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# STRICT U-IDEALS AND U-SUMMANDS IN BANACH SPACES

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**Abstract**

For a strict u-ideal $X$ in a Banach space $Y$ we show that the set of points in the dual unit ball $B_{X^*}$, strongly exposed by points in the range $TY$ of the unconditional extension operator $T$ from $Y$ into the bidual $X^{**}$ of $X$, is contained in the weak* denting points in $B_{X^*}$. We also prove that a u-embedded space is a u-summand if and only if it contains no copy of $c_0$ if and only if it is weakly sequentially complete.

## 1. Introduction

The concepts of a u-ideal and a u-summand were introduced and deeply studied by Godefroy, Kalton, and Saphar in their seminal paper [7]. These structures have also been further studied in later papers (cf. e.g. [2], [3], [12], and [1]).

Let $X$ be a closed subspace of a Banach space $Y$. If the annihilator $X^{\perp}$ of $X$ is the kernel of a norm one projection $P$ on the dual $Y^*$ of $Y$, $X$ is said to be an ideal in $Y$. If $X$ is an ideal in $Y$ and $P$ an associated projection on $Y^*$ with kernel $X^{\perp}$, we can, following [7], define an operator $T$ from $Y$ to $X^{**}$ by

$$(1) \qquad (i_X)^* y^*(Ty) = Py^*(y) \qquad \text{for every} \quad y \in Y, y^* \in Y^*.$$

(Here $i_X$ denotes the canonical embedding of $X$ into $Y$.) It is straightforward to show that this $T$ is in $\mathscr{E}(Y, X^{**})$, the set of all bounded linear operators of norm one from $Y$ into $X^{**}$ being the identity on $X$. (We will refer to $\mathscr{E}(Y, X^{**})$ as the set of extension operators from $Y$ into $X^{**}$). Vice versa if we start with some $T$ in $\mathscr{E}(Y, X^{**})$, then (1) defines a norm one projection $P$ on $Y^*$ with kernel $X^{\perp}$. Thus $X$ is an ideal in $Y$ exactly when $\mathscr{E}(Y, X^{**})$ is non-empty. An ideal $X$ in $Y$ is said to be a strict ideal if there is an isometric $T \in \mathscr{E}(Y, X^{**})$.

A space $X$ is called a u-summand in $Y$ if $X$ is complemented in $Y$ by a projection $P$ with $\|I_Y - 2P\| = 1$. (Here $I_Y$ denotes the identity operator on $Y$). It is useful to note that $X$ is a u-summand in $Y$ if and only if there is a subspace $Z$ of $Y$ so that $X \oplus Z = Y$ and $\|x + z\| = \|x - z\|$ for every $x \in X$ and $z \in Z$. When $X^{\perp}$ is a u-summand in $Y^*$, we say that $X$ is a u-ideal in $Y$.

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The associated projection $P$ on $Y^*$ with kernel $X^\perp$ and with $\|I_{Y^*} - 2P\| = 1$, is called a u-projection on $Y^*$. The $T$ in (1) corresponding to this u-projection $P$ is called an unconditional extension operator. There is only one u-projection on $Y^*$ with kernel $X^\perp$ or equivalently only one unconditional extension operator $T \in \mathscr{E}(Y, X^{**})$ (see [7, Lemma 3.1] or [2, Proposition 2.2]). A strict u-ideal $X$ in $Y$ is a u-ideal where the unconditional extension operator $T \in \mathscr{E}(Y, X^{**})$ is isometric. In the important case when $X$ is a u-ideal in $X^{**}$, we simply say that $X$ is u-embedded.

Let $X$ be a strict u-ideal in $Y$ with (isometric) unconditional extension operator $T \in \mathscr{E}(Y, X^{**})$. In Section 2 we show that the set $T - str.exp.B_{X^*}$ of points in $B_{X^*}$, strongly exposed by some $Ty$ where $\|y\| = 1$, is contained in the weak* denting points in $B_{X^*}$. See Section 2 for definitions of weak* denting and strongly exposed points.

