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LOGARITHMIC CONVEXITY OF AREA INTEGRAL MEANS FOR ANALYTIC FUNCTIONS

CHUNJIE WANG and KEHE ZHU

Abstract

We show that the L^2 integral mean on $r\mathbb{D}$ of an analytic function in the unit disk $\mathbb{D}$ with respect to the weighted area measure $(1-|z|^2)^\alpha\, dA(z)$, where $-3 \le \alpha \le 0$, is a logarithmically convex function of r on $(0, 1)$. We also show that the range $[-3, 0]$ for α is best possible.

1. Introduction

Let $\mathbb{D}$ denote the unit disk in the complex plane $\mathbb{C}$ and let $H(\mathbb{D})$ denote the space of all analytic functions in $\mathbb{D}$. For any real number α let

$$dA_\alpha(z) = (1-|z|^2)^\alpha\, dA(z),$$

where dA is area measure on $\mathbb{D}$.

For any $f \in H(\mathbb{D})$ and $0 < p < \infty$ we consider the weighted area integral means

$$M_{p,\alpha}(f, r) = \frac{\int_{r\mathbb{D}} |f(z)|^p\, dA_\alpha(z)}{\int_{r\mathbb{D}} dA_\alpha(z)}, \qquad 0 < r < 1.$$

It was proved in [6] that the function $r \mapsto M_{p,\alpha}(f, r)$ is strictly increasing for $r \in (0, 1)$, unless f is constant. It was also proved in [6] that for $\alpha \le -1$, the function $r \mapsto M_{p,\alpha}(f, r)$ is bounded on $(0, 1)$ if and only if f belongs to the Hardy space H^p; and for $\alpha > -1$, the function $r \mapsto M_{p,\alpha}(f, r)$ is bounded on $(0, 1)$ if and only if f belongs to the weighted Bergman space

$$A^p_\alpha = H(\mathbb{D}) \cap L^p(\mathbb{D}, dA_\alpha).$$

See [1] for the theory of Hardy spaces and [2] for the general theory of Bergman spaces in the unit disk.

The classical Hardy convexity theorem asserts that the integral means

$$M_p(f, r) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p\, dt,$$

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as a function of r on $[0, 1)$, is not only increasing but also logarithmically convex. In other words, the function $r \mapsto \log M_p(f, r)$ is convex in $\log r$. See [1] again.

Motivated by Hardy's convexity theorem and by some circumstantial evidence, Xiao and Zhu boldly proposed the following conjecture in [6]: the function $r \mapsto \log M_{p,\alpha}(f, r)$ is convex in $\log r$ when $\alpha \le 0$ and concave in $\log r$ when $\alpha > 0$.

In this paper we prove the above conjecture when $-3 \le \alpha \le 0$ and $p = 2$. The cases $\alpha = 0$ and $\alpha = -1$ are direct consequences of Hardy's convexity theorem and a theorem of Taylor in [4]; these cases were addressed in [6]. We also show that the range $[-3, 0]$ for α is best possible.

2. The case of monomials

We first consider the case when $f(z) = z^k$ is a monomial. Despite the simplicity of these functions, the verification of the logarithmic convexity of $M_{p,\alpha}(z^k, r)$ is highly nontrivial. We begin with two lemmas concerning logarithmic convexity of positive functions. The proofs are elementary.

LEMMA 2.1. *Suppose f is twice differentiable on* $(0, 1)$. *Then $f(x)$ is convex in* $\log x$ *if and only if $f(x^2)$ is convex in* $\log x$.

LEMMA 2.2. *Suppose f is positive and twice differentiable on* $(0, 1)$. *Then the function* $\log f(x)$ *is convex in* $\log x$ *if and only if*

$$D(f(x)) =: \frac{f'(x)}{f(x)} + x\frac{f''(x)}{f(x)} - x\left(\frac{f'(x)}{f(x)}\right)^2$$

is nonnegative on $(0, 1)$.

