y^\sharp$, it follows that $H^1(\{Y_1, Y_2\}, \mathscr{O}) = 0$. Hence there are $h_1 \in \mathscr{O}(Y_1)$ and $h_2 \in \mathscr{O}(Y_2)$ such that

$$h_2 - h_1 = 1/(f_2 - \beta).$$

We define the meromorphic function m on $Y^\sharp$ by setting

$$m = \begin{cases} h_1 + 1/(f_2 - \beta) & \text{on } Y_1, \\ h_2 & \text{on } Y_2. \end{cases}$$

Consider the sets X_1, X_2 in X defined by $X_1 = \pi^{-1}(\{f_1 \neq \alpha\})$ and $X_2 = \pi^{-1}(\{f_2 \neq \beta\} \cap Y^\sharp)$. Clearly $\{X_1, X_2\}$ form an open covering of X.

Let $\widehat{m}$ be the lifting of m to $\pi^{-1}(Y^\sharp)$; $\widehat{f_i} = f_i \circ \pi$, $i = 1, 2$. One verifies that $\widehat{m}/(\widehat{f_1} - \alpha)$ is holomorphic on $X_1 \cap X_2$. Thus, granting the hypothesis, there are $g_1' \in \mathcal{M}^\star(X_1)$ and $g_2 \in \mathcal{M}^\star(X_2)$ such that

$$\exp \frac{\widehat{m}}{\widehat{f_1} - \alpha} = \frac{g_1'}{g_2}.$$

Therefore $g_1 := g_1' \exp(-\widehat{h}_1/(\widehat{f_1} - \alpha))$ is meromorphic on $\pi^{-1}(Y_1) \cap X_1$.

Let $H_1 := H(\epsilon)\backslash\{z_1 = \alpha\}$ and $H_2 := H(\epsilon)\backslash\{z_2 = \beta\}$. Let v_1 and v_2 be the meromorphic functions on H_1 and H_2 induced by g_1 and g_2 respectively (via σ). By Proposition 3 from [18], v_1 and v_2 extend to meromorphic invertible functions $\widetilde{v}_1$ and $\widetilde{v}_2$ on $\widehat{H}(\epsilon) \setminus \{z_1 = \alpha\}$ and $\widehat{H}(\epsilon) \setminus \{z_2 = \beta\}$. It then follows $\widetilde{v}_1/\widetilde{v}_2 = \exp(1/((z_1 - \alpha)(z_2 - \beta))$ which contradicts Lemma 2, whence the proof of Proposition 1.

REMARK 5. If we allow singularities for Y one obtains that (X, π) is locally Stein over any regular point of Y, that is every point $y \in \mathrm{Reg}(Y)$ admits an open neighborhood V such that $\pi^{-1}(V)$ is Stein.

Furthermore, the method from above cannot be used to solve the analogous question for open sets in normal Stein surfaces. Take Y to be the normal cone $Y = \{(x, y, z) \in \mathbb{C}^3; x^3 + y^3 + z^3 = 0\}$ and $X := Y \setminus \{(0, 0, 0)\}$. Then for any Hartogs pair $(\widehat{\Omega}, \Omega)$ in $\mathbb{C}^2$ and any holomorphic map $f : \widehat{\Omega} \longrightarrow Y$ with $f(\Omega) \subset X$, it follows that $f(\widehat{\Omega}) \subset X$. This results easily from [9] because X verifies the disc theorem.

6. Proof of Theorem 1

We want to apply Theorem 2 so we claim that for each $f \in \mathcal{O}^\sharp(Y)$ the following holds (for the notation, see §3):

Let X' be a connected component of the smooth complex hypersurface of X given as the zero set of $f \circ \pi$. Then one has:

a′) $H^2(X', \mathcal{O}) = 0$ for $i \geq 2$.

b′) $\mathrm{Pic}^0(X') \subset \mathrm{Im}(\delta_{X'})$.

(Of course condition a′) is void if $n = 2$.) Then the proof is completed by using Theorem 2 and induction over $n = \dim(X)$; so that it is here where we need $n \geq 3$ so we have to start our induction with the case $n = 2$ which is settled by Proposition 1.

