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ASYMPTOTIC APPROXIMATIONS OF EIGEN-FUNCTIONS FOR REGULAR STURM-LIOUVILLE PROBLEMS WITH EIGENVALUE PARAMETER IN THE BOUNDARY CONDITION FOR INTEGRABLE POTENTIAL

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Abstract

In this paper we obtain asymptotic estimates of eigenfunctions for regular Sturm-Liouville problems having the eigenparameter in the boundary condition without smoothness conditions on the potential.

1. Introduction

In this paper we consider the boundary value problem

$$\tau y := -y'' + qy = \lambda y, q(t) \qquad \text{is integrable on} \quad [a, b], \tag{1}$$

$$a_1 y(a) - a_2 y'(a) = \lambda(a_3 y(a) - a_4 y'(a)), \qquad a_1, a_2, a_3, a_4 \in \mathsf{R}, \tag{2}$$

$$y(b)\cos\beta + y'(b)\sin\beta = 0, \qquad \beta \in [0, \pi) \tag{3}$$

where λ is a real parameter. It is shown by Walter [16] that this problem is a self-adjoint problem if the relation

$$\delta := \begin{vmatrix} a_3 & a_1 \\ a_4 & a_2 \end{vmatrix} > 0$$

holds. Problems of this type arise upon separation of variables in the one dimensional wave and heat equations. The cases $a_4 = 0$ ($\beta = 0$, $\beta \neq 0$) and $a_4 \neq 0$ ($\beta = 0$, $\beta \neq 0$) are considered in [5] where $q(t)$ is continuous in $[a, b]$. Fulton's approach in [5] is based on making use of solutions of (1) and employing the residue calculus to give more direct proof of the convergence properties of the eigenfunction expansion of (1)–(3). Some other related results can be found in [1], [6], [8], [9], [10], [11], [12], [13], [14], [15], [16].

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The aim of this paper is to obtained asymptotic approximations for the eigenfunctions of (1)–(3) including the above mentioned cases ($a_4 = 0$, $a_4 \neq 0$) where $q(t)$ is integrable on $[a, b]$. We use similar approach in [4], [7] to derive these approximations by making use of solutions of (1) defined with the initial conditions and asymptotic approximations of the eigenvalues of (1)–(3) obtained in [2].

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2. The Results

We define two solutions, $\Psi(x, \lambda)$ and $\Phi(x, \lambda)$ of (1)–(3) with the initial conditions

$$\psi(a, \lambda) = a_4\lambda - a_2,\ \Psi'(a, \lambda) = a_3\lambda - a_1, \tag{4}$$

and

$$\Phi(b, \lambda) = \sin\beta,\ \Phi'(b, \lambda) = -\cos\beta. \tag{5}$$

THEOREM 2.1. *Let* $\Psi(x, \lambda)$ *be the solution of* (1) *satisfying* (4). *Then*

(i) $a_4 \neq 0$

$$\Psi(x, \lambda) = \frac{a_4\lambda - a_2}{\cos\{\tan^{-1}[F_1(a, \lambda)]\}} \exp\left(\int_a^x S(t, \lambda)\, dt\right) \times \cos\left\{\tan^{-1}[F_1(a, \lambda)] + \int_a^x T(t, \lambda)\, dt\right\}, \tag{6}$$

where

$$F_1(a, \lambda) := \frac{1}{T(a, \lambda)}\left[S(a, \lambda) - \frac{a_3\lambda - a_1}{a_4\lambda - a_2}\right]. \tag{7}$$

(ii) $a_4 = 0$

$$\Psi(x, \lambda) = -\frac{a_2}{\cos\{\cot^{-1}[F_2(a, \lambda)]\}} \exp\left(\int_a^x S(t, \lambda)\, dt\right) \times \cos\left\{\cot^{-1}[F_2(a, \lambda)] + \int_a^x T(t, \lambda)\, dt\right\}, \tag{8}$$

where

$$F_2(a,\lambda) := \frac{a_2 T(a,\lambda)}{a_3\lambda - a_1 + a_2 S(a,\lambda)}. \tag{9}$$

THEOREM 2.2. *Let* $\Phi(x,\lambda)$ *be the solution of* (1) *satisfying* (5). *Then*

(i) $\beta \neq 0$

$$\Phi(x,\lambda) = \frac{\sin\beta}{\exp\left(\int_a^b S(t,\lambda)\,dt\right)\cos\{\tan^{-1}[F_3(b,\lambda)]\}}\exp\left(\int_a^x S(t,\lambda)\,dt\right) \times \cos\left\{\tan^{-1}[F_3(b,\lambda)] - \int_x^b T(t,\lambda)\,dt\right\}, \tag{10}$$

where

$$F_3(b,\lambda) := \frac{S(b,\lambda) + \cot\beta}{T(b,\lambda)} \tag{11}$$

(ii) $\beta = 0$

$$\Phi(x,\lambda) = \frac{1}{\exp\left(\int_a^b S(t,\lambda)\,dt\right)T(b,\lambda)}\exp\left(\int_a^x S(t,\lambda)\,dt\right) \times \cos\left\{\frac{\pi}{2} - \int_x^b T(t,\lambda)\,dt\right\}. \tag{12}$$

THEOREM 2.3. *Solution of* (1) *corresponding to* (4) *satisfies, as* $\lambda \longrightarrow \infty$

(i) $a_4 \neq 0$

$$\Psi(x,\lambda) = a_4\lambda\cos\left[\lambda^{\frac{1}{2}}(x-a)\right] + \lambda^{\frac{1}{2}}\left[a_3 + \frac{a_4}{2}\int_a^x q(t)\,dt\right] \times \sin\left[\lambda^{\frac{1}{2}}(x-a)\right] + O\left(\lambda^{\frac{1}{2}}\eta(\lambda)\right), \tag{13}$$