PROPOSITION 2.3. *Suppose $k \ge 0$, $-2 \le \alpha \le 0$, and $0 < p < \infty$. Then the function* $\log M_{p,\alpha}(z^k, r)$ *is convex in* $\log r$.

PROOF. The case $\alpha = 0$ follows from the classical Hardy convexity theorem and a theorem of Taylor in [4]; see [6]. For the rest of the proof we assume that $\alpha < 0$.

By polar coordinates and an obvious change of variables, we have

$$M_{p,\alpha}(z^k, r) = \frac{\int_0^{r^2} t^{pk/2}(1-t)^\alpha\,dt}{\int_0^{r^2}(1-t)^\alpha\,dt}.$$

For any nonnegative parameter λ we define

$$f_\lambda(x) = \int_0^x t^\lambda(1-t)^\alpha\,dt, \qquad 0 < x < 1. \tag{1}$$

To prove Proposition 2.3, by Lemmas 2.1 and 2.2, we need only to show

$$\Delta(\lambda, x) =: \frac{f_\lambda'}{f_\lambda} + x\frac{f_\lambda''}{f_\lambda} - x\left(\frac{f_\lambda'}{f_\lambda}\right)^2 - \left[\frac{f_0'}{f_0} + x\frac{f_0''}{f_0} - x\left(\frac{f_0'}{f_0}\right)^2\right] \geq 0 \tag{2}$$

for any $\lambda \in [0, \infty)$ and $x \in (0, 1)$. Here and throughout the paper, the derivatives $f_\lambda'(x)$ and $f_\lambda''(x)$ are taken with respect to x. Since $\Delta(0, x) = 0$, the desired result will follow if we can show that for any fixed $x \in (0, 1)$, the function $\lambda \mapsto \Delta(\lambda, x)$ is increasing on $[0, \infty)$.

To simplify notation, we are going to write $h = f_\lambda(x)$ and use h', h'', h''' to denote the various derivatives of $f_\lambda(x)$ with respect to x. On the other hand, the derivative of various functions with respect to λ will be written as $\partial/\partial\lambda$.

Since

$$h = \int_0^x t^\lambda (1-t)^\alpha \, dt, \tag{3}$$

we immediately obtain

$$h' = x^\lambda (1-x)^\alpha, \qquad h'' = (\lambda - \lambda x - \alpha x)x^{\lambda-1}(1-x)^{\alpha-1}. \tag{4}$$

We also have

$$\begin{aligned} h''' = x^{\lambda-2}(1-x)^{\alpha-2}\big[&(-\lambda + 2\lambda\alpha - \alpha + \alpha^2 + \lambda^2)x^2 \\ &+ (-2\lambda\alpha + 2\lambda - 2\lambda^2)x + (\lambda^2 - \lambda)\big]. \end{aligned}$$

On the other hand, it is easy to check that

$$\frac{\partial h}{\partial \lambda} = \int_0^x t^\lambda (1-t)^\alpha \, \log t \, dt,$$

and

$$\frac{\partial h'}{\partial \lambda} = \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial \lambda}\right) = h' \log x,$$

and

$$\frac{\partial h''}{\partial \lambda} = \frac{h'}{x} + h'' \log x.$$

In what follows we will use the notation $A \sim B$ to denote that A and B have the same sign. This differs from the customary meaning of $\sim$ but will make our presentation much easier.

Rewrite

$$\Delta(\lambda, x) = \frac{h'}{h} + x\frac{h''}{h} - x\left(\frac{h'}{h}\right)^2 - \left[\frac{f_0'}{f_0} + x\frac{f_0''}{f_0} - x\left(\frac{f_0'}{f_0}\right)^2\right].$$