In Section 3 we show that if $X$ is a strict ideal in $Y$ and we assume that the associated projection $P$ on $Y^*$ with kernel $X^\perp$ satisfies $\|I_{Y^*} - \lambda P\| = a < \lambda$ for $1 < \lambda \leq 2$, then the characteristic $r(\ker Ty)$ (see Section 3 for the definition) of the kernel $\ker Ty$ of $Ty$ is $\leq 1/2$ when $y \in Y \setminus \{0\}$. As a consequence of this we obtain that $r(Z) \leq 1/2$ for every proper closed subspace of the dual of a strictly u-embedded space $X$.

In Section 4 we show that if $X$ contains no copy of $\ell_1$ then the equality $\|I_{X^{***}} - 2\pi_{X^{***}}\| = 1$ holds if and only if the equality $\|I_{Ba(X)^*} - 2\pi_{Ba(X)^*}\| = 1$ holds. Here $\pi_{X^{***}}$ and $\pi_{Ba(X)^*}$ denote respectively the canonical projections on $X^{***}$ and $Ba(X)^*$ where $Ba(X)$ is the Banach space of Baire-one functions in $X^{**}$.

In Section 5 we show that a u-embedded space is a u-summand if and only if it contains no copy of $c_0$ if and only if it is weakly sequentially complete, hence extending a result only known to hold for order continuous Banach lattices (being u-embedded spaces) [7, Example (1) p. 26].

The notation is mostly standard. When some notation or term is used which we do not think is standard or self explanatory, we explain its meaning there and then.

## 2. Unique extension and strict u-ideals

Let us first recall some definitions that we need below. A slice of $B_X$ is a subset $S(B_X, x^*, \varepsilon)$ of $B_X$ defined by

$$S(B_X, x^*, \varepsilon) = \Big\{x \in B_X : x^*(x) > \sup_{y \in B_X} x^*(y) - \varepsilon\Big\},$$

where $x^* \in X^* \setminus \{0\}$ and $\varepsilon > 0$. If $X$ is a dual space we can speak of a weak* slice when $x^*$ is weak* continuous. A point $x$ in $B_X$ is called a denting point, and we write $x \in dent.B_X$, if there is a sequence of slices $S_n$ of $B_X$ with $x \in S_n$, for all $n$, and $\operatorname{diam}(S_n) \to 0$ where $\operatorname{diam}(S_n)$ is the diameter of $S_n$. If we can choose a fixed $x^* \in X^* \setminus \{0\}$ and the slices $S_n$ to be of the form $S_n = S(B_X, x^*, 1/n)$, then $x$ is called a strongly exposed point of $B_X$, and we write $x \in str.exp.B_X$. When $X$ is a dual space, we say that $x$ in $B_X$ is a weak* denting (resp. weak* strongly exposed) point of $B_X$, and we write $x \in w^* - dent.B_X$ (resp. $x \in w^* - str.exp.B_X$), if the $S_n$'s are weak* slices. By definition $w^* - str.exp.B_X \subset w^* - dent.B_X$.

Now let us motivate the results of this section. To begin with note that the class of strictly u-embedded spaces properly contains the much studied class of M-embedded spaces (cf. e.g. [10, Chapter III] and [7, Example (6) p. 29]).

We know that strictly u-embedded spaces have the unique extension property (*UEP*), i.e., $\mathscr{E}(X^{**}, X^{**}) = \{I_{X^{**}}\}$ (see [12, Theorem 2.3 and Remark 2.1]). At the same time the strictly u-embedded spaces are not Hahn-Banach smooth, i.e., they do not in general have the property that every $x^* \in X^*$ has a unique norm-preserving extension to $X^{**}$ [7, Example (6) p. 29]. (Note that by (1) Hahn-Banach smoothness implies the *UEP*) However, recently it was proved that $str.exp.B_{X^*} \subset w^* - dent.B_{X^*}$ [12, Proposition 3.1] for strictly u-embedded spaces $X$ which implies (2) that every $x^* \in str.exp.B_{X^*}$ has a unique norm-preserving extension to $X^{**}$ since this is well known to hold for every $x^* \in w^* - dent.B_{X^*}$. Using the proof of [12, Proposition 3.1] almost verbatim we obtain the following extension.

