1. Introduction

Given the curve

\[ \Gamma(t) = (t^\alpha, t^\beta, t^\gamma), \quad t > 0 \]

in \( \mathbb{R}^3 \), with \( \alpha, \beta, \gamma \) positive exponents, we consider the “fractional integration operator of order \( \sigma \) along \( \Gamma \)”

\[ T^\sigma f(x) = \int_0^{+\infty} f(x - \Gamma(t))t^{\sigma-1} \, dt, \quad x \in \mathbb{R}^3 \]

defined for \( \Re \sigma > 0 \).

We are interested in studying the boundedness properties of \( T^\sigma \) as an operator from \( L^p(\mathbb{R}^3) \) to \( L^q(\mathbb{R}^3) \). In order to have positive results, we shall suppose that \( \alpha, \beta, \gamma \) are distinct. Therefore we may assume \( \alpha < \beta < \gamma \). The homogeneity of the operator \( T^\sigma \) with respect to the dilations

\[ \delta \cdot (x_1, x_2, x_3) = (\delta^\alpha x_1, \delta^\beta x_2, \delta^\gamma x_3) \]

implies the condition

\[ \frac{1}{p} - \frac{1}{q} = \frac{\Re \sigma}{Q} \]

where \( Q = \alpha + \beta + \gamma \) is the homogeneous dimension of \( \mathbb{R}^3 \) with respect to the dilations (3). If \( \sigma \in \mathbb{R} \), another necessary condition on \( p \) and \( q \) is due to the fact that \( T^\sigma \) dominates the operator \( T_0 \) which is defined by limiting the integration in (2) to a bounded interval of \( \mathbb{R} \) which doesn’t contain the origin. According to what has been proved in [3] and later improved in [4], \( T_0 \) is bounded from \( L^p(\mathbb{R}^3) \) to \( L^q(\mathbb{R}^3) \) when \( (\frac{1}{p}, \frac{1}{q}) \) belongs to the closed trapezoid with vertices \( A = (0, 0), B = (1, 1), C = (\frac{1}{2}, 1), D = (1, \frac{1}{2}) \). As we will prove, this condition is also necessary when \( \sigma \in \mathbb{C} \).

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In the special case of $\sigma = \frac{Q}{6}$, Drury has proved in [2] that the operator $T^\sigma$ is bounded from $L^2(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$ for $(\alpha, \beta, \gamma) = (1, 2, k)$ and $k \geq 4$; subsequently Pan has extended the same estimate to $3 \leq k < 4$, [5]. More recently Pan has shown in [6] that this result holds also for $\beta + \gamma \geq 5\alpha$.

In this paper we will prove the following Theorem.

**Theorem 1.** For $\Re \sigma > 0$, $T^\sigma$ is bounded from $L^p(\mathbb{R}^3)$ to $L^q(\mathbb{R}^3)$ if and only if

(i) $(\frac{1}{p}, \frac{1}{q}) \in ABCD$,

(ii) $\frac{1}{p} - \frac{1}{q} = \frac{\Re \sigma}{Q}$,

where $ABCD$ is the closed trapezoid with vertices

$A = (0, 0), B = (1, 1), C = (\frac{3}{2}, 1), D = (1, \frac{1}{2})$.

We consider now the arc-length $ds$ on the arc $\Gamma_0$ obtained by taking $t \in [0, 1]$ and the convolution operator

$$(4) \quad f(x) \rightarrow \int_{\Gamma_0} f(x - \Gamma(t)) \, ds, \quad x \in \mathbb{R}^3.$$ 

In order to express $ds$ it is convenient to renormalize the exponents of the curve $\Gamma$ so to obtain $\alpha = 1$. This can be done without loss of generality, by changing variable $\rho = u$. The operator in (4) becomes essentially

$$(5) \quad Tf(x) = \int_0^1 f(x - (t, t^\beta, t^\nu)) \, dt, \quad x \in \mathbb{R}^3.$$ 

As a consequence of Theorem 1 we have the following result for the operator $T$.

