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IRRATIONALITY MEASURES OF NUMBERS RELATED TO SOME q-BASIC HYPERGEOMETRIC SERIES

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Abstract

We obtain rather good irrationality measures for numbers related to some q-basic hypergeometric series. The method uses Padé approximations combined with the iteration of the appropriate functional equation.

1. Introduction and results

Let K be an algebraic number field of degree $d = [K : \mathsf{Q}]$. For each place w of K we normalize the valuation $|.|_w$ in the usual way and denote $|.|_w^* = \max\{1, |.|_w\}$, $\|.\|_w = |.|_w^{d_w/d}$, $\|.\|_w^* = |.|_w^{*d_w/d}$ with $d_w = [K_w : \mathsf{Q}_w]$, here Q_w and K_w denote the completions of Q and K with respect to $|.|_w$. The height of $\alpha \in K^*$ is defined by

$$h(\alpha) = \prod_w \|\alpha\|_w^*,$$

where the product is over all places w of K. Furthermore, for $\underline{a} = (a_1, \dots, a_m) \in K^m$, let $|\underline{a}|_w = \max\{|a_i|_w\}$, $|\underline{a}|_w^* = \max\{1, |a_i|_w\}$, $\|\underline{a}\|_w = |\underline{a}|_w^{d_w/d}$, $\|\underline{a}\|_w^* = |\underline{a}|_w^{*d_w/d}$, and for $\underline{a} \neq \underline{0}$ define the height $h(\underline{a})$ by

$$h(\underline{a}) = \prod_w \|\underline{a}\|_w^*.$$

In the following we fix a place v of K and let a, b, q and ξ denote nonzero elements of K satisfying

$$(1) \qquad |q|_v < 1, \ \ |\xi|_v < 1, \ \ aq^n \neq 1, \ \ bq^n \neq 1, \qquad n = 0, 1, \dots.$$

We are interested in the q-basic hypergeometric series

$$(2) \qquad f(z) = f(a, b, z) = \sum_{n=0}^{\infty} \frac{(a, q)_n (b, q)_n}{(q, q)_n} z^n,$$

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where $(a,q)_0 = 1$, $(a,q)_n = (1-a)(1-aq)\dots(1-aq^{n-1})$, $n \geq 1$. Under the conditions (1) the values $f(\xi)$ and $f(q\xi)$ are defined in K_v. We note that there are a lot of works considering arithmetic properties of different type of q-series, see e.g. [14] for a survey of such results, and [3], [4], [5], [6], [8], [10], [11], [12] and [15] for some more recent results, but only a few study functions (2), see [7] and [10]. In [7] Chirskii considered the case $K = \mathsf{Q}$ with finite p and proved the following result.

THEOREM 1.1. *If $q = p^m$ with a prime $p, m \in \mathsf{N}$, $a, b \in \mathsf{Z}$, and $\xi = cp^t$ with a nonzero $c \in \mathsf{Z}$, $t \in \mathsf{N}$, $t > m + \log_p(2|abc|)$, then the p-adic numbers $f(a,b,\xi)$ and $f(aq,b,\xi)$ are linearly independent over Q.*

Since

$$(1-a)f(aq,b,z) = f(a,b,z) - af(a,b,qz),$$

Theorem 1.1 gives also linear independence of p-adic numbers $f(\xi)$ and $f(q\xi)$. The above theorem was generalized and made quantitative in [10], where the authors study the solutions of functional equations

$$(3) \qquad (qz)^s F(q^2z) = -T(z)F(qz) + S(z)F(z), \qquad s \geq 1,$$

with polynomials T and S, containing as a special case the functional equation

$$(4) \qquad abzf(q^2z) = ((a+b)z-1)f(qz) + (1-z)f(z)$$

satisfied by $f(z)$. If we define

$$(5) \qquad \lambda = \lambda(v,q) = \frac{\log h(q)}{\log \|q\|_v},$$

then $\lambda \leq -1$, and [10] gives linear independence of $f(\xi)$ and $f(q\xi)$, if $\lambda = -1$. As a special case of [10, Theorem 2] we have

THEOREM 1.2. *Let the assumptions (1) be satisfied. If $\lambda = -1$, then there exist positive constants c_0 and H_0 depending on a, b, q, ξ and v such that for all nonzero $\underline{A} = (A_0, A_1) \in K^2$*

$$|A_0 f(\xi) + A_1 f(q\xi)|_v > H^{-\mu - c_0/\sqrt{\log H}},$$

where $\mu = 60,91d/d_v$ and $H = \max\{h(\underline{A}), H_0\}$.