PROPOSITION 2.1. *Let $X$ be a strict u-ideal in $Y$ and let $T \in \mathscr{E}(Y, X^{**})$ be the associated unconditional extension operator. Then $T - str.exp.B_{X^*} \subset w^* - dent.B_{X^*}$. In particular every $x^* \in T - str.exp.B_{X^*}$ has a unique norm-preserving extension to $X^{**}$.*

PROOF. Let $x^* \in T - str.exp.B_{X^*}$ and let $Ty$ be a strongly exposing functional for $x^*$. Let $\varepsilon > 0$ and choose $\delta_0 > 0$ such that $\{u^* \in B_{X^*} : Ty(u^*) > 1 - \sqrt{\delta_0}\} \subset B_{X^*}(x^*, \varepsilon)$ and $1 + \varepsilon\delta_0 > 2\sqrt{\delta_0}(1 + \varepsilon)$.

Let $\delta \in (0, \delta_0)$. Then $1 + \varepsilon\delta > 2\sqrt{\delta}(1 + \varepsilon)$, which is equivalent to $2(1 - \delta)/(1 + \varepsilon\delta) - 2 + \sqrt{\delta} > 0$. Choose $\eta > 0$ with $0 < \eta < 2(1 - \delta)/(1 + \varepsilon\delta) - 2 + \sqrt{\delta}$ and $\{u^* \in B_{X^*} : Ty(u^*) > 1 - \eta\} \subset B_{X^*}(x^*, \varepsilon\delta/(1 + \varepsilon\delta))$.

Since $X$ is a strict u-ideal in $Y$ we have $1 = \inf_{x \in S_X} \|y - 2x\|$ [12, Theorem 2.4]. Choose $x \in S_X$ such that $\|y - 2x\| < 1 + \eta$. Choose $u^* \in B_{X^*}$ such that $u^*(x) = 1$. Then

$$1 + \eta > \|y - 2x\| = \|Ty - 2x\| \geq u^*(2x - Ty) = 2 - Ty(u^*).$$

Thus $Ty(u^*) > 1 - \eta$. It follows that $\|u^* - x^*\| < \varepsilon\delta/(1 + \varepsilon\delta)$.

Let $u = x/x^*(x)$. Then $x^*(x) \geq u^*(x) - \|x^* - u^*\| > 1/(1 + \varepsilon\delta)$ so $\|u\| = 1/x^*(x) \leq 1 + \varepsilon\delta$. If $z^* \in B_{X^*}$ and $z^*(u) > 1 - \delta$, then $z^*(x) =$

$z^*(u)x^*(x) > (1-\delta)x^*(x)$. Hence

$$1+\eta > \|Ty-2x\| \geq z^*(2x-Ty) \geq 2(1-\delta)x^*(x) - Ty(z^*),$$

and $Ty(z^*) > 2(1-\delta)x^*(x)-1-\eta \geq \frac{2(1-\delta)}{1+\varepsilon\delta}-1-\eta$. But then $Ty(z^*) > 1-\sqrt{\delta}$, from which it follows that $\|z^*-x^*\| < \varepsilon$. Thus $x^*$ is contained in the weak* slices of arbitrarily small diameter, i.e. $x^*$ is weak* denting.

The last part follows from (2).

Recall that the space $Ba(X)$ of Baire-one functions in $X^{**}$ consists of members $x^{**} \in X^{**}$ which are weak* limits of sequences in $X$. If $X$ is a strict u-ideal in $Y$, $X^*$ need not have the Radon-Nikodým property (*RNP*) unlike when $Y = X^{**}$ [13, Proposition 4.1]). For example if we put $X = c_0 \oplus_\infty \ell_1$, then $X$ is a strict u-ideal in $Ba(X)$ [1, Remark 4.5], but of course $X^* = \ell_1 \oplus_1 \ell_\infty$ does not have the *RNP*. But

Corollary 2.2. *Let $X$ be a strict u-ideal in $Y$ and let $T \in \mathscr{E}(Y, X^{**})$ be the associated unconditional extension operator. If $X^*$ has the RNP, then*

$$\begin{aligned}\overline{\text{conv}}^{\|\cdot\|}(w^*-str.exp.B_{X^*}) &= \overline{\text{conv}}^{\|\cdot\|}(T-str.exp.B_{X^*})\\ &= \overline{\text{conv}}^{\|\cdot\|}(w^*-dent.B_{X^*}).\end{aligned}$$