**Corollary 2.** Let $T$ be the operator defined in (5). Then

(i) if $\beta + \gamma \geq 5$ the typeset for $T$ is the closed trapezoid with vertices $A = (0, 0), B = (1, 1), C' = (\frac{Q-2}{Q}, \frac{Q-3}{Q}), D' = (\frac{3}{Q}, \frac{2}{Q})$;

(ii) if $\beta + \gamma < 5$ the typeset for $T$ is the whole trapezoid with vertices $A = (0, 0), B = (1, 1), C = (\frac{3}{2}, \frac{1}{2}), D = (1, \frac{1}{2})$.

This paper is organized as follows. In the next section we present an estimate on the decay of certain oscillatory integrals that will be used in the proof of Theorem 1. In the third section we prove Theorem 1 with the method used by Christ in [1] which consists in introducing a Littlewood-Paley decomposition adapted to the homogeneity of the curve (1).
2. Notation and preliminary estimates

Definition 3. Given a function $f$ on $\mathbb{R}^3$ and $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \mathbb{R}^3$, we put
$$D_{(\varepsilon_1, \varepsilon_2, \varepsilon_3)} f(x_1, x_2, x_3) = f(\varepsilon_1 x_1, \varepsilon_2 x_2, \varepsilon_3 x_3), \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$  

Definition 4. Given a vector-valued function $f = \{f_j\} \in L^p(I')$ where $j$ varies over $\mathbb{Z}$, we define
$$\|f\|_{L^p(I')} = \left\| \left( \sum_{j \in \mathbb{Z}} |f_j|^p \right)^{\frac{1}{p}} \right\|_{L^q(\mathbb{R}^3)}$$
and we denote by $\| \cdot \|_{p,q,r}$ the norm of an operator from $L^p(I')$ to $L^q(I')$.

Now let $I \subset \mathbb{R}$ be a closed interval and let $\mathcal{C}$ be a curve in $\mathbb{R}^3$ defined by
$$\mathcal{C} : t \in I \longmapsto (\psi_1(t), \psi_2(t), \psi_3(t)) = \psi(t) \in \mathbb{R}^3,$$  
where $\psi_1, \psi_2, \psi_3$ are smooth real-valued functions and let $\omega$ be a smooth function with compact support in $I$. For $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ we define the oscillatory integral
$$I(\xi) = \int_I e^{-i\xi \cdot \psi(t)} \omega(t) \, dt.$$  

We have the following result:

Lemma 5. Suppose that for every $t \in I$, the vectors $\psi'(t), \psi''(t), \psi'''(t)$ span $\mathbb{R}^3$ and that $\psi'_1(t) \neq 0, \psi'_2(t) \neq 0, \psi'_3(t) \neq 0$ for all $t \in I$. Then
$$I(\xi) = O(\xi^{-\frac{3}{2}}) \quad \text{as } |\xi| \to \infty.$$  
Moreover there exists a constant $C > 1$ such that
$$I(\xi) = O(|\xi|^{-N}) \quad \text{as } |\xi| \to \infty$$  
for all $N \geq 0$, in the regions of the space $\xi_1, \xi_2, \xi_3$ where
$$|\xi_1| > C(|\xi_2| + |\xi_3|)$$
$$|\xi_2| > C(|\xi_1| + |\xi_3|)$$
$$|\xi_3| > C(|\xi_1| + |\xi_2|).$$  

The proof of Lemma 5 is an application of van der Corput’s lemma and therefore we omit the details.
3. Proof of Theorem 1

For \( \Re \sigma > 0 \), let \( K^\sigma(x_1, x_2, x_3) \) be the kernel on \( \mathbb{R}^3 \) defined by

\[
\langle K^\sigma, f \rangle = \int_0^{+\infty} f(t^\alpha, t^\beta, t^\gamma) t^{\sigma-1} dt
\]

so that

\[
T^\sigma f(x_1, x_2, x_3) = f * K^\sigma(x_1, x_2, x_3).
\]

