UNIFORM DOMAINS AND UNIFORM DOMAIN DECOMPOSITION PROPERTY IN REAL NORMED VECTOR SPACES

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Abstract

Let $E$ be a real normed vector space with $\dim(E) \geq 2$, $D$ a proper subdomain of $E$. In this paper we characterize uniform domains in $E$ in terms of the uniform domain decomposition property. In addition, we discuss the relation between quasiballs and domains with the quasiball decomposition property in $\mathbb{R}^n$.

1. Introduction and Main Results

Throughout the paper, we assume that $E$ is a real normed vector space with $\dim(E) \geq 2$ and the norm of a vector $z \in E$ is denoted by $|z|$. For any two points $z_1, z_2$ in $E$, the distance between them is denoted by $|z_1 - z_2|$. $D$ is always assumed to be a proper domain in $E$ and $B(x_0, r) = \{x \in E : |x - x_0| < r\}$, the open ball centered at $x_0$ of radius $r > 0$. Similarly, for the closed balls and spheres, we use the notations $\overline{B}(x_0, r)$ and $\partial B(x_0, r)$.

We now introduce two basic concepts: uniform domains and John domains.

**Definition 1.1.** A proper domain $D$ in $E$ is called **uniform** in the norm metric provided there exists a constant $c$ with the property that each pair of points $z_1, z_2$ in $D$ can be joined by a rectifiable arc $\gamma$ in $D$ satisfying (cf. [18] and [20])

\[
\begin{align*}
\text{(1)} & \quad \min_{j=1,2} \ell(\gamma[z_j, z]) \leq c d_D(z) \text{ for all } z \in \gamma, \text{ and} \\
\text{(2)} & \quad \ell(\gamma[z_1, z_2]) \leq c |z_1 - z_2|.
\end{align*}
\]

Here $\ell(\gamma)$ denotes the arclength of $\gamma$, $\gamma[z_j, z]$ the part of $\gamma$ between $z_j$ and $z$. The distance from $z$ to the boundary $\partial D$ of $D$ in $E$ is denoted by $d_D(z)$.

$D$ is said to be a **John domain** if it satisfies the first condition in above but not necessarily the second one (see [16]).

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John [10], Martio and Sarvas [15] were the first who introduced John domains and uniform domains in $\mathbb{R}^2$, respectively. Now, there are plenty of alternative characterizations for uniform and John domains (see [4], [5], [6], [8], [11], [14], [18]), and their importance along with some special domains throughout the function theory is well documented, see [5], [11], [16], [18]. Moreover, uniform domains in $E$ enjoy numerous geometric and function theoretic features in many areas of modern mathematical analysis, see [1], [2], [3], [4], [8], [9], [18].

We refer to the books of Väisälä [17] and Vuorinen [21] for the definition of $K$-quasiconformal ($K$-qc) homeomorphism of $\mathbb{R}^n$ and for basic facts regarding quasiconformal (qc) mappings.

A Jordan curve $\gamma$ in $\mathbb{R}^2 = \mathbb{R}^2 \cup \{\infty\}$ is called a $K$-qc circle (or simply qc circle) if there is a $K$-qc mapping $f$ of $\mathbb{R}^2$ onto itself such that $\gamma = f(\partial B^2)$, and $f(B^2)$ is called a $K$-quasidisk (or simply quasidisk), where $B^2$ denotes the unit disk in $\mathbb{R}^2$. We say that a domain $D \subset \mathbb{R}^2$ is a $K$-quasiball (or simply quasiball) if there exists a $K$-qc mapping $f$ of $\mathbb{R}^2$ onto itself such that $D = f(B^n)$, where $B^n$ denotes the unit ball in $\mathbb{R}^n$.

As a characterization of qc circles, Martio and Sarvas [15] proved that a Jordan domain in $\mathbb{R}^2$ is uniform if and only if its boundary is a qc circle. After that, Gehring and Hag [7, Theorems 3.10 and 4.1] proved that a finitely connected domain $D$ in $\mathbb{R}^2$ is uniform if and only if there is a constant $K$ such that each component of $\partial D$ is either a point or a $K$-qc circle. As a further generalization, Gehring and Osgood proved

**Theorem A** ([8, Theorem 5]). A domain $D$ in $\mathbb{R}^2$ is a uniform domain if and only if it is quasiconformally decomposable.