Proof. Choose $x^* \in w^*-dent.B_{X^*}$. Since $\overline{\text{conv}}^{w^*}(w^*-str.exp.B_{X^*}) = B_{X^*}$ [5], we can find a net $(x_\alpha)$ in $\overline{\text{conv}}^{\|\cdot\|}(w^*-str.exp.B_{X^*})$ such that $x^*_\alpha \to x^*$ weak*. Since $x^*$ is a weak* denting point we get that $\|x^*-x^*_\alpha\| \to 0$. Thus $\overline{\text{conv}}^{\|\cdot\|}(w^*-str.exp.B_{X^*}) = \overline{\text{conv}}^{\|\cdot\|}(w^*-dent.B_{X^*})$. Since $w^*-str.exp.B_{X^*} \subset T-str.exp.B_{X^*} \subset w^*-dent.B_{X^*}$ the result follows.

Remark 2.3. Note that if the conditions in Corollary 2.2 hold and $\overline{\text{conv}}^{\|\cdot\|}(w^*-str.exp.B_{X^*}) = B_{X^*}$, then $X$ has the unique ideal property in $Y$. This is e.g. the case when $X$ is strictly u-embedded (see [13, Proposition 4.1]).

In the general case when $X$ is a strict u-ideal in $Y$, far less is known than in the special case when $Y = X^{**}$. For example it is not known whether a strict u-ideal $X$ in $Y$ has the unique ideal property, i.e., $\mathscr{E}(Y, X^{**})$ consists of a singleton. In some special cases one can however prove that a strict u-ideal has the unique ideal property.

Proposition 2.4. *If $X$ is a strict ideal in $Y$ with corresponding $T \in \mathscr{E}(Y, X^{**})$ and the norm on $X$ is Fréchet differentiable on $X \setminus \{0\}$, then $X$ has the unique ideal property in $Y$.*

Proof. Suppose $X$ is a strict ideal in $Y$ with corresponding $T \in \mathscr{E}(Y, X^{**})$ and the norm on $X$ is Fréchet differentiable on $X \setminus \{0\}$. Then by the smoothness condition it follows that $X$ is nicely smooth, i.e. for every $x^{**} \in X^{**}$ we have $\cap_{x\in X} B_{X^{**}}(x, \|x^{**}-x\|) = \{x^{**}\}$ [9, Examples p. 118]. Since $T$ is isometric, we have for every $y \in Y$ that $\{Ty\} = \cap_{x\in X} B_{X^{**}}(x, \|Ty-x\|) = \cap_{x\in X} B_{X^{**}}(x, \|y-x\|)$. Now if $S \in \mathscr{E}(Y, X^{**})$ we get that $\|Sy - x\| \le \|y - x\|$ for every $x \in X$ and $y \in Y$. Thus $Sy = Ty$, as wanted.

## 3. Norming subspaces and strict (u-)ideals

Recall that a subspace $Z$ of the dual $X^*$ of $X$ is $r$-norming for $X$ if $\sup_{x^*\in S_{X^*}\cap Z} |x^*(x)| \ge r\|x\|$ for every $x \in X$. The characteristic $r(Z)$ of $Z$ is defined to be the greatest such constant $r$. In [7, Propositions 2.7 and 5.2] Godefroy et al. showed that the characteristic of every proper closed subspace of the dual of a strictly u-embedded space $X$ is $\le 1/2$ when $X$ does not contain $\ell_1$ (See also [8, Lemma 4.1] for M-embedded spaces). This result, but with no restrictions on $X$, follows from the proof of Proposition 3.1 below. The proposition is inspired by [7, Proposition 2.7].