Since the function inside the brackets is independent of λ, we have

$$\begin{aligned}\frac{\partial\Delta}{\partial\lambda} &= \frac{1}{h^2}\left(h\frac{\partial h'}{\partial\lambda} + xh\frac{\partial h''}{\partial\lambda} - 2xh'\frac{\partial h'}{\partial\lambda}\right) - \frac{1}{h^3}\frac{\partial h}{\partial\lambda}(hh' + xhh'' - 2x(h')^2) \\ &= \frac{1}{h^2}(hh'\log x + hh' + xhh''\log x - 2x(h')^2\log x) \\ &\quad - \frac{1}{h^3}\frac{\partial h}{\partial\lambda}(hh' + xhh'' - 2x(h')^2) \\ &= \frac{h'}{h} + \frac{1}{h^3}\left(h\log x - \frac{\partial h}{\partial\lambda}\right)(hh' + xhh'' - 2x(h')^2).\end{aligned}$$

We proceed to show that $\partial\Delta(\lambda, x)/\partial\lambda > 0$ for $\lambda > 0$, $x \in (0, 1)$, and $-2 \le \alpha < 0$. To this end, we fix $\lambda > 0$ and regard the expression

$$\frac{\partial\Delta}{\partial\lambda} = \frac{h'}{h} + \frac{1}{h^3}\left(h\log x - \frac{\partial h}{\partial\lambda}\right)(h' + xh'')\left(h - \frac{2x(h')^2}{h' + xh''}\right)$$

as a function of x. It is clear that $\alpha < 0$ and $\lambda > 0$ imply that

$$h' + xh'' \sim \lambda + 1 - (\lambda + 1 + \alpha)x > 0$$

for all $x \in (0, 1)$.

Let us consider the following two functions (with λ fixed again):

$$d_1(x) = h\log x - \frac{\partial h}{\partial\lambda},$$

and

$$d_2(x) = h - \frac{2x(h')^2}{h' + xh''} = h - \frac{2x^{\lambda+1}(1-x)^{\alpha+1}}{\lambda + 1 - (\lambda + 1 + \alpha)x}.$$

Since $d_1'(x) = h/x > 0$, we have $d_1(x) \ge d_1(0) = 0$. By direct computations,

$$\begin{aligned}d_2'(x) &= x^\lambda(1-x)^\alpha - \frac{2x^\lambda(1-x)^\alpha(\lambda + 1 - (\lambda + 2 + \alpha)x)}{\lambda + 1 - (\lambda + 1 + \alpha)x} \\ &\qquad - \frac{2(\lambda + 1 + \alpha)x^{\lambda+1}(1-x)^{\alpha+1}}{(\lambda + 1 - (\lambda + 1 + \alpha)x)^2} \\ &\sim -(\lambda+1)^2 + 2(\lambda^2 + 2\lambda + 1 + \lambda\alpha)x - (\lambda + 1 + \alpha)^2x^2 \\ &=: e_2(x).\end{aligned}$$

Note that $e_2(0) = -(\lambda+1)^2 < 0$, $e_2(1) = -\alpha(2+\alpha) \ge 0$, $d_2(0) = 0$, and $d_2(1) > 0$. Here $d_2(1) = +\infty$ when $-2 < \alpha \le -1$. It is easy to check that

$e_2'(x) > 0$ on $(0, 1)$. In the case when $\alpha = -2$, we have $e_2(x) < e_2(1) = 0$ on $(0, 1)$. Thus $d_2(x)$ is decreasing on $(0, 1)$, so that $d_2(x) < d_2(0) = 0$ on $(0, 1)$. In the other case, $e_2(x)$ has exactly one zero in $(0, 1)$, say c, so that $e_2(x) < 0$ for $x \in (0, c)$ and $e_2(x) > 0$ for $x \in (c, 1)$. Thus $d_2(x)$ is decreasing on $(0, c)$ and increasing on $(c, 1)$. This implies that $d_2(x)$ has exactly one zero in $(0, 1)$. Either way, there exists $x^* \in (0, 1]$ such that $d_2(x) > 0$ when $x^* \le x < 1$ and $d_2(x) < 0$ when $0 < x < x^*$.