Note that $\lambda = -1$ e.g. if $K = \mathsf{Q}$ and in the archimedean case $q = Q^{-1}$, $Q \in \mathsf{Z}$, or in the p-adic case $q = \pm p^m$, $m \geq 1$.

Our main result in this paper is the following improvement of the above results.

THEOREM 1.3. *Assume that a, b, q and* ξ *are nonzero elements of* K *satisfying* (1). *If* λ *satisfies*

$$-\lambda < \frac{2+\sqrt{2}}{1+\sqrt{2}}, \tag{6}$$

then the numbers $f(\xi)$ *and* $f(q\xi)$ *belonging to* K_v *are linearly independent over* K, *and there exist positive constants* c_0 *and* H_0 *depending on* a, b, q, ξ *and* v *such that for all nonzero* $\underline{A} = (A_0, A_1) \in K^2$ *we have*

$$|A_0 f(\xi) + A_1 f(q\xi)|_v > |\underline{A}|_v^* H^{-\mu - c_0/\sqrt{\log H}}$$

where $H = \max\{h(\underline{A}), H_0\}$ *and*

$$\mu = \frac{d}{d_v} \frac{2+\sqrt{2}}{2+\sqrt{2}+\lambda(1+\sqrt{2})}.$$

In particular, we have $\mu = (2+\sqrt{2})d/d_v$ *if* $\lambda = -1$.

As an immediate corollary we have

COROLLARY 1.4. *If the assumptions of Theorem 1.3 are valid and* $-\lambda < (2+\sqrt{2})/(1+\sqrt{2})$, *then* $f(\xi)/f(q\xi)$ *is not an element of* K.

In the archimedean case Theorem 1.3 implies

COROLLARY 1.5. *Let* $a, b, q = Q^{-1}$, $Q \in \mathbb{Z}$ *and* ξ *be nonzero rationals satisfying* (1). *Then the numbers* $f(\xi)$ *and* $f(q\xi)$ *are linearly independent over* $\mathbb{Q}$ *and for all nonzero* $\underline{A} = (A_0, A_1) \in \mathbb{Z}^2$ *we have*

$$|A_0 f(\xi) + A_1 f(q\xi)| > H^{-(1+\sqrt{2})-c_0/\sqrt{\log H}}$$

where $H = \max\{|A_0|, |A_1|, H_0\}$ *and* c_0, H_0 *are as in Theorem 1.3.*

COROLLARY 1.6. *Under the assumptions of Corollary 1.5* $f(\xi)/f(q\xi)$ *is irrational and has the irrationality measure* $2+\sqrt{2}$.

As a final corollary we give

COROLLARY 1.7. *Let* p *be a prime and let* a, b, ξ *and* $q = \pm p^m$ *be nonzero rationals satisfying* (1). *Then the* p*-adic numbers* $f(\xi)$ *and* $f(q\xi)$ *are linearly independent over* $\mathbb{Q}$ *and for all nonzero* $\underline{A} = (A_0, A_1) \in \mathbb{Z}^2$ *we have*

$$|A_0 f(\xi) + A_1 f(q\xi)|_p > H^{-(2+\sqrt{2})-c_0/\sqrt{\log H}}$$

where $H = \max\{|A_0|, |A_1|, H_0\}$ *and* c_0, H_0 *are as in Theorem 1.3.*

In his proof of Theorem 1.1 Chirskii [7] constructs explicit Padé approximations for the functions $f(a, b, z)$ and $f(aq, b, z)$. To prove Theorem 1.3 we shall improve these approximations by using iteration of the functional equation (4). This kind of iteration process is used often successfully in the consideration of q-series, and in this case we can apply [10], where the iteration of (3) is given in details. We may also say that the improvement in comparision to [10] is obtained by replacing linear forms before iteration, obtained by Siegel's lemma in [1], by Padé approximations given in [7]. It would be of interest to study if this kind of combination of explicit Padé approximations and the iteration of the appropriate functional equation could be applied to get new results also on some other q-series.