PROPOSITION 3.1. *Let $X$ be a strict ideal in $Y$ with corresponding ideal projection $P$ on $Y^*$ and associated isometric extension operator $T \in \mathscr{E}(Y, X^{**})$. Suppose $1 < \lambda \le 2$ and that $\|I_{Y^*} - \lambda P\| = a < \lambda$. Then $r(\ker Ty) \le a\lambda^{-1}$ for every $y \in Y \setminus \{0\}$.*

PROOF. Let $\varepsilon > 0$, $y \in S_Y$, and put $M = \ker Ty \subset X^*$. From [7, Lemma 2.2] there is a net $(x_\alpha)$ in $B_X$ which tends to $Ty$ weak* in $X^{**}$ and with $\limsup_\alpha \|y - \lambda x_\alpha\| \le a$. We have

$$\lambda \sup_{x^*\in S_M} |x^*(x_\alpha)| \le \|Ty - \lambda x_\alpha\| \le \|y - \lambda x_\alpha\|.$$

Since $\|Ty\| = \|y\| = 1$, there is an $\alpha_1$ such that $1 \ge \|x_{\alpha_1}\| > 1 - \varepsilon$ and $\|y - \lambda x_{\alpha_1}\| < a + \varepsilon$. Thus $\sup_{x^*\in S_M} \left|x^*\left(\frac{x_{\alpha_1}}{\|x_{\alpha_1}\|}\right)\right| \le \frac{a+\varepsilon}{\lambda(1-\varepsilon)}$. Since $\varepsilon$ is arbitrary we are done.

COROLLARY 3.2. *Let $X$ be a strict ideal in $X^{**}$ with corresponding ideal projection $P$ on $X^{***}$. Suppose $1 < \lambda \le 2$ and that $\|I_{X^{***}} - \lambda P\| = a < \lambda$. Then for every proper closed subspace $Z$ of $X^*$ we have $r(Z) \le a\lambda^{-1}$.*

PROOF. Since every proper closed subspace of $X^*$ is contained in $\ker x^{**}$ for some $x^{**} \in X^{**} \setminus \{0\}$, it suffices to prove that for every $x^{**} \in X^{**} \setminus \{0\}$ we have $r(\ker x^{**}) \le a\lambda^{-1}$. But this follows by arguing as in the proof of Proposition 3.1.

COROLLARY 3.3. *Let $X$ be strictly u-embedded. Then for every proper closed subspace $Z$ of $X^*$ we have $r(Z) \le 1/2$.*

Recall from the previous section that strictly u-embedded spaces have the *UEP*. Here is an alternative proof of this fact.

COROLLARY 3.4. *Strictly u-embedded spaces have the UEP.*

PROOF. This follows from [8, Proposition 2.5] since $r(Z) \leq 1/2$ for every proper closed subspace $Z$ of $X^*$.

## 4. Strict u-ideals in the space of Baire-one functions

In [12] Lima and Lima characterized when a Banach space $X$ is strictly u-embedded both in terms of the canonical projection $\pi_{X^{***}}$ on $X^{***}$ onto $X^*$ and in terms of subspaces of $X$. (Note that $\pi_{X^{***}} = k_{X^*}(k_X)^*$ where $k_X$ is the canonical embedding of $X$ into $X^{**}$ defined by $k_X x(x^*) = x^*(x)$.)

THEOREM 4.1 ([12, Theorems 2.8 and 2.9]). *Let X be a Banach space. Then the following statements are equivalent.*

a) $\|I_{X^{***}} - 2\pi_{X^{***}}\| = 1$.

b) *X is strictly u-embedded.*

c) *For every separable subspace Z in X, Z is strictly u-embedded.*

The aim of this section is to show that when $X$ does not contain a copy of $\ell_1$, then we can add to this theorem the statement that $\|I_{Ba(X)^*} - 2\pi_{Ba(X)^*}\| = 1$. Here $\pi_{Ba(X)^*}$ is the canonical projection on $Ba(X)^*$ defined by $\pi_{Ba(X)^*} = i_{X^*}(i_X)^*$ where $i_X : X \to Ba(X)$ and $i_{X^*} : X^* \to Ba(X)^*$ denote the canonical embeddings. Note that $i_{X^*}x^*(x^{**}) = x^{**}(x^*)$ for every $x^* \in X^*$ and $x^{**} \in Ba(X)$. In the proof of Proposition 4.2 we will use that the $T$ in $\mathscr{E}(Ba(X), X^{**})$ associated with $\pi_{Ba(X)^*}$ is just the canonical embedding $i_{Ba(X)}$ of $Ba(X)$ into $X^{**}$.