We prove first the necessity of conditions (i) and (ii). As we have already observed, condition (ii) is necessary for \( \sigma \in \mathbb{C} \) and condition (i) is necessary for \( \sigma \in \mathbb{R} \). Therefore it remains only to prove the necessity of condition (i) for \( \sigma \in \mathbb{C} \). Given \( 0 < \epsilon < 1 \), let \( \varphi \) be a smooth positive real-valued function in \( \mathbb{R}^3 \) compactly supported in a ball of radius \( \epsilon \) centered at the point \( x_0 = (1, 1, 1) \) and identically one in a ball of radius \( \frac{\epsilon}{2} \) centered at \( x_0 \). We may write

\[
\varphi(x) = \int e^{-i\xi \cdot x} (\mathcal{F}^{-1}\varphi)(\xi) \, d\xi
\]

\[
= \int \check{\delta}_\epsilon(\mathcal{F}^{-1}\varphi)(\xi) \, d\xi.
\]

Then if we define

\[
K^\sigma \varphi = \int \check{\delta}_\epsilon K^\sigma (\mathcal{F}^{-1}\varphi)(\xi) \, d\xi,
\]

we have

\[
\|K^\sigma \varphi\|_{pq} \leq \int \|\check{\delta}_\epsilon K^\sigma\|_{pq} |\mathcal{F}^{-1}\varphi(\xi)| \, d\xi
\]

\[
\leq \|K^\sigma\|_{pq} \|\mathcal{F}^{-1}\varphi\|_1
\]

\[
\leq C\|K^\sigma\|_{pq}.
\]

Since

\[
\Re(t^{\alpha-1}) = t^{\Re\sigma-1}\cos(\Im \sigma \log t),
\]

then if \( \epsilon \) has been chosen sufficiently small and \( f \in L^p(\mathbb{R}^3) \) is a positive real-valued function, we obtain

\[
\|K^{\Re\sigma} \varphi * f\|_q \leq C\|K^\sigma \varphi * f\|_q
\]

\[
\leq C\|K^\sigma \varphi\|_{pq} \|f\|_p.
\]

By combining (7) and (8) we get
\[ \|K^\text{Res}_\varphi\|_{pq} \leq C\|K^\sigma\|_{pq}. \]

The necessity of condition (i) then follows from the fact that the kernel \(K^\text{Res}_\varphi\) defines a convolution operator which is bounded only on the closed trapezoid \(ABCD\).

To prove the sufficiency of the conditions (i) and (ii) we first observe that since
\[ \|T^\sigma\|_{pq} \leq \|T^\text{Res}\|_{pq}, \]
we may assume \(\sigma \in \mathbb{R}\) and \(\sigma > 0\).

Then we make a dyadic decomposition of the kernel \(K^\sigma\). Let \(\omega(t)\) be a smooth function on \(\mathbb{R}^+\), supported on \(\{t : \frac{1}{2} < t < 4\}\) and such that \(\sum_{j \in \mathbb{Z}} \omega(2^j t) = 1\) for \(t > 0\). We define the kernels
\[
(9) \quad \langle K^\sigma_j, f \rangle = \int f(t^\alpha, t^\beta, t^\gamma)\omega(2^j t) t^{\sigma-1} dt.
\]
Every kernel \(K^\sigma_j\) may be obtained from the kernel \(K^\sigma_0\) by dilation and with a multiplicative factor in fact, by changing variable in (9), we have
\[
\langle K^\sigma_j, f \rangle = 2^{-\sigma j} \langle K^\sigma_0, D_{(2^{-\sigma j}, 2^{-\sigma j}, 2^{-\sigma j})} f \rangle.
\]

Then we define the operators
\[
T^\sigma f(x_1, x_2, x_3) = \int f(x_1 - t^\alpha, x_2 - t^\beta, x_3 - t^\gamma)\omega(2^j t) t^{\sigma-1} dt
\]
so that \(T^\sigma = \sum_{j \in \mathbb{Z}} T^\sigma_j\).