(ii) $a_4 = 0$

$$\Psi(x,\lambda) = a_3\lambda^{\frac{1}{2}}\sin\left[\lambda^{\frac{1}{2}}(x-a)\right] - \left[a_2 + \frac{a_3}{2}\int_a^x q(t)\,dt\right] \times \cos\left[\lambda^{\frac{1}{2}}(x-a)\right] + O(\eta(\lambda)) \tag{14}$$

where $\eta(\lambda) := \sup_{a\le t\le b} F(t,\lambda)$ *with*

$$F(t,\lambda) := \begin{cases} \dfrac{\left|\int_t^b e^{2i\lambda^{\frac{1}{2}}x} q(x)\,dx\right|}{\int_t^b |q(x)|\,dx} & \text{if } \int_t^b |q(x)|\,dx \neq 0, \\ 0 & \text{if } \int_t^b |q(x)|\,dx = 0. \end{cases}$$

We note that $\eta(\lambda) \longrightarrow 0$ *as* $\lambda \to \infty$ [2].

THEOREM 2.4. *Solution of* (1) *corresponding to* (5) *satisfies, as* $\lambda \longrightarrow \infty$

(i) $\beta \neq 0$

$$(15)\quad \Phi(x,\lambda) = \sin\beta\cos\left[\lambda^{\frac{1}{2}}(b-x)\right] + \lambda^{-\frac{1}{2}}\left[\cos\beta + \frac{\sin\beta}{2}\int_x^b q(t)\,dt\right] \times \sin\left[\lambda^{\frac{1}{2}}(b-x)\right] + O\left(\lambda^{-\frac{1}{2}}\eta(\lambda)\right),$$

(ii) $\beta = 0$

$$(16)\quad \Phi(x,\lambda) = \lambda^{-\frac{1}{2}}\sin\left[\lambda^{\frac{1}{2}}(b-x)\right] - \frac{1}{2}\lambda^{-1}\left[\int_x^b q(t)\,dt\right] \times \cos\left[\lambda^{\frac{1}{2}}(b-x)\right] + O\left(\lambda^{-1}\eta(\lambda)\right).$$

Our results include the following four distinct cases concerning a_4 and β as pointed out in [5]. These are $a_4 \neq 0$, $\beta \neq 0$; $a_4 \neq 0$, $\beta = 0$; $a_4 = 0$, $\beta \neq 0$ and $a_4 = 0$, $\beta = 0$.

THEOREM 2.5. *The asymptotic formula for the eigenfunctions of* (1) *and* (4) *satisfies, as* $n \longrightarrow \infty$

(i) $a_4 \neq 0$, $\beta \neq 0$

$$(17)\quad \Psi(x,\lambda_n) = a_4\frac{(n+1)^2\pi^2}{(b-a)^2}\cos\left[\frac{(n+1)\pi(x-a)}{b-a}\right] + \frac{(n+1)\pi}{b-a} \times \left[a_3 + \frac{a_4}{2}\int_a^x q(t)\,dt\right]\sin\left[\frac{(n+1)\pi(x-a)}{b-a}\right] + O(n\eta(n)),$$

(ii) $a_4 \neq 0$, $\beta = 0$

$$(18)\quad \Psi(x,\lambda_n) = a_4\frac{(2n+3)^2\pi^2}{4(b-a)^2}\cos\left[\frac{(2n+3)\pi(x-a)}{2(b-a)}\right] + \frac{(2n+3)\pi}{2(b-a)} \times \left[a_3 + \frac{a_4}{2}\int_a^x q(t)\,dt\right]\sin\left[\frac{(2n+3)\pi(x-a)}{2(b-a)}\right] + O(n\eta(n)),$$

(iii) $a_4 = 0,\ \beta \neq 0$

$$\text{(19)}\quad \Psi(x,\lambda_n) = a_3 \frac{(2n+3)\pi}{2(b-a)} \sin\left[\frac{(2n+3)\pi(x-a)}{2(b-a)}\right] - \left[a_2 + \frac{a_3}{2}\int_a^x q(t)\,dt\right]\cos\left[\frac{(2n+3)\pi(x-a)}{2(b-a)}\right] + O(\eta(n)),$$

(iv) $a_4 = 0,\ \beta = 0$

$$\text{(20)}\quad \Psi(x,\lambda_n) = a_3 \frac{(n+2)\pi}{b-a} \sin\left[\frac{(n+2)\pi(x-a)}{b-a}\right] - \left[a_2 + \frac{a_3}{2}\int_a^x q(t)\,dt\right]\cos\left[\frac{(n+2)\pi(x-a)}{b-a}\right] + O(\eta(n)).$$

THEOREM 2.6. *The asymptotic formula for the eigenfunctions of* (1) *and* (5) *satisfies, as* $n \longrightarrow \infty$

(i) $a_4 \neq 0,\ \beta \neq 0$

$$\text{(21)}\quad \Phi(x,\lambda_n) = \sin\beta\cos\left[\frac{(n+1)\pi(b-x)}{b-a}\right] + \frac{b-a}{(n+1)\pi} \times \left[\cos\beta + \frac{\sin\beta}{2}\int_x^b q(t)\,dt\right]\sin\left[\frac{(n+1)\pi(b-x)}{b-a}\right] + O(n^{-1}\eta(n)),$$

(ii) $a_4 \neq 0,\ \beta = 0$

$$\text{(22)}\quad \Phi(x,\lambda_n) = \frac{2(b-a)}{(2n+3)\pi}\sin\left[\frac{(2n+3)\pi(b-x)}{2(b-a)}\right] - \frac{2(b-a)^2}{(2n+3)^2\pi^2} \times \left[\int_x^b q(t)dt\right]\cos\left[\frac{(2n+3)\pi(b-x)}{2(b-a)}\right] + O(n^{-2}\eta(n)),$$