2. Preliminaries

By defining

$$\alpha_1 = a, \quad \alpha_2 = b, \quad \alpha_{k+2} = \alpha_k q, \qquad k = 1, 2, \ldots,$$

we get a sequence (α_k) with

$$\alpha_{2k-1} = aq^{k-1}, \quad \alpha_{2k} = bq^{k-1}, \qquad k = 1, 2, \ldots. \tag{7}$$

If

$$f_n(z) = f(\alpha_{n+1}, \alpha_{n+2}, z), \qquad n = 0, 1, \ldots, \tag{8}$$

then $f_0(z) = f(a, b, z)$, $f_1(z) = f(aq, b, z)$, and

$$\alpha_{n+1}(1 - \alpha_{n+2}) z f_{n+2}(z) = f_{n+1}(z) - f_n(z), \qquad n = 0, 1, \ldots, \tag{9}$$

see [7]. Furthermore, if

$$\begin{aligned} R_0(z) &= f_0(z), \\ R_1(z) &= (1 - \alpha_1) f_1(z), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ R_n(z) &= \alpha_1 \ldots \alpha_{n-1}(1 - \alpha_1) \ldots (1 - \alpha_n) z^{n-1} f_n(z), \end{aligned} \tag{10}$$

$n = 2, 3, \ldots$, then, by (9),

$$\begin{aligned} R_2(z) = R_1(z) - (1 - \alpha_1) R_0(z),\ & R_{n+2}(z) \\ &= R_{n+1}(z) - \alpha_n (1 - \alpha_{n+1}) z R_n(z), \end{aligned} \tag{11}$$

$n = 1, 2, \ldots$.

We now suppose that the assumptions of Theorem 1.3 are valid and define n_0 to be the smallest integer such that $|aq^n|_v < 1$ and $|bq^n|_v < 1$ for all $n \geq n_0$. We also denote by $c_1, c_2, \ldots$ effectively computable positive constants independent of n.

LEMMA 2.1. *For all $n \geq n_0$, we have*

$$|R_{2n}(\xi)|_v \leq c_1 |a|_v^n |b|_v^{n-1} |q|_v^{(n-1)^2} |\xi|_v^{2n-1},$$
$$|R_{2n+1}(\xi)|_v \leq c_2 |a|_v^n |b|_v^n |q|_v^{n(n-1)} |\xi|_v^{2n}.$$

PROOF. By the definition (8),

$$(12) \qquad \begin{aligned} f_{2n}(\xi) &= \sum_{j=0}^{\infty} \frac{(\alpha_{2n+1}, q)_j (\alpha_{2n+2}, q)_j}{(q, q)_j} \xi^j \\ &= \sum_{j=0}^{\infty} \frac{\prod_{k=0}^{j-1}(1 - aq^{n+k}) \prod_{k=0}^{j-1}(1 - bq^{n+k})}{\prod_{k=1}^{j}(1 - q^k)} \xi^j. \end{aligned}$$

Thus $|f_{2n}(\xi)|_v = 1$ for finite v, if $n \geq n_0$. This together with (10) gives the first bound of Lemma 2.1 for finite p.

We next consider an infinite v. If $n \geq n_0$, then

$$\begin{aligned} 0 &< \left| \frac{\prod_{k=0}^{j-1}(1 - aq^{n+k}) \prod_{k=0}^{j-1}(1 - bq^{n+k})}{\prod_{k=1}^{j}(1 - q^k)} \right|_v \\ &\leq \frac{\prod_{k=0}^{j-1}(1 + |a|_v |q|_v^{n+k}) \prod_{k=0}^{j-1}(1 + |b|_v |q|_v^{n+k})}{\prod_{k=1}^{j}(1 - |q|_v^k)} \\ &\leq \frac{\prod_{k=0}^{\infty}(1 + |a|_v |q|_v^{n_0+k}) \prod_{k=0}^{\infty}(1 + |b|_v |q|_v^{n_0+k})}{\prod_{k=1}^{\infty}(1 - |q|_v^k)} =: c_3. \end{aligned}$$

Furthermore,

$$\left| \prod_{k=0}^{n-1}(1 - aq^k) \prod_{k=0}^{n-2}(1 - bq^k) \right|_v \leq \prod_{k=0}^{\infty}(1 + |a|_v |q|_v^k) \prod_{k=0}^{\infty}(1 + |b|_v |q|_v^k) =: c_4.$$

Now the first inequality of Lemma 2.1 follows by (10).