The Fourier multiplier corresponding to the operator \(T^\sigma_j\) is
\[
\widehat{K}^\sigma_j(\xi_1, \xi_2, \xi_3) = 2^{-\sigma j} \widehat{K}^\sigma_0(2^{-\sigma j} \xi_1, 2^{-\sigma j} \xi_2, 2^{-\sigma j} \xi_3)
\]
and by Lemma 5, we know that the multiplier
\[
\widehat{K}^\sigma_0(\xi_1, \xi_2, \xi_3) = \int_\mathbb{R} e^{-i(\xi_1 t_1 + \xi_2 t_2 + \xi_3 t_3)} e^{-i(t^\alpha, t^\beta, t^\gamma)} \omega(t) t^{\sigma-1} dt
\]
decays slowly in the complementary set of the region of the space \(\xi_1, \xi_2, \xi_3\) defined by (6). We say that this set contains the singular directions for \(K^\sigma_0\) and we decompose it into four subsets:
\[ \Gamma_0^1 = \{ (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : a|\xi_3| < |\xi_2| < b|\xi_3|, \ |\xi_1| < \delta|\xi_3| \} \]

\[ \Gamma_0^2 = \{ (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : a|\xi_3| < |\xi_1| < b|\xi_3|, \ |\xi_2| < \delta|\xi_3| \} \]

\[ \Gamma_0^3 = \{ (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : a|\xi_2| < |\xi_1| < b|\xi_2|, \ |\xi_3| < \delta|\xi_2| \} \]

\[ \Gamma_0^4 = \{ (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : a|\xi_3| < |\xi_1| < b|\xi_3|, \ a|\xi_3| < |\xi_2| < b|\xi_3| \} \]

where \( a, b, \delta \) are positive constants and \( \delta \) is sufficiently small.

Since the region \( \bigcup_{i=1}^{4} \Gamma_0^i \) contains the singular directions for \( \hat{K}_0^\sigma \), then the singular directions for \( \hat{K}_0^\sigma \) are contained in the union of the cones \( \Gamma_j^i, i = 1, \ldots, 4 \), where

\[ \Gamma_j^i = \{ (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : (2^{-\alpha_j} \xi_1, 2^{-\beta_j} \xi_2, 2^{-\gamma_j} \xi_3) \in \Gamma_0^i \}. \]

For fixed \( i = 1, \ldots, 4 \), as \( j \) varies in \( \mathbb{Z} \), the sets \( \Gamma_j^i \) are essentially disjoint.

Now, for \( i = 1, \ldots, 4 \), we introduce a \( C^\infty \) partition of unity \( \{ \eta_j^i \} \) in \( \mathbb{R}^3 \) minus the coordinate planes with \( \eta_j^i \) homogeneous of degree zero (with respect to the Euclidean dilation group), supported in a set like \( \Gamma_j^i \), but a little bit larger and identically one in the set \( \Gamma_j^i \). Moreover let \( \eta_j^i \) be such that

\[ \eta_j^i(\xi_1, \xi_2, \xi_3) = \eta_0^i(2^{-\alpha_j} \xi_1, 2^{-\beta_j} \xi_2, 2^{-\gamma_j} \xi_3). \]

Let \( Q_j^i, i = 1, \ldots, 4 \), be the operator with \( \eta_j^i \) as Fourier multiplier and let \( \phi \) be a smooth function on \( \mathbb{R}^3 \), compactly supported and identically one in a neighborhood of the origin. Finally let \( P_j \) be the operator with \( \phi(2^{-\alpha_j} \xi_1, 2^{-\beta_j} \xi_2, 2^{-\gamma_j} \xi_3) \) as Fourier multiplier. Following Christ [1], for fixed \( \sigma \) we write

\[ T^\sigma = \sum_{j \in \mathbb{Z}} T_j^\sigma \]

\[ = \sum_{j \in \mathbb{Z}} T_j^\sigma P_j + \sum_{j \in \mathbb{Z}} T_j^\sigma (I - P_j) \]

\[ = \sum_{j \in \mathbb{Z}} T_j^\sigma P_j + \sum_{j \in \mathbb{Z}} T_j^\sigma (I - P_j) Q_j^1 + \sum_{j \in \mathbb{Z}} T_j^\sigma (I - P_j) (I - Q_j^1) \]

\[ \ldots \]

\[ = \sum_{j \in \mathbb{Z}} T_j^\sigma P_j + \sum_{j \in \mathbb{Z}} T_j^\sigma (I - P_j) Q_j^1 + \sum_{j \in \mathbb{Z}} T_j^\sigma (I - P_j) (I - Q_j^1) Q_j^2 \]

\[ + \sum_{j \in \mathbb{Z}} T_j^\sigma (I - P_j) \left( \prod_{i=1}^{2} (I - Q_j^i) \right) Q_j^3 \]
\[ + \sum_{j \in \mathbb{Z}} T_j^\sigma (I - P_j) \left( \prod_{i=1}^{3} (I - Q_i^j) \right) Q_j^3 \]
\[ + \sum_{j \in \mathbb{Z}} T_j^\sigma (I - P_j) \prod_{i=1}^{4} (I - Q_i^j) \]

where I is the identity operator.