(iii) $a_4 = 0,\ \beta \neq 0$

$$\text{(23)}\quad \Phi(x,\lambda_n) = \sin\beta\cos\left[\frac{(2n+3)\pi(b-x)}{2(b-a)}\right] + \frac{2(b-a)}{(2n+3)\pi} \times \left[\cos\beta + \frac{\sin\beta}{2}\int_x^b q(t)\,dt\right]\sin\left[\frac{(2n+3)\pi(b-x)}{2(b-a)}\right] + O(n^{-1}\eta(n)),$$

(iv) $a_4 = 0,\ \beta = 0$

$$(24)\quad \Phi(x,\lambda_n) = \frac{b-a}{(n+2)\pi}\sin\left[\frac{(n+2)\pi(b-x)}{b-a}\right] - \frac{(b-a)^2}{2(n+2)^2\pi^2} \times \left[\int_x^b q(t)\,dt\right]\cos\left[\frac{(n+2)\pi(b-x)}{b-a}\right] + O(n^{-2}\eta(n)).$$

For $a = 0$, $b = \pi$ we get the following corollaries:

COROLLARY 2.7. *The eigenfunctions* $\Psi(x,\lambda_n)$ *of* (1) *corresponding to* (4) *satisfy, as* $n \longrightarrow \infty$

(i) $a_4 \neq 0,\ \beta \neq 0$

$$(25)\quad \Psi(x,\lambda_n) = a_4(n+1)^2\cos[(n+1)x] + (n+1) \times \left[a_3 + \frac{a_4}{2}\int_0^x q(t)\,dt\right]\sin[(n+1)x] + O(n\eta(n)),$$

(ii) $a_4 \neq 0,\ \beta = 0$

$$(26)\quad \Psi(x,\lambda_n) = a_4\frac{(2n+3)^2}{4}\cos\left[\frac{(2n+3)x}{2}\right] + \frac{(2n+3)}{2} \times \left[a_3 + \frac{a_4}{2}\int_0^x q(t)\,dt\right]\sin\left[\frac{(2n+3)x}{2}\right] + O(n\eta(n)),$$

(iii) $a_4 = 0,\ \beta \neq 0$

$$(27)\quad \Psi(x,\lambda_n) = a_3\frac{(2n+3)}{2}\sin\left[\frac{(2n+3)x}{2}\right] - \left[a_2 + \frac{a_3}{2}\int_0^x q(t)\,dt\right]\cos\left[\frac{(2n+3)x}{2}\right] + O(\eta(n)),$$

(iv) $a_4 = 0,\ \beta = 0$

$$(28)\quad \Psi(x,\lambda_n) = a_3(n+2)\sin[(n+2)x] - \left[a_2 + \frac{a_3}{2}\int_0^x q(t)\,dt\right]\cos[(n+2)x] + O(\eta(n)).$$

COROLLARY 2.8. *The eigenfunctions* $\Phi(x,\lambda_n)$ *of* (1) *corresponding to* (5) *satisfy, as* $n \longrightarrow \infty$

(i) $a_4 \neq 0$, $\beta \neq 0$

$$(29)\quad \Phi(x,\lambda_n) = \sin\beta\cos[(n+1)(\pi-x)] + \frac{1}{(n+1)} \times \left[\cos\beta + \frac{\sin\beta}{2}\int_x^\pi q(t)\,dt\right]\sin[(n+1)(\pi-x)] + O(n^{-1}\eta(n)),$$

(ii) $a_4 \neq 0$, $\beta = 0$

$$(30)\quad \Phi(x,\lambda_n) = \frac{2}{(2n+3)}\sin\left[\frac{(2n+3)(\pi-x)}{2}\right] - \frac{2}{(2n+3)^2} \times \left[\int_x^\pi q(t)\,dt\right]\cos\left[\frac{(2n+3)(\pi-x)}{2}\right] + O(n^{-2}\eta(n)),$$

(iii) $a_4 = 0$, $\beta \neq 0$

$$(31)\quad \Phi(x,\lambda_n) = \sin\beta\cos\left[\frac{(2n+3)(\pi-x)}{2}\right] + \frac{2}{(2n+3)} \times \left[\cos\beta + \frac{\sin\beta}{2}\int_x^\pi q(t)\,dt\right]\sin\left[\frac{(2n+3)(\pi-x)}{2}\right] + O(n^{-1}\eta(n)),$$

(iv) $a_4 = 0$, $\beta = 0$

$$(32)\quad \Phi(x,\lambda_n) = \frac{1}{(n+2)}\sin[(n+2)(\pi-x)] - \frac{1}{2(n+2)^2} \times \left[\int_x^\pi q(t)\,dt\right]\cos[(n+2)(\pi-x)] + O(n^{-2}\eta(n)).$$

Now we give an example to illustrate our results. Let $q(t) = t^{-\frac{1}{2}} - 2\pi^{-\frac{1}{2}}$. Using (25)–(32), following estimates for $\Psi(x,\lambda_n)$ and $\Phi(x,\lambda_n)$ are obtained: $a_4 \neq 0$, $\beta \neq 0$

$$\Psi(x,\lambda_n) = a_4(n+1)^2\cos[(n+1)x] + (n+1) \times \left[a_3 + a_4(x^{\frac{1}{2}} - \pi^{-\frac{1}{2}}x)\right]\sin[(n+1)x] + O(n\eta(n)),$$

$$\Phi(x,\lambda_n) = \sin\beta\cos[(n+1)(\pi-x)] + \frac{1}{(n+1)} \times \left[\cos\beta + \sin\beta(-x^{\frac{1}{2}} + \pi^{-\frac{1}{2}}x)\right]\sin[(n+1)(\pi-x)] + O(n^{-1}\eta(n));$$