If $x^* \le x < 1$, the condition $d_2(x) > 0$ implies that $\partial\Delta/\partial\lambda > 0$. If $0 < x < x^*$, the condition $d_2(x) < 0$ implies that $hh' + xhh'' - 2x(h')^2 < 0$, from which we deduce that

$$\frac{\partial \Delta}{\partial \lambda} \sim -\frac{h^2 h'}{hh' + xhh'' - 2x(h')^2} - h \log x + \frac{\partial h}{\partial \lambda} =: \delta(x).$$

Again, it follows from direct computations that

$$\begin{aligned}
\delta'(x) &= -\frac{2h(h')^2 + h^2 h''}{hh' + xhh'' - 2x(h')^2} \\
&\quad + \frac{h^2 h'(2hh'' + xhh''' - 3xh'h'' - (h')^2)}{(hh' + xhh'' - 2x(h')^2)^2} - \frac{h}{x} \\
&= \frac{h^2}{x(hh' + xhh'' - 2x(h')^2)^2} \\
&\qquad \cdot \big[-\big((h')^2 + xh'h'' + 2x^2(h'')^2 - x^2 h'h'''\big)h + x(h')^2(h' + xh'')\big] \\
&\sim \big(-(\lambda+1)^2 + (2\lambda^2 + 4\lambda + 2 + 2\lambda\alpha + \alpha)x - (\lambda + 1 + \alpha)^2 x^2\big)\, h \\
&\qquad + x^{\lambda+1}(1-x)^{\alpha+1}(\lambda + 1 - (\lambda + 1 + \alpha)x) \\
&=: \delta_1(x).
\end{aligned}$$

Here

$$\delta'(x) = \left(\frac{hh'}{hh' + xhh'' - 2x(h')^2}\right)^2 \cdot \frac{\delta_1(x)}{x(1-x)^2}. \tag{5}$$

Continuing the computations, we have

$$\begin{aligned}
\delta_1'(x) &= [2\lambda^2 + 4\lambda + 2 + 2\lambda\alpha + \alpha - 2(\lambda + 1 + \alpha)^2 x]h \\
&\qquad - 2(\lambda + 1 + \alpha)x^{\lambda+1}(1-x)^{\alpha+1}, \\
\delta_1''(x) &= -2(\lambda + 1 + \alpha)^2 h + [-\alpha + 2(\lambda + 1 + \alpha)x]x^{\lambda}(1-x)^{\alpha}, \\
\delta_1'''(x) &= -\alpha(\lambda + (\lambda + 2 + \alpha)x)x^{\lambda-1}(1-x)^{\alpha-1}.
\end{aligned}$$

Since $\alpha < 0$, $\lambda > 0$, and $\lambda + 2 + \alpha > 0$, we have $\delta_1'''(x) > 0$ for all $x \in (0, 1)$.

It is easy to see that $\delta_1''(0) = \delta_1'(0) = \delta_1(0) = 0$. With details deferred to after the proof, we also have $\delta'(0) = 0$. It then follows from elementary calculus that the functions $\delta_1''(x)$, $\delta_1'(x)$, $\delta_1(x)$, and $\delta'(x)$ are all positive on $(0, x^*)$. This shows that $\partial\Delta(\lambda, x)/\partial\lambda > 0$ for $0 < x < x^*$. Combining this with our earlier conclusion on $[x^*, 1)$, we obtain $\partial\Delta(\lambda, x)/\partial\lambda > 0$ for $x \in (0, 1)$. In particular, for any fixed $x \in (0, 1)$, the function $\lambda \mapsto \Delta(\lambda, x)$ is increasing for $\lambda \in [0, \infty)$. This completes the proof of the proposition.

In the previous paragraph, we claimed that $\delta'(0) = 0$. We deferred the details to here. L'Hopital's rule gives us

$$\lim_{x\to 0} \frac{h}{x} = 0, \qquad \lim_{x\to 0} \frac{xh'}{h} = \lim_{x\to 0} \frac{h' + xh''}{h'} = \lambda + 1.$$

Consequently,

$$\lim_{x\to 0} \frac{hh'}{hh' + xhh'' - 2x(h')^2} = \lim_{x\to 0} \frac{1}{\frac{h'+xh''}{h'} - 2\frac{xh'}{h}} = -\frac{1}{\lambda + 1}.$$

It follows from the definition of $\delta_1(x)$ that $\delta_1(x)/x \to 0$ as $x \to 0$. Therefore, by (5)) we have $\delta'(0) = 0$.