We consider first the operator \( \sum_{j \in \mathbb{Z}} T_j^\sigma P_j \). Each term \( T_j^\sigma P_j \) has a smooth, compactly supported Fourier multiplier given by
\[
2^{-j\alpha} K_0^\sigma (2^{-\alpha} \xi_1, 2^{-\beta} \xi_2, 2^{-\gamma} \xi_3) \phi(2^{-\alpha} \xi_1, 2^{-\beta} \xi_2, 2^{-\gamma} \xi_3)
\]
therefore it is a convolution operator with a Schwartz kernel. Let \( H_j(x_1, x_2, x_3) \) be such a kernel. By homogeneity we have
\[
H_j(x_1, x_2, x_3) = 2^{(Q-\sigma)j} H_0(2^{\alpha j} x_1, 2^{\beta j} x_2, 2^{\gamma j} x_3)
\]

where \( H_0 \) is the convolution kernel corresponding to the operator \( T_0^\sigma P_0 \). Let \( \rho(x_1, x_2, x_3) \) be a homogeneous norm with respect to the dilations (3), we prove that the sum over all \( j \) of the absolute values of the convolution kernels of \( T_j^\sigma P_j \) is bounded by a constant times the kernel \( \rho(x_1, x_2, x_3) \)^{Q-\sigma}. We consider the series
\[
(11) \sum_{j \in \mathbb{Z}} |H_j(x_1, x_2, x_3)| = \sum_{j \in \mathbb{Z}} 2^{(Q-\sigma)j} |H_0(2^{\alpha j} x_1, 2^{\beta j} x_2, 2^{\gamma j} x_3)|
\]
\[ = \sum_{2^j \rho(x_1, x_2, x_3) \leq 1} 2^{(Q-\sigma)j} |H_0(2^{\alpha j} x_1, 2^{\beta j} x_2, 2^{\gamma j} x_3)| + \sum_{2^j \rho(x_1, x_2, x_3) > 1} 2^{(Q-\sigma)j} |H_0(2^{\alpha j} x_1, 2^{\beta j} x_2, 2^{\gamma j} x_3)|.
\]

For \( N \in \mathbb{N} \) we denote by \( \|H_0\|_{(N)} \) the norm of \( H_0 \) in \( \mathcal{S}(\mathbb{R}^3) \) given by
\[
\|H_0\|_{(N)} = \sum_{|\nu| \leq N} \sup_{x \in \mathbb{R}^3} (1 + |x|)^N |\partial^\nu H_0(x)|
\]
where \( \nu \) is a multi-index.

Observe that from hypotheses (i) and (ii) we get \( Q - \sigma > 0 \). In the first sum of (11) we put
\[
|H_0(2^{\alpha j} x_1, 2^{\beta j} x_2, 2^{\gamma j} x_3)| \leq \|H_0\|_{(0)},
\]
so that
Since \( \rho(x_1, x_2, x_3) \leq C|(x_1, x_2, x_3)|^\frac{3}{2} \) when \( |(x_1, x_2, x_3)| \geq 1 \), in the second sum of (11) we put
\[
|H_0(2^\gamma x_1, 2^\beta x_2, 2^\gamma x_3)| \leq \frac{\|H_0\|_{(M)}}{1 + \|(2^\gamma x_1, 2^\beta x_2, 2^\gamma x_3)\|^M} \leq \frac{C}{[2\rho(x_1, x_2, x_3)]^\sigma}
\]
for a positive integer \( M \) such that \( M\sigma > Q - \sigma \). In this way we have
\[
\sum_{2^\rho(x_1, x_2, x_3) \geq 1} 2^{Q-\sigma}|H_0(2^\gamma x_1, 2^\beta x_2, 2^\gamma x_3)| \leq C[\rho(x_1, x_2, x_3)]^{-Q+\sigma}.
\]
By combining (12) and (13) we get
\[
\sum_{j \in \mathbb{Z}} |H_j(x_1, x_2, x_3)| \leq C[\rho(x_1, x_2, x_3)]^{-Q+\sigma}.
\]
Therefore by applying the Hardy-Littlewood-Sobolev’s Theorem, it follows that the operator \( \sum_{j \in \mathbb{Z}} T_j^* P_j \) is bounded from \( L^p \) to \( L^q \) whenever \( \frac{1}{p} - \frac{1}{q} = \sigma Q \).