Now, to proceed with the proof of the claim, let $g = f \circ \pi$. Then $g \in \mathcal{O}^\sharp(X)$. Consider a connected component Y' of $Z_f = \{y \in Y; f(y) = 0\}$ such that

$\pi(X') \subset Y'$. Let $\pi' := \pi|_{X'}$. Clearly (X', π') is a domain over the connected Stein manifold Y' of dimension $n-1$. Moreover, as f has multiplicity one on Y', the same is true for g over X'. Therefore the multiplication by g induces a short exact sequence over X:

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_{X'} \longrightarrow 0$$

which implies readily that $H^i(X', \mathcal{O})$ vanishes for $i \geq 2$, whence a′), and that the restriction map $H^1(X, \mathcal{O}) \longrightarrow H^1(X', \mathcal{O})$ is surjective.

Observe now that there is a natural restriction $\rho : \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X')$; moreover, from a standard commutative diagram obtained from the exponential sequence and because the restriction $H^1(X, \mathcal{O}) \longrightarrow H^1(X', \mathcal{O})$ is surjective, ρ induces a surjective mapping $\rho^0 : \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}^0(X')$.

LEMMA 6. *There is a canonical restriction map*

$$\beta : \mathrm{Div}(X) \longrightarrow \mathrm{Div}(X')$$

making the following diagram commutative

$$\begin{array}{ccc} \mathrm{Div}(X) & \xrightarrow{\delta_X} & \mathrm{Pic}(X) \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\rho} \\ \mathrm{Div}(X') & \xrightarrow{\delta_{X'}} & \mathrm{Pic}(X') \end{array}$$

Here we would like to clarify a point concerning divisors and their restrictions, which might cause confusions. To a divisor D of $\mathrm{Div}(X)$ we associate canonically an invertible subsheaf $\mathcal{O}_X(D)$ of $\mathcal{M}_X$. Now, if $\pi : Y \longrightarrow X$ is a morphism from another complex space Y into X, the pull-back $\pi^\star\mathcal{O}_X(D)$ of $\mathcal{O}_X(D)$ is an invertible sheaf on Y, hence defines a linear equivalence class of divisors on Y (improperly) denoted by $\pi^\star D$. Only the linear equivalence class of $\pi^\star D$ is well-defined in general; however, when Y is reduced and D is a divisor (U_i, f_i) whose support does not contain an image of a irreducible component of Y, the collection $(\pi^{-1}(U_i), f_i \circ \pi)$ defines a divisor $\pi^\star D$ in that class. In particular, it makes sense to restrict a Cartier divisor to a subvariety not contained in its support, and to restrict a Cartier divisor *class* to any subvariety. Thus one should be careful when considering restrictions of divisors to subspaces.

However, due to the particular form of the hypersurface X', in the setting we are dealing with it is possible to define a natural restriction

$$\beta : \mathrm{Div}(X) \longrightarrow \mathrm{Div}(X')$$

enjoying the properties stated in the above lemma.

Let D be a divisor on X defined by $\{(U_i, m_i)\}_{i \in I}$. Clearly we may assume that each $U_i \cap X'$ is connected (the empty set is connected!). It is readily seen that each m_i can be uniquely written as $m_i = m_i' f^{\nu_i}$ for some $\nu_i \in \mathbb{Z}$ and $m_i' \in \mathcal{M}^\star(U_i)$ and the intersection of the pole set as well as the zero set of m_i' with X' has complex dimension less than $n-1$. This implies $\nu_i = \nu_j$ for those indices i and j such that U_i and U_j intersect X' in a non empty set. Therefore $m_i'|_{U_i'} \in \mathcal{M}^\star(U_i')$, where $U_i' = U_i \cap X'$. It follows that $\{(U_i', m_i'|_{U_i'})\}$ defines a Cartier divisor D' in $\mathrm{Div}(X')$ which is the desired image of $\beta(D)$ so that the restriction map β is defined.

Furthermore, it is not difficult to see that the diagram of the lemma is commutative.

Now, to proceed with the proof of the claim, since by hypothesis $\mathrm{Pic}^0(X)$ is contained in the image of δ_X, the surjectivity of ρ^0 and the lemma from above imply immediately that $\mathrm{Pic}^0(X')$ is contained in the image of the map $\delta_{X'}$. Thus the claim, whence the proof of the theorem.