Note that the statement $\|I_{Ba(X)^*} - 2\pi_{Ba(X)^*}\| = 1$ implies that $X$ is a strict u-ideal in $Ba(X)$ [7, Lemma 2.2]. If $X$ is separable also the converse of this holds [1].

We need a preliminary result inspired by [7, Proposition 2.3].

PROPOSITION 4.2. *Let X be a Banach space and suppose K is a compact subset of* $\mathsf{R}$ *and* $a > 0$. *Then the following conditions are equivalent.*

a) $\|I_{Ba(X)^*} - \lambda\pi_{Ba(X)^*}\| \leq a$ *for every* $\lambda \in K$.

b) *Whenever* $\varepsilon > 0$, $x^{**} \in Ba(X)$ *and* $(x_n)$ *is a sequence in X such that* $x_n \to x^{**}$ *weak* in* $X^{**}$ *then there exists* $u \in \operatorname{conv}(x_n)_{n\geq 1}$ *with* $\|x^{**} - \lambda u\| < a\|x^{**}\| + \varepsilon$ *for every* $\lambda \in K$.

c) *For every* $x^{**} \in Ba(X)$, *there is a sequence* $(x_n)$ *in X such that* $x_n \to x^{**}$ *weak* in* $X^{**}$ *and with* $\limsup_n \|x^{**} - \lambda x_n\| \leq a\|x^{**}\|$ *for every* $\lambda \in K$.

PROOF. a) $\Rightarrow$ b) follows from [7, Lemma 2.2 (1) $\Rightarrow$ (2)] by putting $A = \mathrm{conv}(x_n)_{n\geq 1}$ since $T \in \mathscr{E}(Ba(X), X^{**})$ associated with $\pi_{Ba(X)^*}$ is $i_{Ba(X)}$.

b) $\Rightarrow$ c). Let $x^{**} \in Ba(X)$, remember that $i_{Ba(X)}(x^{**}) = x^{**}$, and choose a sequence $(x_n)$ in $X$ such that $x_n$ tends to $x^{**}$ weak* in $X^{**}$. Put $A_1 = \mathrm{conv}(x_n)_{n\geq 1}$. Then $x^{**}$ is in the weak* closure of $A_1$ and by [7, Lemma 2.2 (2)] we can find $u_1 \in A_1$ such that $\|x^{**} - \lambda u_1\| < a\|x^{**}\| + 2^{-1}$ for every $\lambda \in K$. Now, let $A_2 = \mathrm{conv}(x_k)_{k\geq N_2}$ where $N_2$ is the largest index of $x_k$ where $x_k$ is in the convex combination of $u_1$. Still $x^{**}$ is in the weak* closure of $A_2$ and by [7, Lemma 2.2 (2)] we can find $u_2 \in A_2$ such that $\|x^{**} - \lambda u_2\| < a\|x^{**}\| + 2^{-2}$ for every $\lambda \in K$. By repeating this process, we obtain a sequence $(u_n)$ in $X$ such that $u_n \to x^{**}$ weak* in $X^{**}$ with $\limsup_n \|x^{**} - \lambda u_n\| \leq a\|x^{**}\|$ for every $\lambda \in K$.

c) $\Rightarrow$ a) follows from [7, Lemma 2.2 (3) $\Rightarrow$ (1)].

Let us now state and prove our result.

PROPOSITION 4.3. *Let $X$ be a Banach space not containing $\ell_1$. Then the following are equivalent.*

a) $\|I_{X^{***}} - 2\pi_{X^{***}}\| = 1$.

b) $\|I_{Ba(X)^*} - 2\pi_{Ba(X)^*}\| = 1$.