We may repeat the previous arguments also for the operator
\[
\sum_{j \in \mathbb{Z}} T_j^*(I - P_j) \prod_{i=1}^4 (I - Q_j^i)
\]
because every term \( T_j^*(I - P_j) \prod_{i=1}^4 (I - Q_j^i) \) has a Fourier multiplier which belongs to the Schwartz class. This is because the multiplier of the operator \( T_j^* \) decays rapidly with all its derivatives outside the regions \( \Gamma_j^i, i = 1, \ldots, 4 \).

So the operator \( \sum_{j \in \mathbb{Z}} T_j^*(I - P_j) \prod_{i=1}^4 (I - Q_j^i) \) is bounded from \( L^p \) to \( L^q \) when \( \frac{1}{p} - \frac{1}{q} = \sigma Q \). Therefore it remains only to study the central terms of (10). We treat explicitly only the operator \( \sum_{j \in \mathbb{Z}} T_j^*(I - P_j) Q_j^1 \), since the other ones are analogous.

Let \( \{\tilde{\eta}_j^1\} \) another \( C^\infty \) partition of unity in \( \mathbb{R}^3 \) minus the coordinate planes with \( \tilde{\eta}_j^1 \) homogeneous of degree zero (with respect to the Euclidean dilation group), supported in the set \( \Gamma_j^1 \) ulteriorly enlarged and identically one on the support of \( \eta_j^1 \). Moreover let \( \tilde{\eta}_j^1 \) be such that \( \tilde{\eta}_j^1(\xi_1, \xi_2, \xi_3) = \tilde{\eta}_j^1(2^{-\nu_j^1} \xi_1, 2^{-\beta_j} \xi_2, 2^{-\gamma_j} \xi_3) \). Let \( \bar{Q}_j^1 \) be the operator with \( \tilde{\eta}_j^1 \) as Fourier multiplier, then
\[
Q_j^1 \circ \bar{Q}_j^1 = Q_j^1
\]
for all \( j \in \mathbb{Z} \).
We prove that

\[
\left\| \sum_{j \in \mathbb{Z}} T_j^\alpha (I - P_j) Q_j^1 f \right\|_q \leq C \left\{ \left\| T_j^\alpha (I - P_j) Q_j^1 f \right\|_{L^q(\mathbb{F})} \right. \\
\left. + \left\| P_j T_j^\alpha Q_j^1 f \right\|_{L^q(\mathbb{F})} \right\}
\]

for any \( f \in \mathcal{S}'(\mathbb{R}^3) \).

Since (14) holds we have

\[
\left\| \sum_{j \in \mathbb{Z}} T_j^\alpha (I - P_j) Q_j^1 f \right\|_q = \left\| \sum_{j \in \mathbb{Z}} Q_j^1 T_j^\alpha (I - P_j) Q_j^1 f \right\|_q.
\]

Moreover the linear operator \( \sum_{j \in \mathbb{Z}} \epsilon_j Q_j^1 \) is bounded on \( L^q(\mathbb{R}^3) \), uniformly in all choices of \( \epsilon_j = \pm 1 \). This is because the multiplier corresponding to the operator \( \sum_{j \in \mathbb{Z}} \epsilon_j Q_j^1 \) is a Marcinkiewicz multiplier. Therefore by applying the Littlewood-Paley inequality, we get (15). Since \( P_j \) is a convolution operator with a Schwartz kernel and the vector-valued Hardy-Littlewood maximal function is bounded on \( L^q(\mathbb{F}) \) for \( 1 < q < +\infty \), we get from (15) the following estimates