$a_4 \neq 0,\ \beta = 0$

$$\Psi(x,\lambda_n) = a_4\frac{(2n+3)^2}{4}\cos\left[\frac{(2n+3)x}{2}\right] + \frac{(2n+3)}{2} \times \left[a_3 + a_4(x^{\frac{1}{2}} - \pi^{-\frac{1}{2}}x)\right]\sin\left[\frac{(2n+3)x}{2}\right] + O(n\eta(n)),$$

$$\Phi(x,\lambda_n) = \frac{2}{(2n+3)}\sin\left[\frac{(2n+3)(\pi-x)}{2}\right] - \frac{2}{(2n+3)^2} \times \left(-2x^{\frac{1}{2}} + 2\pi^{-\frac{1}{2}}x\right)\cos\left[\frac{(2n+3)(\pi-x)}{2}\right] + O(n^{-2}\eta(n));$$

$a_4 = 0,\ \beta \neq 0$

$$\Psi(x,\lambda_n) = a_3\frac{(2n+3)}{2}\sin\left[\frac{(2n+3)x}{2}\right] - \left[a_2 + a_3(x^{\frac{1}{2}} - \pi^{-\frac{1}{2}}x)\right]\cos\left[\frac{(2n+3)x}{2}\right] + O(\eta(n)),$$

$$\Phi(x,\lambda_n) = \sin\beta\cos\left[\frac{(2n+3)(\pi-x)}{2}\right] + \frac{2}{(2n+3)} \times \left[\cos\beta + \sin\beta(-x^{\frac{1}{2}} + \pi^{-\frac{1}{2}}x)\right]\sin\left[\frac{(2n+3)(\pi-x)}{2}\right] + O(n^{-1}\eta(n));$$

$a_4 = 0,\ \beta = 0$

$$\Psi(x,\lambda_n) = a_3(n+2)\sin[(n+2)x] - \left[a_2 + a_3(x^{\frac{1}{2}} - \pi^{-\frac{1}{2}}x)\right]\cos[(n+2)x] + O(\eta(n)),$$

$$\Phi(x,\lambda_n) = \frac{1}{(n+2)}\sin[(n+2)(\pi-x)] - \frac{1}{(n+2)^2}\left(-x^{\frac{1}{2}} + \pi^{-\frac{1}{2}}x\right) \times \cos[(n+2)(\pi-x)] + O(n^{-2}\eta(n)).$$

3. The Method

We associate with (1) the Riccati equation

$$v' = -\lambda + q - v^2. \tag{33}$$

We define

$$S(t,\lambda) := \mathrm{Re}\{v(t,\lambda)\}, \tag{34}$$

$$T(t,\lambda) := \operatorname{Im}\{v(t,\lambda)\}. \tag{35}$$

It is shown in [3] that any real-valued solution of (1) is in the form

$$y(t,\lambda) = R(t,\lambda)\cos(\theta(t,\lambda)) \tag{36}$$

with
$$\frac{R'(t,\lambda)}{R(t,\lambda)} = S(t,\lambda), \qquad \theta'(t,\lambda) = T(t,\lambda).$$

We consider (33) on $[a,b]$ and set

$$v(t,\lambda) := i\lambda^{\frac{1}{2}} + \sum_{n=1}^{\infty} v_n(t,\lambda). \tag{37}$$

Substitution of (37) into (33) and rearrangement then gives

$$\begin{aligned} v_1' + 2i\lambda^{\frac{1}{2}} v_1 + v_2' + 2i\lambda^{\frac{1}{2}} v_2 + \sum_{n=3}^{\infty}\big(v_n' + 2i\lambda^{\frac{1}{2}} v_n\big) \\ = q - v_1^2 - \sum_{n=3}^{\infty}\left(v_{n-1}^2 + 2v_{n-1}\sum_{m=1}^{n-2} v_m\right). \end{aligned}$$

We choose the v_n so that

$$\begin{aligned} &v_1' + 2i\lambda^{\frac{1}{2}} v_1 = q, \\ &v_2' + 2i\lambda^{\frac{1}{2}} v_2 = -v_1^2, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &v_n' + 2i\lambda^{\frac{1}{2}} v_n = -\left(v_{n-1}^2 + 2v_{n-1}\sum_{m=1}^{n-2} v_m\right) \end{aligned} \tag{38}$$

for $n = 3, 4, \dots$, and

$$\begin{aligned} v_1(t,\lambda) &= -e^{-2i\lambda^{\frac{1}{2}} t}\int_t^b e^{2i\lambda^{\frac{1}{2}} x} q(x)\,dx, \\ v_2(t,\lambda) &= e^{-2i\lambda^{\frac{1}{2}} t}\int_t^b e^{2i\lambda^{\frac{1}{2}} x} v_1^2(x,\lambda)\,dx, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ v_n(t,\lambda) &= e^{-2i\lambda^{\frac{1}{2}} t}\int_t^b e^{2i\lambda^{\frac{1}{2}} x}\left(v_{n-1}^2 + 2v_{n-1}\sum_{m=1}^{n-2} v_m\right)dx. \end{aligned} \tag{39}$$

It is proven in [3] that $\sum_{n=1}^{\infty} v_n'(t,\lambda)$ is uniformly absolutely convergent and the series $i\lambda^{\frac{1}{2}} + \sum_{n=1}^{\infty} v_n(t,\lambda)$ is thus a solution of (33) and

$$S(t,\lambda) = \operatorname{Re} \sum_{n=1}^{\infty} v_n(t,\lambda), \tag{40}$$

$$T(t,\lambda) = \lambda^{\frac{1}{2}} + \operatorname{Im} \sum_{n=1}^{\infty} v_n(t,\lambda). \tag{41}$$