PROOF. a) $\Rightarrow$ b). This is immediate from [7, Proposition 2.3 (1) $\Rightarrow$ (2)] and Proposition 4.2 b) $\Rightarrow$ a).

b) $\Rightarrow$ a). By Theorem 4.1 it suffices to prove that every separable subspace $Z$ in $X$ is strictly u-embedded. To this end let $z^{**} \in Ba(Z)$ and choose a sequence $(z_n)$ in $Z$ such that $z_n \to z^{**}$ weak* in $Z^{**}$. Note that $x^{**} = (i_Z)^{**}z^{**} \in Ba(X)$ ($i_Z : Z \to X$ is the embedding of $Z$ into $X$) and $z_n \to x^{**}$ weak* in $X^{**}$ by the weak* continuity of $(i_Z)^{**}$. Now using the assumption and Proposition 4.2 there is some $u \in \mathrm{conv}(z_n)_{n\geq 1}$ such that $\|z^{**} - 2u\| = \|x^{**} - 2u\| < \|x^{**}\| + \varepsilon = \|z^{**}\| + \varepsilon$. Again by Proposition 4.2 it follows that $\|I_{Ba(Z)^*} - 2\pi_{Ba(Z)^*}\| = 1$. Finally, since $Z$ is separable and does not contain $\ell_1$, we have $Ba(Z) = Z^{**}$ by a result of Odell and Rosenthal [15] so $\pi_{Ba(Z)^*} = \pi_{Z^{***}}$ and we are done.

## 5. U-ideals which are u-summands

In [7] the following important result about u-ideals was proved.

THEOREM 5.1. *Let $X$ be a u-ideal in $Y$. If $X$ contains no copy of $c_0$, then $X$ is a u-summand in $Y$.*

The projection $P$ on $c_0 \oplus_1 \mathsf{R}$ defined by $P(x, y) = x$ makes $c_0$ a u-summand (in particular a u-ideal) in $c_0 \oplus_1 \mathsf{R}$. Thus the converse of Theorem 5.1 is not true. However, if we put $Y = X^{**}$, one can still ask if the converse holds.

Actually Godefroy et al. pointed out [7, Example (1) p. 26] that the converse in fact holds for order-continuous Banach lattices (being u-embedded spaces). The next result shows that this is indeed true for all Banach spaces.

THEOREM 5.2. *Let $X$ be a u-ideal in $X^{**}$. Then the following statements are equivalent.*

a) *$X$ is a u-summand in $X^{**}$.*

b) *$X$ contains no copy of $c_0$.*

c) *$X$ is weakly sequentially complete.*

PROOF. a) $\Rightarrow$ c). Let $P : X^{**} \to X^{**}$ be the u-projection onto $X$ and note that $J = 2P - I_{X^{**}}$ is an isometry onto $X^{**}$ such that $JX = X$. By [6, Proposition 9] $J$ is a homeomorphism from $(Ba(X), \text{weak}^*)$ onto $(Ba(X), \text{weak}^*)$. Thus, if $x^{**} \in Ba(X)$ and $x^{**} = \text{weak}^* - \lim x_n$, then $2Px^{**} - x^{**} = \text{weak}^* - \lim(2Px_n - x_n) = \text{weak}^* - \lim x_n = x^{**}$. Hence $x^{**} = Px^{**} \in X$.

c) $\Rightarrow$ b) follows since weak sequential completeness is inherited by subspaces and $c_0$ is not weakly sequentially complete.

b) $\Rightarrow$ a) is Theorem 5.1.

REMARK 5.3. In the more general case when $X^{**}$ is replaced by $Y$ in Theorem 5.2, the implications c) $\Rightarrow$ b) $\Rightarrow$ a) follow from Theorem 5.1. However, none of the reverse implications hold. For a) $\not\Rightarrow$ b) we have already given a counterexample. For b) $\not\Rightarrow$ c) put e.g. $X = \ell_1 \oplus_1 J$ where $J$ is the space of James [11]. Then $X$ contains no copy of $c_0$ and is certainly not weakly sequentially complete since $J$ is not. Moreover, by a result of Maurey [14], there is an $x^{**} \in X^{**}$ such that $\|x^{**} - x\| = \|x^{**} + x\|$ for all $x \in X$. It follows that $X$ is a u-summand in $Z = \text{span}(X, \{x^{**}\})$.

Recall [4] that a closed subspace $X$ of a Banach space $Y$ is said to be very non-constrained (*VNC*) in $Y$ if for all $y \in Y$,

$$\bigcap_{x \in X} B_Y(x, \|y - x\|) = \{y\}.$$

A *VNC* subspace in its bidual is often referred to as a nicely smooth space. Nicely smooth spaces have been studied in e.g. [9] and [8].