\[
\left\| \sum_{j \in \mathbb{Z}} T_j^\alpha (I - P_j) Q_j^1 f \right\|_q \leq C \left\{ \left\| T_j^\alpha (I - P_j) Q_j^1 f \right\|_{L^q(\mathbb{F})} \right. \\
\left. + \left\| P_j T_j^\alpha Q_j^1 f \right\|_{L^q(\mathbb{F})} \right\} \leq C \left\{ \left\| T_j^\alpha Q_j^1 f \right\|_{L^q(\mathbb{F})} + C_1 \left\| MT_j^\alpha Q_j^1 f \right\|_{L^q(\mathbb{F})} \right. \\
\left. \leq C \left\{ \left\| T_j^\alpha Q_j^1 f \right\|_{L^q(\mathbb{F})} \right. \right. \\
\left. \left. + \left\| T_j^\alpha Q_j^1 f \right\|_{L^q(\mathbb{F})} \right\} \right. \\
\left. \leq C \left\{ \left\| T_j^\alpha \right\|_{p, q, 2} \left\| f_j \right\|_{L^p(\mathbb{F})} \right. \right. \\
\left. \left. \leq \left\| \left\{ T_j^\alpha \right\} \right\|_{p, q, 2} \left\| f \right\|_{L^p(\mathbb{F})} \right. \right.
\]

where \( f_j = Q_j^1 f \).

Since also the linear operator \( \sum_{j \in \mathbb{Z}} \epsilon_j Q_j^1 \) is bounded on \( L^p(\mathbb{R}^3) \), uniformly in all choices of \( \epsilon_j = \pm 1 \), because its Fourier multiplier is a Marcinkiewicz multiplier, by the Littlewood-Paley theory we get

\[
\left\| \left\{ Q_j^1 f \right\} \right\|_{L^p(\mathbb{F})} \leq C \left\| f \right\|_{L^p(\mathbb{F})}
\]

By combining (16) and (17) we have

\[
\left\| \sum_{j \in \mathbb{Z}} T_j^\alpha (I - P_j) Q_j^1 f \right\|_q \leq C \left\{ \left\| T_j^\alpha \right\|_{p, q, 2} \left\| f \right\|_{L^p(\mathbb{F})} \right. \right. \\
\left. \left. \leq \left\| \left\{ T_j^\alpha \right\} \right\|_{p, q, 2} \left\| f \right\|_{L^p(\mathbb{F})} \right. \right.
\]

By repeating the previous considerations, we get an analogous estimate also for the remaining operators in (10).

Now we consider exponents \( p \leq 2 \) because the other ones may be treated...
by duality. For $0 < \theta < 1$, we write $\frac{1}{\theta} = \frac{\theta}{p} + \frac{1-\theta}{\infty}$. Given the vector-valued operator

$$\{ T_j^\gamma \} : L^p(\mathbb{R}^3) \to L^q(\mathbb{R}^3)$$

$$\{ g_j \} \mapsto \{ T_j^\gamma g_j \},$$

by Holder’s inequality we have that

$$\| \{ T_j^\gamma \} \|_{p,q,\infty} \leq \| \{ T_j^\gamma \} \|_{p,q,p}^{\theta} \| \{ T_j^\gamma \} \|_{p,q,\infty}^{1-\theta}. \tag{19}$$

Therefore, by the previous estimates on the operators in (10) and by (19), we have proved that

$$\| T^\gamma \|_{pq} \leq C[1 + \| \{ T_j^\gamma \} \|_{p,q,p}^{\theta} \| \{ T_j^\gamma \} \|_{p,q,\infty}^{1-\theta}]. \tag{20}$$