PROPOSITION 2.4. *Suppose $k \geq 0$, $-3 \leq \alpha \leq 0$, and $p = 2$. Then the function* $\log M_{2,\alpha}(z^k, r)$ *is convex in* $\log r$.

PROOF. By Proposition 2.3, the result already holds in the case $-2 \leq \alpha \leq 0$. So for the rest of the proof we assume that $-3 \leq \alpha < -2$.

We still consider the functions $\Delta(\lambda, x)$ and $\partial\Delta/\partial\lambda$. But this time we restrict our attention to $0 < x < 1$ and $\lambda_0 \leq \lambda < \infty$, where $\lambda_0 = -(\alpha + 2) > 0$. Our strategy is to show that $\Delta(\lambda_0, x) > 0$ and $\partial\Delta(\lambda, x)/\partial\lambda > 0$ for all $x \in (0, 1)$ and $\lambda \in (\lambda_0, \infty)$. This will then imply that $\Delta(\lambda, x) \geq \Delta(\lambda_0, x) > 0$ for all $\lambda \geq \lambda_0$ and $x \in (0, 1)$. In particular, we will have $\Delta(pk/2, x) > 0$ for all $k \geq 1$ and $x \in (0, 1)$, because in this case $p = 2$ and $\lambda_0 \in (0, 1]$.

For $\lambda = \lambda_0$, we have

$$h = h(x) = \int_0^x t^{-2-\alpha}(1 - t)^\alpha \, dt.$$

Changing variables from t to $1/s$, we easily obtain

$$h(x) = -\frac{1}{\alpha + 1}\left(\frac{1}{x} - 1\right)^{\alpha+1}.$$

Using the D-notation from Lemma 2.2 we get $D(h(x)) = -(\alpha+1)/(1-x)^2$ and

$$D(f_0(x)) = (\alpha+1)(1-x)^{\alpha-1}\,\frac{1-x-\alpha x-(1-x)^{\alpha+1}}{[1-(1-x)^{\alpha+1}]^2}.$$

It follows that

$$\begin{aligned}\Delta(\lambda_0, x) &= D(h(x)) - D(f_0(x))\\ &\sim [1-(1-x)^{\alpha+1}]^2 + (1-x)^{\alpha+1}[1-x-\alpha x-(1-x)^{\alpha+1}]\\ &= 1-(1+x+\alpha x)(1-x)^{\alpha+1}\\ &=: \delta_3(x).\end{aligned}$$

It is easy to check that $\delta_3'(x) > 0$ for $0 < x < 1$. Thus $\delta_3(x) > \delta_3(0) = 0$ and hence $\Delta(\lambda_0, x) > 0$ for $0 < x < 1$.

To finish the proof of the proposition, we indicate how to adapt the proof of Proposition 2.3 to show that $\partial\Delta(\lambda, x)/\partial\lambda > 0$ for $\lambda_0 < \lambda < \infty$ and $0 < x < 1$. So for the rest of this proof, we are going to use the notation from the proof of Proposition 2.3.

First, observe that the assumptions $\lambda > \lambda_0$ and $-3 \le \alpha < -2$ give $e_2'(x) > 0$ on $(0, 1)$, so that $e_2(x) \le e_2(1) = -\alpha(2+\alpha) < 0$ on $(0, 1)$. Thus $d_2(x)$ is decreasing on $(0, 1)$. But $d_2(0) = 0$, so $d_2(x)$ is always negative on $(0, 1)$. Use $x^* = 1$ in the proof of Proposition 2.3 and continue from there until the equation

$$\delta_1'''(x) = -\alpha\,[\lambda + (\lambda+2+\alpha)x]\,x^{\lambda-1}(1-x)^{\alpha-1}.$$

The assumptions $-3 \le \alpha < -2$ and $\lambda > \lambda_0$ imply that $\delta_1'''(x) > 0$ for all $x \in (0, 1)$. The rest of the proof of Proposition 2.3 remains valid here. This completes the proof of Proposition 2.4.