It is also proven in [3] that there exist a sequence $\{k_n\}$ of real numbers with

$$|v_n(t,\lambda)| \le k_n \eta(\lambda)^n, \tag{42}$$

(Lemma 2.2, [3]). It is shown in [7] that any nontrivial real-valued solution, z, of (1) can be expressed as

$$z(x,\lambda) = c_1 \exp\left(\int_a^x S(t,\lambda)\,dt\right) \cos\left\{c_2 + \int_a^x T(t,\lambda)\,dt\right\}, \tag{43}$$

with

$$\begin{aligned} z'(x,\lambda) = {} & c_1 S(x,\lambda) \exp\left(\int_a^x S(t,\lambda)\,dt\right) \cos\left\{c_2 + \int_a^x T(t,\lambda)\,dt\right\} \\ & - c_1 \exp\left(\int_a^x S(t,\lambda)\,dt\right) \sin\left(c_2 + \int_a^x T(t,\lambda)\,dt\right) T(x,\lambda) \end{aligned} \tag{44}$$

where $S(x,\lambda)$ and $T(x,\lambda)$ are given by (34) and (35), respectively.

4. Proof of the Results

We first give the following lemma:

LEMMA 4.1. *As* $\lambda \longrightarrow \infty$

(i)

$$\frac{S(x,\lambda)}{T(x,\lambda)} = -\lambda^{-\frac{1}{2}} \sin\left(2\lambda^{\frac{1}{2}} x + \zeta_x\right) + O\left(\lambda^{-\frac{1}{2}} \eta(\lambda)^2\right), \tag{45}$$

(ii)

$$\frac{a_3\lambda - a_1}{(a_4\lambda - a_2)T(a,\lambda)} = \frac{a_3}{a_4}\lambda^{-\frac{1}{2}} + O(\lambda^{-1}\eta(\lambda)), \tag{46}$$

$$\frac{a_2 T(a,\lambda)}{a_3\lambda - a_1 + a_2 S(a,\lambda)} = \frac{a_2}{a_3}\lambda^{-\frac{1}{2}} + O(\lambda^{-1}\eta(\lambda)), \tag{47}$$

(iii)

(48) $$\int_a^x S(t,\lambda)\,dt = \frac{1}{2\lambda^{\frac12}}\left\{\cos\left(2\lambda^{\frac12}x+\zeta_x\right)-\cos\left(2\lambda^{\frac12}a+\zeta_a\right)\right\}+O\left(\lambda^{-\frac12}\eta(\lambda)^2\right),$$

(iv)

(49) $$\int_a^x T(t,\lambda)\,dt = \lambda^{\frac12}(x-a) - \frac{1}{2\lambda^{\frac12}}\left\{\sin\left(2\lambda^{\frac12}x+\zeta_x\right)-\sin\left(2\lambda^{\frac12}a+\zeta_a\right)+\int_a^x q(t)\,dt\right\}+O\left(\lambda^{-\frac12}\eta(\lambda)^2\right),$$

where

(50) $$\sin\zeta_t = \int_t^b q(t)\cos\left(2\lambda^{\frac12}t\right)dt,$$

(51) $$\cos\zeta_t = \int_t^b q(t)\sin\left(2\lambda^{\frac12}t\right)dt.$$

PROOF.

(i) It is shown in [2] that

(52) $$S(x,\lambda) = -\sin\left(2\lambda^{\frac12}x+\zeta_x\right)+O(\eta(\lambda)^2),$$

(53) $$T(x,\lambda) = \lambda^{\frac12}-\cos\left(2\lambda^{\frac12}x+\zeta_x\right)+O(\eta(\lambda)^2).$$

Using this last two equalities we get

$$\begin{aligned}\frac{S(x,\lambda)}{T(x,\lambda)} &= \frac{-\sin\left(2\lambda^{\frac12}x+\zeta_x\right)+O(\eta(\lambda)^2)}{\lambda^{\frac12}-\cos\left(2\lambda^{\frac12}x+\zeta_x\right)+O(\eta(\lambda)^2)}\\ &= -\frac{\sin\left(2\lambda^{\frac12}x+\zeta_x\right)+O(\eta(\lambda)^2)}{\lambda^{\frac12}\left[1-\lambda^{-\frac12}\cos\left(2\lambda^{\frac12}x+\zeta_x\right)+O\left(\lambda^{-\frac12}\eta(\lambda)^2\right)\right]}\\ &= -\lambda^{-\frac12}\sin\left(2\lambda^{\frac12}x+\zeta_x\right)+O\left(\lambda^{-\frac12}\eta(\lambda)^2\right).\end{aligned}$$

(ii) From (52) and (53) we obtain

$$\begin{aligned}\frac{a_3\lambda-a_1}{(a_4\lambda-a_2)T(a,\lambda)} &= \frac{a_3\lambda-a_1}{(a_4\lambda-a_2)\left[\lambda^{\frac12}-\cos\left(2\lambda^{\frac12}a+\zeta_a\right)+O(\eta(\lambda)^2)\right]}\\ &= \frac{a_3}{a_4}\lambda^{-\frac12}+O(\lambda^{-1}\eta(\lambda))\end{aligned}$$

and

$$\frac{a_2 T(a,\lambda)}{a_3\lambda - a_1 + a_2 S(a,\lambda)} = \frac{a_2\big\{\lambda^{\frac{1}{2}} - \cos\big(2\lambda^{\frac{1}{2}}a + \zeta_a\big) + O(\eta(\lambda)^2)\big\}}{a_3\lambda - a_1 + a_2\big\{-\sin\big(2\lambda^{\frac{1}{2}}a + \zeta_a\big) + O(\eta(\lambda)^2)\big\}}$$
$$= \frac{a_2}{a_3}\lambda^{-\frac{1}{2}} + O(\lambda^{-1}\eta(\lambda)).$$