The bound for $|R_{2n+1}(\xi)|_v$ is obtained in a similar way.

LEMMA 2.2. *At least one of the numbers $f(\xi)$ and $f(q\xi)$ is different from zero.*

PROOF. Assume that $f(\xi) = f(q\xi) = 0$. Then $f(q^n\xi) = 0$ for all $n \geq 0$ by (1) and (4), and we obtain a contradiction $0 = \lim_{n\to\infty} f(q^n\xi) = f(0) = 1$. This proves Lemma 2.2.

3. Approximation polynomials

By using the equations (10) and (11) we may write

$$R_n(z) = P_{n,0}(z)R_0(z) + P_{n,1}(z)R_1(z), \qquad n = 0, 1, \ldots, \tag{13}$$

where $P_{n,i}$ are polynomials satisfying

$$\begin{matrix} P_{0,0}(z) = 1 \\ P_{0,1}(z) = 0 \end{matrix}, \qquad \begin{matrix} P_{1,0}(z) = 0 \\ P_{1,1}(z) = 1 \end{matrix}, \qquad \begin{matrix} P_{2,0}(z) = \alpha_1 - 1 \\ P_{2,1}(z) = 1 \end{matrix},$$

and, for all $n \geq 1$, the recursions

$$P_{n+2,i}(z) = P_{n+1,i}(z) - \alpha_n(1-\alpha_{n+1})zP_{n,i}(z), \qquad i = 0, 1. \tag{14}$$

Here $\deg P_{2n,i}(z) \leq n-1$, $\deg P_{2n+1,i}(z) \leq n$, $\operatorname{ord} R_{2n}(z) = 2n-1$, and $\operatorname{ord} R_{2n+1}(z) = 2n$. Thus the polynomials $P_{n,i}$ are Padé approximations for R_0 and R_1. For these polynomials we have the following non-vanishing result, see Lemma 1 of [7].

LEMMA 3.1. *If the assumptions of Theorem 1.3 are valid, then*

$$\delta(\xi, n) := \begin{vmatrix} P_{n,0}(\xi) & P_{n,1}(\xi) \\ P_{n+1,0}(\xi) & P_{n+1,1}(\xi) \end{vmatrix} \neq 0$$

for all $n \geq 0$.

PROOF. By the above definition of $P_{n,i}$ we have $\delta(\xi, 0) = 1$, $\delta(\xi, 1) = (1-\alpha_1)$, and $\delta(\xi, n) = \alpha_{n-1}(1-\alpha_n)\xi\delta(\xi, n-1)$, $n \geq 2$. This proves Lemma 3.1.

LEMMA 3.2. *If the assumptions of Theorem 1.3 are valid, then, for all places w,*

$$\max(\|P_{2n,i}(\xi)\|_w, \|P_{2n+1,i}(\xi)\|_w) \leq C_1(w)^n \|q\|_w^{*n(n-1)},$$

where

$$C_1(w) = 4^{\delta_w}\|a\|_w^*\|b\|_w^*\|\xi\|_w^*,$$

and here $\delta_w = 0$, if w is finite, and $\delta_w = 1$ for infinite w.

PROOF. Clearly $|P_{0,i}(\xi)|_w \leq 1$, $|P_{1,i}(\xi)|_w \leq 1$, $|P_{2,i}(\xi)|_w \leq 2^{\delta_w}|a|_w^*$, $|P_{3,i}(\xi)|_w \leq 2^{2\delta_w}|a|_w^*|b|_w^*|\xi|_w^*$. The recursion (14) gives

$$P_{2n+2,i}(\xi) = P_{2n+1,i}(\xi) - bq^{n-1}(1-aq^n)\xi P_{2n,i}(\xi)$$

and

$$P_{2n+3,i}(\xi) = P_{2n+2,i}(\xi) - aq^n(1-bq^n)\xi P_{2n+1,i}(\xi),$$

and therefore we can see, by induction, that

$$|P_{2n,i}(\xi)|_w \le 2^{(2n-1)\delta_w}|a|_w^{*n}|b|_w^{*(n-1)}|\xi|_w^{*(n-1)}|q|_w^{*(n-1)^2},$$
$$|P_{2n+1,i}(\xi)|_w \le 2^{2n\delta_w}|a|_w^{*n}|b|_w^{*n}|\xi|_w^{*n}|q|_w^{*n(n-1)}.$$

This proves Lemma 3.2.