Finally in this section we show that the range $-3 \le \alpha \le 0$ in the case $p = 2$ is best possible.

PROPOSITION 2.5. *Suppose* $\alpha \notin [-3, 0]$ *and* $p = 2$. *Then there exist positive integers* k *such that the function* $\log M_{2,\alpha}(z^k, r)$ *is not convex in* $\log r$ *for* $r \in (0, 1)$.

PROOF. Once again we consider the function $\Delta(\lambda, x)$. We are going to show that if $\alpha \notin [-3, 0]$ then $\Delta(pk/2, x) < 0$ for certain positive integers k and x sufficiently close to 1.

First consider the case in which $\alpha > 0$. In this case,

$$\begin{aligned}\Delta(\lambda, x) &= \left(\frac{h'}{h} - \frac{f_0'}{f_0}\right) - x\left(\left(\frac{h'}{h}\right)^2 - \left(\frac{f_0'}{f_0}\right)^2\right) + x\left(\frac{h''}{h} - \frac{f_0''}{f_0}\right) \\ &\sim \left[(1-x)\left(\frac{x^\lambda}{h} - \frac{1}{f_0}\right) - x(1-x)^{\alpha+1}\left(\frac{x^{2\lambda}}{h^2} - \frac{1}{f_0^2}\right)\right] \\ &\quad + \frac{x}{hf_0}\left[(\lambda - \lambda x - \alpha x)x^{\lambda-1} f_0 + \alpha h\right] \\ &=: S_1(\lambda, x) + S_2(\lambda, x).\end{aligned}$$

The assumption $\alpha > 0$ implies that the integrals

$$h(1) = \int_0^1 t^\lambda (1-t)^\alpha \, dt, \qquad f_0(1) = \int_0^1 (1-t)^\alpha \, dt,$$

are finite and positive numbers. It follows that $\lim_{x\to 1} S_1(\lambda, x) = 0$, and

$$\lim_{x\to 1}[(\lambda - \lambda x - \alpha x)x^{\lambda-1} f_0 + \alpha h] = -\alpha \int_0^1 (1-t^\lambda)(1-t)^\alpha \, dt < 0.$$

We deduce that $\Delta(\lambda, x) < 0$ when x is sufficiently close to 1. Consequently, if $\alpha > 0$, then for any $0 < p < \infty$ and any $k > 0$, the function $\log M_{p,\alpha}(z^k, r)$ is not convex in $\log r$ for $r \in (0, 1)$.

Next we consider the case in which $\alpha < -3$. In this case, we rewrite

$$\Delta(\lambda, x) = \left(\frac{h'}{h} - \frac{f_0'}{f_0}\right) - x\left(\frac{h'}{h} - \frac{f_0'}{f_0}\right)^2 + x\frac{(1-x)^{3(\alpha+1)}}{(\alpha+1)hf_0^2}\Big[T_1(\lambda, x) + T_2(\lambda, x)\Big],$$

where

$$T_1(\lambda, x) = \frac{\lambda x^{\lambda-1}}{(1-x)^{\alpha+2}}\frac{f_0}{(1-x)^{\alpha+1}} + \frac{\alpha}{(1-x)^{\alpha+2}}\frac{h - x^\lambda f_0}{(1-x)^{\alpha+2}},$$

and

$$T_2(\lambda, x) = \frac{(\alpha+2)h - (\lambda - \lambda x + \alpha x + 2x)x^{\lambda-1} f_0}{(1-x)^{\alpha+3}}.$$