Since $T_j^\gamma$ is a convolution operator with a positive kernel we get

$$\| \{ T_j^\gamma \} \|_{p,q,\infty} \leq \| T^\gamma \|_{pq}. \tag{21}$$

Furthermore, by Minkowski’s integral inequality we get, for $x \in \mathbb{R}^3$

$$\| \{ T_j^\gamma g_j \} \|_{L^p(\mathbb{R}^3)} = \left( \int \left( \sum_{j \in \mathbb{Z}} |T_j^\gamma g_j(x)|^p \right)^{\frac{q}{q'}} \right)^{\frac{1}{q}}$$

$$\leq \left( \int \left( \sum_{j \in \mathbb{Z}} |T_j^\gamma g_j(x)|^q \right)^{\frac{p}{p'}} \right)^{\frac{1}{p}}$$

$$\leq \left( \sum_{j \in \mathbb{Z}} \left( \int |T_j^\gamma g_j(x)|^q \right)^{\frac{p}{q'}} \right)^{\frac{1}{p}}$$

$$\leq \sup_{j \in \mathbb{Z}} \| T_j^\gamma \|_{pq} \left( \sum_{j \in \mathbb{Z}} \| g_j \|_p^p \right)^{\frac{1}{p}}$$

$$= \sup_{j \in \mathbb{Z}} \| T_j^\gamma \|_{pq} \| \{ g_j \} \|_{L^p(\mathbb{R}^3)}.$$

that is

$$\| \{ T_j^\gamma \} \|_{p,q,p} \leq \sup_{j \in \mathbb{Z}} \| T_j^\gamma \|_{pq}. \tag{22}$$

Then we shall prove that $T_j^\gamma, j \in \mathbb{Z}$, are uniformly bounded from $L^p(\mathbb{R}^3)$ to $L^q(\mathbb{R}^3)$.

By homogeneity we have
\[ T_j^\sigma(x_1, x_2, x_3) = 2^{-j\sigma}D_{(2^{-j\sigma}, 2^{j\sigma})} \left( T_0^\sigma D_{(2^{-j\sigma}, 2^{j\sigma})} f \right)(x_1, x_2, x_3) \]

and since \( \frac{1}{p} - \frac{1}{q} = \frac{\sigma}{2} \) we get

\[
\| T_j^\sigma f \|_q = 2^{-j\sigma} \| D_{(2^{-j\sigma}, 2^{j\sigma})} \left( T_0^\sigma D_{(2^{-j\sigma}, 2^{j\sigma})} f \right) \|_q \\
= 2^{-j(\sigma + \frac{\sigma}{2})} \| T_0^\sigma D_{(2^{-j\sigma}, 2^{j\sigma})} f \|_p \\
\leq 2^{-j(\sigma + \frac{\sigma}{2})} \| T_0^\sigma \|_{pq} \| D_{(2^{-j\sigma}, 2^{j\sigma})} f \|_p \\
= 2^{-j(\sigma + \frac{\sigma}{2})} \| T_0^\sigma \|_{pq} \| f \|_p \\
= \| T_0^\sigma \|_{pq} \| f \|_p.
\]

But

\[ T_0^\sigma f(x_1, x_2, x_3) = \int_{\frac{1}{2}}^1 f(x_1 - t^\alpha, x_2 - t^\beta, x_3 - t^\gamma) \omega(t)t^{\sigma - 1} \, dt \]

is a convolution operator with a finite measure supported on the curve \( \Gamma(t) \) for \( t \in [\frac{1}{2}, 4] \). Therefore by Pan’s theorem [4], \( T_0^\sigma \) is bounded on the whole trapezoid \( ABCD \) and so \( T_j^\sigma, j \in \mathbb{Z}, \) are uniformly bounded from \( L^p(\mathbb{R}^3) \) to \( L^q(\mathbb{R}^3) \) when \( \frac{1}{p} - \frac{1}{q} = \frac{\sigma}{2} \).

By combining (19) with (20), (21), (22), we get

\[
\| T_j^\sigma \|_{pq} \leq C \left[ 1 + \| T_0^\sigma \|_{pq}^{1 - \frac{\sigma}{2}} \right].
\]

Now, by replacing on both sides of (24) the operator \( T^\sigma \) by the operator

\[
T_N^\sigma = \sum_{|j| \leq N} T_j^\sigma
\]

for a finite \( N \), we may repeat the previous arguments and prove that

\[
\| T_N^\sigma \|_{pq} \leq C
\]

where \( C \) is a positive constant independent of \( N \). By taking the limit of (25) and (26) as \( N \to +\infty \) we conclude the proof.

REFERENCES