4. Improved approximations

In this section we shall use iteration of (4) to improve linear forms $R_n(z)$ in (13). For this we write (4) to the form (3)

$$\begin{aligned} qzf(q^2z) = {} & \frac{q}{ab}((a+b)z-1)f(qz) + \frac{q}{ab}(1-z)f(z) \\ & =: -T(z)f(qz) + S(z)f(z). \end{aligned}$$

It is proved in Section 4 of [10] that the iteration of this functional equation gives, for each $k = 1, 2, \ldots$, polynomials T_k and S_k in $K[z]$ such that

$$z^{k+1}q^{\binom{k+2}{2}}f(q^{k+2}) =: T_k(z)f(qz) + S_k(z)f(z) \tag{15}$$

and the following lemma holds.

LEMMA 4.1. *For all w*

$$\max\{\|T_k(\xi)\|_w\,,\,\|S_k(\xi)\|_w\} \le C_2(w)^k\,\|q\|_w^{*k^2/2}\,,$$

where

$$C_2(w) = 8^{\delta_w}\|\underline{\Lambda}\|_w^*\|\xi\|_w^{*3}\|q\|_w^*, \qquad \underline{\Lambda} = \frac{q}{ab}(1, a+b).$$

Furthermore the determinant

$$\gamma(\xi,k) := \begin{vmatrix} S_{k-1}(\xi) & T_{k-1}(\xi) \\ S_k(\xi) & T_k(\xi) \end{vmatrix} \ne 0.$$

The claim concerning the determinant follows immediately from the recursion

$$\gamma(\xi,k) = -\xi q^k S(q^k\xi)\gamma(\xi,k-1),$$

$k = 1, 2, \ldots, \gamma(\xi, 0) = -S(\xi) \ne 0$, see [10].

We next construct improved approximations by applying the above iterations to (13). Let

$$r_{n,k}(z) = R_n(q^{k+1}z). \tag{16}$$

Then

$$\begin{aligned}
&z^{k+1}q^{\binom{k+2}{2}}\begin{pmatrix} r_{2n,k}(z) \\ r_{2n+1,k}(z)\end{pmatrix}\\
&\quad = z^{k+1}q^{\binom{k+2}{2}}\begin{pmatrix} P_{2n,0}(q^{k+1}z) & P_{2n,1}(q^{k+1}z) \\ P_{2n+1,0}(q^{k+1}z) & P_{2n+1,1}(q^{k+1}z)\end{pmatrix}\begin{pmatrix} R_0(q^{k+1}z) \\ R_1(q^{k+1}z)\end{pmatrix}\\
&\quad = \begin{pmatrix} P_{2n,0}(q^{k+1}z) & P_{2n,1}(q^{k+1}z) \\ P_{2n+1,0}(q^{k+1}z) & P_{2n+1,1}(q^{k+1}z)\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & -a\end{pmatrix}\\
&\qquad\quad \cdot \begin{pmatrix} zq^{k+1}S_{k-1}(z) & zq^{k+1}T_{k-1}(z) \\ S_k(z) & T_k(z)\end{pmatrix}\begin{pmatrix} f(z) \\ f(qz)\end{pmatrix}\\
&\quad =: \begin{pmatrix} p_{2n,0,k}(z) & p_{2n,1,k}(z) \\ p_{2n+1,0,k}(z) & p_{2n+1,1,k}(z)\end{pmatrix}\begin{pmatrix} f(z) \\ f(qz)\end{pmatrix}.
\end{aligned}$$

Here

$$\begin{aligned}
\Delta_{n,k} &:= \begin{vmatrix} p_{2n,0,k}(\xi) & p_{2n,1,k}(\xi) \\ p_{2n+1,0,k}(\xi) & p_{2n+1,1,k}(\xi)\end{vmatrix}\\
&= -a\xi q^{k+1}\delta(q^{k+1}\xi, 2n)\gamma(\xi, k) \neq 0
\end{aligned} \tag{17}$$

by Lemmas 3.1 and 4.1.