Observe that the condition $\alpha < -3$ implies that

$$h - x^\lambda f_0 = \int_0^x (t^\lambda - x^\lambda)(1-t)^\alpha \, dt \to -\infty$$

as $x \to 1$, and we can use L'Hospital's Rule to obtain the limits

$$\lim_{x\to 1} \frac{(1-x)^{\alpha+1}}{h} = \lim_{x\to 1} \frac{(1-x)^{\alpha+1}}{f_0} = -(\alpha+1), \tag{6}$$

and

$$\lim_{x\to 1} \frac{h - x^\lambda f_0}{(1-x)^{\alpha+2}} = \lim_{x\to 1} \frac{-\lambda x^{\lambda-1} f_0}{-(\alpha+2)(1-x)^{\alpha+1}} = -\frac{\lambda}{(\alpha+1)(\alpha+2)}, \tag{7}$$

and

$$\lim_{x\to 1}\left(\frac{h'}{h} - \frac{f_0'}{f_0}\right) = \lim_{x\to 1} \frac{(1-x)^{2\alpha+2}}{hf_0} \cdot \frac{x^\lambda f_0 - h}{(1-x)^{\alpha+2}} = \lambda\,\frac{\alpha+1}{\alpha+2}.$$

It follows from (6) and (7) that

$$\lim_{x\to 1} \frac{(1-x)^{3(\alpha+1)}}{(\alpha+1)hf_0^2} = -(\alpha+1)^2,$$

and $T_1(\lambda, x) \to 0$ as $x \to 1$. Since $(\alpha+1) f_0 = 1 - (1-x)^{\alpha+1}$ and $\alpha < -3$, it follows from L'Hopital's rule and elementary manipulations that

$$\lim_{x\to 1} T_2(\lambda, x) = -\frac{\lambda(\lambda-1)}{(\alpha+1)(\alpha+3)}.$$

Therefore,

$$\begin{aligned}\lim_{x\to 1} \Delta(\lambda, x) &= \lambda\,\frac{\alpha+1}{\alpha+2} - \left(\lambda\frac{\alpha+1}{\alpha+2}\right)^2 + \lambda(\lambda-1)\frac{\alpha+1}{\alpha+3}\\ &= \frac{\lambda(\alpha+1)(\lambda+2+\alpha)}{(\alpha+2)^2(\alpha+3)}.\end{aligned}$$

If $p = 2$ and $k = 1$, then for $\lambda = pk/2 = 1$ we have

$$\lim_{x\to 1} \Delta(\lambda, x) = \frac{\alpha+1}{(\alpha+2)^2} < 0.$$

This shows that $\Delta(\lambda, x) < 0$ for x sufficiently close to 1. Thus the function $\log M_{2,\alpha}(z, r)$ is not convex in $\log r$.

3. The case of $p = 2$ and arbitrary f

In this section we prove the logarithmic convexity of $M_{p,\alpha}(f, r)$ when $p = 2$ and $-3 \le \alpha \le 0$. Basically, the problem is reduced to the case of monomials because of the following well-known result; see [3].

LEMMA 3.1. *Suppose* $\{h_k(x)\}$ *is a sequence of positive and twice differentiable functions on* $(0, 1)$ *such that the function* $H(x) = \sum_{k=0}^{\infty} h_k(x)$ *is also twice differentiable on* $(0, 1)$. *If for each k the function* $\log h_k(x)$ *is convex in* $\log x$, *then* $\log H(x)$ *is also convex in* $\log x$.

We now obtain the main result of the paper.

THEOREM 3.2. *Suppose* f *is analytic in* D *and* $-3 \le \alpha \le 0$. *Then the function* $r \mapsto \log M_{2,\alpha}(f, r)$ *is convex in* $\log r$. *Moreover, the range* $-3 \le \alpha \le 0$ *is best possible.*

PROOF. Suppose $f(z) = \displaystyle\sum_{k=0}^{\infty} a_k z^k$. It follows from integration in polar coordinates that

$$M_{2,\alpha}(f, r) = \sum_{k=0}^{\infty} |a_k|^2 M_{2,\alpha}(z^k, r).$$

By Proposition 2.4, each function $h_k(r) = |a_k|^2 M_{2,\alpha}(z^k, r)$ has the property that $\log h_k(r)$ is convex in $\log r$. So by Lemma 3.1, the function $\log M_{2,\alpha}(f, r)$ is convex in $\log r$.