(iii) We now evaluate $\int_a^x S(t,\lambda)\,dt = \int_a^x \mathrm{Re}(v(t,\lambda))\,dt$. From (39) and (42), we see that

$$\begin{aligned}\int_a^x S(t,\lambda)\,dt &= \int_a^b S(t,\lambda)\,dt - \int_x^b S(t,\lambda)\,dt\\
&= -\int_a^b \cos 2\lambda^{\frac{1}{2}}t\left(\int_t^b \cos 2\lambda^{\frac{1}{2}}xq(x)\,dx\right)dt\\
&\quad - \int_a^b \sin 2\lambda^{\frac{1}{2}}t\left(\int_t^b \sin 2\lambda^{\frac{1}{2}}xq(x)\,dx\right)dt\\
&\quad + \int_x^b \cos 2\lambda^{\frac{1}{2}}t\left(\int_t^b \cos 2\lambda^{\frac{1}{2}}xq(x)\,dx\right)dt\\
&\quad + \int_x^b \sin 2\lambda^{\frac{1}{2}}t\left(\int_t^b \sin 2\lambda^{\frac{1}{2}}xq(x)\,dx\right)dt\\
&\quad + O(\lambda^{-\frac{1}{2}}\eta(\lambda)^2).\end{aligned}$$

Using a change of the order of integration and integration by parts

$$\begin{aligned}\int_a^x S(t,\lambda)dt &= -\int_a^b \cos 2\lambda^{\frac{1}{2}}xq(x)\left[\frac{\sin 2\lambda^{\frac{1}{2}}(x-a)}{2\lambda^{\frac{1}{2}}}\right]dx\\
&\quad - \int_a^b \sin 2\lambda^{\frac{1}{2}}xq(x)\left[-\frac{\cos 2\lambda^{\frac{1}{2}}(x-a)}{2\lambda^{\frac{1}{2}}}\right]dx\\
&\quad - \frac{\sin 2\lambda^{\frac{1}{2}}x}{2\lambda^{\frac{1}{2}}}\int_x^b \cos 2\lambda^{\frac{1}{2}}xq(x)\,dx\\
&\quad + \frac{\cos 2\lambda^{\frac{1}{2}}x}{2\lambda^{\frac{1}{2}}}\int_x^b \sin 2\lambda^{\frac{1}{2}}xq(x)\,dx + O\big(\lambda^{-\frac{1}{2}}\eta(\lambda)^2\big)\\
&= \frac{1}{2\lambda^{\frac{1}{2}}}\big\{\cos\big(2\lambda^{\frac{1}{2}}x + \zeta_x\big) - \cos\big(2\lambda^{\frac{1}{2}}a + \zeta_a\big)\big\}\\
&\quad + O(\lambda^{-\frac{1}{2}}\eta(\lambda)^2).\end{aligned}$$

The last equality follows from some algebraic calculations.

(iv) Similarly we evaluate $\int_a^x T(t,\lambda)\,dt = \int_a^x \mathrm{Im}(v(t,\lambda))\,dt$. From (39) and (42), we see that

$$\begin{aligned}
\int_a^x T(t,\lambda)\,dt &= \int_a^b T(t,\lambda)\,dt - \int_x^b T(t,\lambda)\,dt \\
&= \int_a^x \lambda^{\frac{1}{2}}\,dt + \int_a^b \sin 2\lambda^{\frac{1}{2}} t\left(\int_t^b \cos 2\lambda^{\frac{1}{2}} x q(x)\,dx\right)dt \\
&\quad - \int_a^b \cos 2\lambda^{\frac{1}{2}} t\left(\int_t^b \sin 2\lambda^{\frac{1}{2}} x q(x)\,dx\right)dt \\
&\quad - \int_x^b \sin 2\lambda^{\frac{1}{2}} t\left(\int_t^b \cos 2\lambda^{\frac{1}{2}} x q(x)\,dx\right)dt \\
&\quad + \int_x^b \cos 2\lambda^{\frac{1}{2}} t\left(\int_t^b \sin 2\lambda^{\frac{1}{2}} x q(x)\,dx\right)dt \\
&\quad + O\big(\lambda^{-\frac{1}{2}}\eta(\lambda)^2\big) \\
&= \lambda^{\frac{1}{2}}(x-a) - \frac{1}{2\lambda^{\frac{1}{2}}}\bigg\{\sin\big(2\lambda^{\frac{1}{2}}x+\zeta_x\big) - \sin\big(2\lambda^{\frac{1}{2}}a+\zeta_a\big) \\
&\quad + \int_a^x q(t)\,dt\bigg\} + O\big(\lambda^{-\frac{1}{2}}\eta(\lambda)^2\big).
\end{aligned}$$

The last equality follows from a change of the order of integration and integration by parts.

PROOF OF THEOREM 2.1.

(i) For $a_4 \neq 0$; from (4), (43) and (44) we obtain

$$\Psi(a,\lambda) = c_1\cos c_2 = a_4\lambda - a_2, \tag{54}$$

$$\Psi'(a,\lambda) = c_1 S(a,\lambda)\cos c_2 - c_1 \sin c_2 T(a,\lambda) = a_3\lambda - a_1. \tag{55}$$

From the last two equalities

$$c_1 = \frac{a_4\lambda - a_2}{\cos c_2}, \tag{56}$$

$$c_2 = \tan^{-1}[F_1(a,\lambda)]. \tag{57}$$

Hence

$$c_1 = \frac{a_4\lambda - a_2}{\cos\big\{\tan^{-1}[F_1(a,\lambda)]\big\}}. \tag{58}$$

Substitution the values of c_1 and c_2 in (43) proves part (i).

(ii) For $a_4 = 0$; again using (54) and (55) we obtain

$$c_1 = -\frac{a_2}{\cos c_2},$$
$$c_2 = \cot^{-1}[F_2(a, \lambda)].$$

So

$$c_1 = -\frac{a_2}{\cos\{\cot^{-1}[F_2(a, \lambda)]\}}.$$

Substitution the values of c_1 and c_2 in (43) proves part (ii).