We now denote

$$\begin{aligned}
r_{2n} &= \xi^{k+1}q^{\binom{k+2}{2}}r_{2n,k}(\xi), & p_{2n,i} &= p_{2n,i,k}(\xi),\\
r_{2n+1} &= \xi^{k+1}q^{\binom{k+2}{2}}r_{2n+1,k}(\xi), & p_{2n+1,i} &= p_{2n+1,i,k}(\xi),
\end{aligned}$$

where k has the value $k = [\rho n]$ with some $\rho > 0$. By using these notations we obtain, for all $n = 0, 1, \ldots$, linear forms

$$\begin{aligned}
r_{2n} &= p_{2n,0}f(\xi) + p_{2n,1}f(q\xi),\\
r_{2n+1} &= p_{2n+1,0}f(\xi) + p_{2n+1,1}f(q\xi),
\end{aligned} \tag{18}$$

with the following properties.

LEMMA 4.2. *The linear forms r_{2n} and r_{2n+1} are linearly independent and, for all places w,*

$$\max\{\|p_{2n,i}\|_w, \|p_{2n+1,i}\|_w\} \le C_3(w)C_4(w)^n \|q\|_w^{*An^2},$$

where

$$A = A(\rho) = 1 + \rho + \rho^2/2, \quad C_3(w) = 4^{\delta_w}\|a\|_w^*, \quad C_4(w) = C_1(w)C_2(w)^\rho.$$

Furthermore, for all $n \ge n_0$, we have

$$\max\{\|r_{2n}\|_v, \|r_{2n+1}\|_v\} \le c_5 C_5(v)^n \|q\|_v^{Bn^2}$$

with $B = B(\rho) = 1 + 2\rho + \rho^2/2$, $c_5 = \max\{c_1, c_2\}$ and

$$C_5(v) = \|a\|_v^*\|b\|_v^*\|\xi\|_v^{*2+\rho}\|q\|_v^{-(2+\rho/2)}.$$

PROOF. The linear independence of r_{2n} and r_{2n+1} follows from (17), and the estimate for $\|p_{2n,i}\|_w$ and $\|p_{2n+1,i}\|_w$ is a consequence of Lemmas 3.2 and 4.1 together with the definition of $p_{2n,i}$ and $p_{2n+1,i}$. Finally, the last estimate is obtained by Lemma 2.1, (16) and (18).

5. Proof of Theorem 1.3 and the corollaries

Let the assumptions of Theorem 1.3 hold and let

$$L = A_0 f(\xi) + A_1 f(q\xi),$$

where $\underline{A} = (A_0, A_1) \in K^2$ is nonzero. By Lemma 2.2 at least one of the numbers $f(\xi)$ and $f(q\xi)$ is different from zero, say $f(\xi) \ne 0$. We then apply Lemma 4.2, which implies that at least one of Δ_{2n} and Δ_{2n+1} is nonzero, if

$$\Delta_n := \begin{vmatrix} A_0 & A_1 \\ p_{n,0} & p_{n,1} \end{vmatrix}.$$

We may assume that $\Delta_{2n} \ne 0$. Then

$$\Delta_{2n} f(\xi) = p_{2n,1}L - A_1 r_{2n} \tag{19}$$

with $\Delta_{2n} f(\xi) \ne 0$. By using Lemma 4.2 with the special choice $\rho = \sqrt{2}$ and considering the equation (19) in a standard way, see e.g. the proof of Theorem 6.1 in [2], the proof of Theorem 3.3 in [9], or the proof of Theorem 1 in [13], we obtain the estimate

$$|L|_v > |\underline{A}|_v^* H^{-\frac{d}{d_v}\frac{B(\sqrt{2})}{B(\sqrt{2})+\lambda A(\sqrt{2})} - c_0/\sqrt{\log H}}$$

giving the truth of Theorem 1.3.

In Corollary 1.5 $\lambda = -1$ and $|\underline{A}|^* = \max\{|A_0|, |A_1|\}$, which gives the truth of Corollary 1.5. This proves Corollary 1.6, too. Also in Corollary 1.7 $\lambda = -1$, and the truth of this corollary follows, since $|\underline{A}|_p^* = 1$.

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