That the range $-3 \le \alpha \le 0$ is best possible follows from Proposition 2.5.

4. Two Examples

It was shown in [6] by an example that when $\alpha > 0$, $\log M_{p,\alpha}(f, r)$ is not always convex in $\log r$. Based on this particular example and some circumstantial evidence, it was further conjectured in [6] that if $\alpha > 0$, the function $\log M_{p,\alpha}(f, r)$ is concave in $\log r$. We show in this section that this is not so. In fact, when $\alpha = 1$ or $\alpha = -4$, we give examples such that the function $\log M_{2,\alpha}(f, r)$ is *neither* convex *nor* concave on $(0, 1)$. These examples also illustrate the somewhat abstract calculations we did in Section 2 with arbitrary monomials.

First, let $p = 2$, $\alpha = 1$, and $f(z) = 1 + z$. It follows from a direct computation that

$$M_{2,1}(1 + z, r) = \frac{2(3 - r^4)}{3(2 - r^2)}.$$

By Lemma 2.1, we just need to consider the convexity of the following function in $\log x$:

$$h(x) = \frac{3 - x^2}{2 - x}, \qquad 0 < x < 1.$$

Using the D-notation from Lemma 2.2, we have

$$D(h(x)) = \frac{2g(x)}{(2 - x)^2(3 - x^2)^2},$$

where
$$g(x) = 9 - 24x + 18x^2 - 6x^3 + x^4.$$

It is easy to check that $g''(x) = 36 - 36x + 12x^2 > 0$ for all $x \in (0, 1)$. Thus $g(x)$ is convex on [0, 1]. Since $g(0) = 9 > 0$ and $g(1) = -2 < 0$, there exists a point $c \in (0, 1)$ such that $g(x) > 0$ for $x \in (0, c)$ and $g(x) < 0$ for $x \in (c, 1)$. Thus $\log h(x)$ is neither convex nor concave in $\log x$.

We note that the functions $z + a$ have also been considered by Xiao and Xu [5] in their recent work on weighted area integral means of analytic functions and other related problems.

Next, consider the case when $p = 2$, $\alpha = -4$, and $f(z) = \sqrt{2}\,z$. It follows from a direct computation that
$$M_{2,-4}\big(\sqrt{2}\,z, r\big) = \frac{3r^2 - r^4}{3 - 3r^2 + r^4}.$$

By Lemma 2.1, we just need to consider the convexity of the following function in $\log x$:
$$h(x) = \frac{3x - x^2}{3 - 3x + x^2}, \qquad 0 < x < 1.$$

Using the D-notation from Lemma 2.2, we have
$$D(h(x)) \sim 18 - 36x + 21x^2 - 4x^3 =: g(x).$$

It is easy to check that $g''(x) = 42 - 24x > 0$ for all $x \in (0, 1)$. Thus $g(x)$ is convex on [0, 1]. Since $g(0) = 18 > 0$ and $g(1) = -1 < 0$, there exists a point $c \in (0, 1)$ such that $g(x) > 0$ for $x \in (0, c)$ and $g(x) < 0$ for $x \in (c, 1)$. Thus $\log h(x)$ is neither convex nor concave in $\log x$.

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CHUNJIE WANG
DEPARTMENT OF MATHEMATICS
HEBEI UNIVERSITY OF TECHNOLOGY
TIANJIN 300401
CHINA
E-mail: wcj@hebut.edu.cn

KEHE ZHU
DEPARTMENT OF MATHEMATICS AND STATISTICS
STATE UNIVERSITY OF NEW YORK, ALBANY
NY 12222
USA
E-mail: kzhu@math.albany.edu