PROOF OF THEOREM 2.2.

(i) For $\beta \neq 0$; from (5), (43) and (44) we obtain

$$\Phi(b, \lambda) = c_1 \exp\left(\int_a^b S(t, \lambda)\, dt\right) \cos\left\{c_2 + \int_a^b T(t, \lambda)\, dt\right\} = \sin\beta, \tag{59}$$

$$\begin{aligned} \Phi'(b, \lambda) = c_1 S(b, \lambda) \exp\left(\int_a^b S(t, \lambda)\, dt\right) \cos\left\{c_2 + \int_a^b T(t, \lambda)\, dt\right\} \\ - c_1 \exp\left(\int_a^b S(t, \lambda)\, dt\right) \sin\left\{c_2 + \int_a^b T(t, \lambda)\, dt\right\} T(b, \lambda) \\ = -\cos\beta. \end{aligned} \tag{60}$$

From (59) and (60)

$$c_1 = \frac{\sin\beta}{\exp\left(\int_a^b S(t, \lambda)\, dt\right) \cos\left\{c_2 + \int_a^b T(t, \lambda)\, dt\right\}}, \tag{61}$$

$$c_2 = \tan^{-1}\left\{\frac{S(b, \lambda) + \cot\beta}{T(b, \lambda)}\right\} - \int_a^b T(t, \lambda)\, dt. \tag{62}$$

Hence

$$c_1 = \frac{\sin\beta}{\exp\left(\int_a^b S(t, \lambda)\, dt\right) \cos\left\{\tan^{-1}\left[\frac{S(b,\lambda)+\cot\beta}{T(b,\lambda)}\right]\right\}}. \tag{63}$$

Substitution of c_1 and c_2 in (43) proves part (i).

(ii) For $\beta = 0$, again from (5), (43) and (44) we obtain

$$\Phi(b, \lambda) = c_1 \exp\left(\int_a^b S(t, \lambda)\, dt\right) \cos\left\{c_2 + \int_a^b T(t, \lambda)\, dt\right\} = 0, \tag{64}$$

$$\begin{aligned}\Phi'(b,\lambda) &= c_1 S(b,\lambda)\exp\Big(\int_a^b S(t,\lambda)\,dt\Big)\cos\Big\{c_2+\int_a^b T(t,\lambda)\,dt\Big\} \\ &\quad - c_1\exp\Big(\int_a^b S(t,\lambda)\,dt\Big)\sin\Big\{c_2+\int_a^b T(t,\lambda)\,dt\Big\}T(b,\lambda) \\ &= -1.\end{aligned} \tag{65}$$

Using (64) we see that

$$c_2 = \frac{\pi}{2} - \int_a^b T(t,\lambda)\,dt. \tag{66}$$

Hence from (65) and (66)

$$c_1 = \frac{1}{\exp\big(\int_a^b S(t,\lambda)\,dt\big)T(b,\lambda)}. \tag{67}$$

Substitution of c_1 and c_2 in (43) proves part (ii).

Proof of Theorem 2.3.

(i) For $a_4 \neq 0$, we evaluate the terms in (6). Firstly, substituting (45) and (46) into $F_1(a,\lambda)$ given by (7) we obtain

$$F_1(a,\lambda) = -\lambda^{-\frac{1}{2}}\left[\sin\big(2\lambda^{\frac{1}{2}}a+\zeta_a\big)+\frac{a_3}{a_4}\right] + O\big(\lambda^{-\frac{1}{2}}\eta(\lambda)^2\big), \tag{68}$$

hence

$$\tan^{-1}[F_1(a,\lambda)] = -\frac{a_3}{a_4}\lambda^{-\frac{1}{2}} + O\big(\lambda^{-\frac{1}{2}}\eta(\lambda)\big). \tag{69}$$

From (57) we know that

$$c_2 = \tan^{-1}[F_1(a,\lambda)] = -\frac{a_3}{a_4}\lambda^{-\frac{1}{2}} + O\big(\lambda^{-\frac{1}{2}}\eta(\lambda)\big) \tag{70}$$

and from that

$$\cos c_2 = 1 - \frac{1}{2}\left(\frac{a_3}{a_4}\right)^2\lambda^{-1} + O(\lambda^{-1}\eta(\lambda)), \tag{71}$$

$$\sin c_2 = -\frac{a_3}{a_4}\lambda^{-\frac{1}{2}} + O\big(\lambda^{-\frac{1}{2}}\eta(\lambda)\big), \tag{72}$$

$$\frac{a_4\lambda - a_2}{\cos c_2} = a_4\lambda + O(1). \tag{73}$$

Using $\int_a^x S(t,\lambda)\,dt$ given by (48) we get

$$\begin{aligned}(74)\quad &\exp\left(\int_a^x S(t,\lambda)\,dt\right)\\ &\qquad = 1 + \frac{1}{2\lambda^{\frac12}}\left\{\cos\left(2\lambda^{\frac12}x+\zeta_x\right) - \cos\left(2\lambda^{\frac12}a+\zeta_a\right)\right\} + O\left(\lambda^{-\frac12}\eta(\lambda)^2\right)\end{aligned}$$

and also, using $\int_a^x T(t,\lambda)\,dt$ given by (49)

$$\begin{aligned}(75)\quad &\sin\left(\int_a^x T(t,\lambda)\,dt\right)\\ &\qquad = \sin\left[\lambda^{\frac12}(x-a) - \frac{1}{2\lambda^{\frac12}}\int_a^x q(t)\,dt\right] + O\left(\lambda^{-\frac12}\eta(\lambda)\right),\end{aligned}$$

$$\begin{aligned}(76)\quad &\cos\left(\int_a^x T(t,\lambda)\,dt\right)\\ &\qquad = \cos\left[\lambda^{\frac12}(x-a) - \frac{1}{2\lambda^{\frac12}}\int_a^x q(t)\,dt\right] + O\left(\lambda^{-\frac12}\eta(\lambda)\right).\end{aligned}$$