In the same circle of ideas we state here a variant of Theorem 1 for the singular case that can be obtained along the same lines with a little more care, namely

THEOREM 4. *Let Y be a Stein space of pure dimension $n \geq 2$ and (X, π) a domain over Y. Assume that $H^k(X, \mathcal{O}) = 0$ for $2 \leq k < n$ and that every topologically trivial holomorphic line bundle over X is associated to some Cartier divisor. Then X is locally Stein over each regular point of Y.*

Notice that when (X, π) is schlicht and Y has at worst Cohen-Macaulay singularities we recover a result in [2]. Also in the surface case the cohomological vanishing condition is superfluous.

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REFERENCES

1. Abe, M., *Holomorphic line bundles on a domain of a two-dimensional Stein manifold*, Ann. Polon. Math. 83 (2004), 269–272.
2. Abe, M., *Holomorphic line bundles and divisors on a domain of a Stein manifold*, Ann. Sc. Norm. Sup. Pisa Cl. Sci 6 (2007), 323–330.
3. Andreotti, A., and Grauert, H., *Théorèmes de finitude pour la cohomologie des espaces complexes*, Bull. Soc. Math. France 90 (1962), 193–259.
4. Ballico, E., *Holomorphic vector bundles on* $\mathbb{C}^2 \setminus \{0\}$, Israel J. Math. 128 (2002), 197–204.
5. Ballico, E., *Cousin I condition and Stein spaces*, Complex Var. Theory Appl. 50 (2005), 23–25.
6. Brenner, H., *A class of counter-examples to the hypersection problem based on forcing equations*, Arch. Math. 82 (2004), 564–569.

7. Coltoiu, M., and Diederich, K., *Open sets with Stein hypersurface sections in Stein spaces*, Ann. of Math. 145 (1997), 175–182.
8. Docquier, F., and Grauert, H., *Levisches Problem und Rungescher Satz für Teilgebiete Steinscher Mannigfaltigkeiten*, Math. Ann. 140 (1960), 94–123.
9. Fornæss, J.-E., and Narasimhan, R., *The Levi problem on complex spaces with singularities*, Math. Ann. 248 (1980), 47–72.
10. Fritzsche, K., and Grauert, H., *From holomorphic functions to complex manifolds*, Grad. Texts Math. 213, Springer, Berlin 2002.
11. Fujita, R., *Domaines sans point critique intérieur sur l'espace projectif complexe*, J. Math. Soc. Japan 15 (1963), 443–473.
12. Grauert, H,, and Remmert, R., *Theory of Stein spaces*, Grundl. Math. Wiss. 236, Springer, Berlin 1979.
13. Grauert, H., *Charakterisierung der holomorph vollständigen Komplexen Räumen*, Math. Ann. 129 (1955), 233–259.
14. Griffiths, P., and Harris, J., *Principles of algebraic geometry*, Wiley, New York 1978.
15. Gunning, R. C., *Introduction to holomorphic functions of several variables 3*, Wadsworth, Monterey 1990.
16. Jennane, B., *Groupes de cohomologie d'un fibré holomorphe à base et à fibre de Stein*, pp. 100–108 in: Séminaire Pierre Lelong–Henri Skoda (Analyse). Années 1978/79, Lect. Notes Math. 822, Springer, Berlin 1980.
17. Kajiwara, J., and Kazama, H., *Two dimensional complex manifold with vanishing cohomology set*, Math. Ann. 204 (1973), 1–12.
18. Kajiwara, J., and Sakai, E., *Generalization of Levi-Oka's theorem concerning meromorphic functions*, Nagoya Math. J. 29 (1967), 75–84.
19. Lelong, P., *Domaines convexes par rapport aux fonctions plurisousharmoniques*, J. Analyse Math. 2 (1952), 178–208.
20. Oka, K., *Domaines finis sans points critiques intérieurs*, Jap. J. Math. 23 (1953), 97–155.
21. Siu, Y.-T., *Non-countable dimensions of cohomology groups of analytic sheaves and domains of holomorphy*, Math. Z. 102 (1967), 17–29.
22. Siu, Y.-T., *Every Stein subvariety admits a Stein neighborhood*, Invent. Math. 38 (1976/77), 89–100.
23. Vâjâitu, V., *Pseudoconvex domains over q-complete manifolds*, Ann. Sc. Norm. Sup. Pisa Cl. Sci. 29 (2000), 503–530.

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