The aim of the last part of this section is to prove Proposition 5.5 and thus extend the result [1, Proposition 4.1] that if $X$ is separable and a u-ideal in $Ba(X)$ with unconditional extension operator $T \in \mathscr{E}(Ba(X), X^{**})$, then $T(Ba(X)) \subset Ba(X)$ whenever $X$ is a *VNC* subspace in $Ba(X)$.

We will use the following result in the proof of Proposition 5.5.

PROPOSITION 5.4 ([4, Lemma 2.10]). *Let $X$ be a closed subspace of a Banach space $Y$. Then the following are equivalent.*

a) *$X$ is a VNC subspace in $Y$.*

b) *For every $y \in Y \setminus \{0\}$, $\ker(y)|_X \subset X^*$ is not 1-norming for $X$.*

We need to recall also the definition of the number $\kappa_u(X)$ (see [7, pp. 22–23]); For each $x^{**} \in X^{**}$ define $\kappa_u(x^{**})$ to be the infimum over all $a$ such that if $x^{**} = \sum_n x_n$ in the weak* topology of $X^{**}$, with $x_n \in X$ and such that for any $n \in \mathbb{N}$ and $\theta_k = \pm 1$ for $1 \le k \le n$, we have $\left\| \sum_{k=1}^n \theta_k x_k \right\| \le a$. Put $\kappa_u(x^{**}) = \infty$ if no such $a$ exists. Recall that $X$ has property (u) if every $x^{**} \in Ba(X)$ has $\kappa_u(x^{**}) < \infty$. In this case it follows from the closed graph theorem that there exists a constant $C$ such that $\kappa_u(x^{**}) \le C\|x^{**}\|$ for all $x^{**} \in Ba(X)$. The least such constant is $\kappa_u(X)$.

PROPOSITION 5.5. *Let $X$ be a separable u-ideal in a Banach space $Y$ with unconditional extension operator $T \in \mathscr{E}(Y, X^{**})$ and suppose $X$ is a VNC subspace in $Ba(X)$. Then $T(Y) \subset Ba(X)$ and $\kappa_u(Ty) \le \|y\|$ for every $y \in Y$.*

PROOF. Since $X$ is separable there is a sequence $(x_i^*)_{i=1}^\infty \subset S_{X^*}$ such that $M = \overline{\operatorname{span}}\{x_i^*\}$ is 1-norming for $X$. Let $y \in Y$ and put

$$A_n = \left\{ x \in X : |Ty(x_i^*) - x(x_i^*)| < \frac{1}{n}, i = 1, 2, \dots, n \right\}.$$

Note that $A_n$ is convex and non-empty and that $Ty \in H_n$, the weak* closure of $A_n$ in $X^{**}$, for each $n$. Since $X$ is a u-ideal in $Y$, by [7, Lemma 3.4], for every $\varepsilon > 0$ there exists $\chi \in \cap_n H_n$ such that $\kappa_u(\chi) \le \|y\| + \varepsilon$. In particular, $\chi \in Ba(X)$. Since $\chi \in \cap_n H_n$, $\chi(f) = Ty(f)$ for all $f \in M$.

Now take an arbitrary $x^* \in X^*$ and put $N = \operatorname{span}\{M, \{x^*\}\}$. The same argument as above produces a Baire-one member $\chi_1 \in \cap_n H_n$ with $\chi_1(f) = Ty(f)$ for all $f \in M$ and $\chi_1(x^*) = Ty(x^*)$.

We now use that $X$ is a *VNC*-subspace of $Ba(X)$. By Proposition 5.4 $\chi_1 = \chi$ since $\ker(\chi - \chi_1)|_X \subset X^*$ contains the norming subspace $M$. Since $\chi(x^*) = \chi_1(x^*) = Ty(x^*)$, we get that $Ty = \chi \in Ba(X)$ and $\kappa_u(Ty) \le \|y\| + \varepsilon$. As $\varepsilon$ is arbitrary, we get $\kappa_u(Ty) \le \|y\|$.

COROLLARY 5.6. *Let $X$ be a separable and weakly sequentially complete Banach space. Then $X$ is a u-summand in a Banach space $Y$ whenever it is a u-ideal in $Y$.*

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