Hence

$$\begin{aligned}(77)\quad &\cos\left[c_2 + \int_a^x T(t,\lambda)\,dt\right] = \cos\left[\lambda^{\frac12}(x-a) - \frac{1}{2\lambda^{\frac12}}\int_a^x q(t)\,dt\right]\\ &\qquad + \frac{a_3}{a_4}\lambda^{-\frac12}\sin\left[\lambda^{\frac12}(x-a) - \frac{1}{2\lambda^{\frac12}}\int_a^x q(t)\,dt\right] + O\left(\lambda^{-\frac12}\eta(\lambda)\right).\end{aligned}$$

Substituting the values of (73), (74) and (77) into (6) and using trigonometric expansions we prove part (i). The other part (ii) can be proved similarly.

PROOF OF THEOREM 2.4.

(i) For $\beta \neq 0$, we evaluate the terms in (10). Using (45) and (53), $F_3(b,\lambda)$ given by (11) is obtained as

$$\begin{aligned}(78)\quad F_3(b,\lambda) &= \frac{S(b,\lambda)+\cot\beta}{T(b,\lambda)} = \frac{\cot\beta + O(\eta(\lambda)^2)}{\lambda^{\frac12} + O(\eta(\lambda)^2)}\\ &= \frac{\cot\beta + O(\eta(\lambda)^2)}{\lambda^{\frac12}\left[1 + O\left(\lambda^{-\frac12}\eta(\lambda)^2\right)\right]} = \lambda^{-\frac12}\cot\beta + O\left(\lambda^{-\frac12}\eta(\lambda)^2\right)\end{aligned}$$

and from that

$$(79)\qquad \tan^{-1}[F_3(b,\lambda)] = \lambda^{-\frac12}\cot\beta + O\left(\lambda^{-\frac12}\eta(\lambda)^2\right),$$

$$(80)\qquad \cos\{\tan^{-1}[F_3(b,\lambda)]\} = 1 - \tfrac12\lambda^{-1}\cot^2\beta + O(\lambda^{-1}\eta(\lambda)^2).$$

Using $\int_a^x S(t,\lambda)\,dt$ given by (48) we get

$$\exp\left(\int_a^b S(t,\lambda)\,dt\right) = 1 + O(\lambda^{-\frac{1}{2}}\eta(\lambda)). \tag{81}$$

Hence

$$\begin{aligned}\frac{\sin\beta}{\exp\left(\int_a^b S(t,\lambda)\,dt\right)\cos\left\{\tan^{-1}[F_3(b,\lambda)]\right\}} \\ = \frac{\sin\beta}{1 + O\left(\lambda^{-\frac{1}{2}}\eta(\lambda)\right)} = \sin\beta + O\left(\lambda^{-\frac{1}{2}}\eta(\lambda)\right).\end{aligned} \tag{82}$$

Also, using $\int_a^x T(t,\lambda)\,dt$ given by (49)

$$\begin{aligned}\int_x^b T(t,\lambda)\,dt = \int_a^b T(t,\lambda)\,dt - \int_a^x T(t,\lambda)\,dt = \lambda^{\frac{1}{2}}(b-x) \\ + \frac{1}{2\lambda^{\frac{1}{2}}}\left\{\sin\left(2\lambda^{\frac{1}{2}}x + \zeta_x\right) - \int_x^b q(t)\,dt\right\} + O\left(\lambda^{-\frac{1}{2}}\eta(\lambda)^2\right).\end{aligned} \tag{83}$$

From the last equality and $\tan^{-1}[F_3(b,\lambda)]$ given by (79) we see that

$$\begin{aligned}&\cos\left\{\tan^{-1}[F_3(b,\lambda)] - \int_x^b T(t,\lambda)\,dt\right\} \\ &= \cos\left[\lambda^{\frac{1}{2}}(b-x) - \frac{1}{2\lambda^{\frac{1}{2}}}\int_x^b q(t)\,dt\right] \\ &\quad + \lambda^{-\frac{1}{2}}\cot\beta\sin\left[\lambda^{\frac{1}{2}}(b-x) - \frac{1}{2\lambda^{\frac{1}{2}}}\int_x^b q(t)\,dt\right] + O\left(\lambda^{-\frac{1}{2}}\eta(\lambda)\right).\end{aligned} \tag{84}$$

Finally, substituting the values of (74), (82) and (84) into (10) and using trigonometric expansions we prove part (i).

(ii) For $\beta = 0$, we evaluate the terms in (12). From (53) and (81) we obtain

$$\frac{1}{\exp\left(\int_a^b S(t,\lambda)\,dt\right)T(b,\lambda)} = \frac{1}{\lambda^{\frac{1}{2}} + O(\eta(\lambda))} = \lambda^{-\frac{1}{2}} + O(\lambda^{-1}\eta(\lambda)). \tag{85}$$

Also, using $\int_x^b T(t,\lambda)\,dt$ given by (83) we see that

$$\begin{aligned}\cos\left[\frac{\pi}{2} - \int_x^b T(t,\lambda)\,dt\right] = \sin\left[\int_x^b T(t,\lambda)\,dt\right] \\ = \sin\left[\lambda^{\frac{1}{2}}(b-x) - \frac{1}{2\lambda^{\frac{1}{2}}}\int_x^b q(t)\,dt\right] + O\left(\lambda^{-\frac{1}{2}}\eta(\lambda)\right).\end{aligned} \tag{86}$$

Finally, substituting the values of (74), (85) and (86) into (12) and using trigonometric expansions we prove part (ii).

To prove Theorem 2.5 and Theorem 2.6 we use eigenvalues obtained by [2] previously together with Theorem 2.3 and Theorem 2.4.

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