Here a domain $D$ in $\mathbb{R}^2$ is said to be quasiconformally decomposable if there exists a constant $K$ with the following property: For each pair of points $z_1, z_2$ in $D$, there exists a subdomain $D_0$ of $D$ such that $z_1, z_2$ are contained in $\overline{D_0}$ and $\partial D_0$ is a $K$-qc circle. Obviously, $D_0$ is a $K$-quasidisk.

We refer to [8] for some applications of Theorem A including a new proof of the injectivity properties of uniform domains in $\mathbb{R}^2$. The situation is very different in $\mathbb{R}^n$. The 3-dimensional analog of Theorem A fails to hold even for simply connected domains, see [13, Example 3.8]. In order to consider the generalization of Theorem A in $\mathbb{R}^n$ or real normed vector spaces $E$, we introduce the following concepts.

**Definition 1.2.** A domain $D$ in $E$ is said to have the uniform domain decomposition property if there exists a positive constant $c$ with the following property: For each pair of points $z_1, z_2$ in $D$, there exists a subdomain $D_0$ of $D$ such that $z_1, z_2 \in D_0$ and $D_0$ is a simply connected $c$-uniform domain.
A domain $D$ in $\mathbb{R}^n$ is said to have the *quasiball decomposition property* if there exists a positive constant $K$ with the following property: For each pair of points $z_1, z_2$ in $D$, there exists a subdomain $D_0$ of $D$ such that $z_1, z_2 \in D_0$ and $D_0$ is a $K$-quasiball.

By proving the Lipschitz continuous first differentiability of quasihyperbolic geodesics in $\mathbb{R}^n$, Martin obtained

**Theorem B** ([13, Theorem 5.1]). Let $D$ be a uniform domain in $\mathbb{R}^n$. Then there is a constant $L$, depending only on the constant of uniformity for $D$, such that for each pair of points $x_1, x_2$ in $D$ there is an $L$-bi-Lipschitz embedding $f : \overline{B}(0, |x_1 - x_2|) \to D$ with $\{x_1, x_2\} \subset f(\overline{B}(0, |x_1 - x_2|))$.

Obviously, Theorem B shows that

**Corollary 1.3.** A domain in $\mathbb{R}^n$ is uniform if and only if it has the uniform domain decomposition property.

It easily follows from [8, Corollary 3] that

**Proposition 1.4.** Let $D$ be a domain in $\mathbb{R}^n$. If $D$ has the quasiball decomposition property, then it has the uniform domain decomposition property.

For a simply connected domain $D$ in $\mathbb{R}^2$, $D$ is uniform if and only if it is a quasidisk [9, Lemma 6.4] if and only if it is a quasiball. In view of Theorem A, it is easy to formulate the following proposition which characterizes uniform domains.

**Proposition 1.5.** For any domain $D$ in $\mathbb{R}^2$, the following are equivalent.
1. $D$ is uniform;
2. $D$ is quasiconformally decomposable;
3. $D$ has the uniform domain decomposition property;
4. $D$ has the quasiball decomposition property.

By [13, Example 3.8], it is natural to consider a suitable generalization of Proposition 1.5 which works for $E$ or $\mathbb{R}^n$. To achieve this goal, in this paper, we mainly consider the following two questions.

**Question 1.6.** Is it true that a domain $D$ in $E$ is uniform if and only if it has the uniform domain decomposition property?

**Question 1.7.** Is it true that a domain $D$ in $\mathbb{R}^n$ is a quasiball if and only if it has the quasiball decomposition property?

In the proof of Theorem A, the authors [8] have utilized the Riemann mapping theorem. In the absence of the Riemann mapping theorem in $E$ when $\text{dim}(E) \geq 3$, it is natural that the methods used in the proof of Theorem A are
no more useful in $E$ when $\dim(E) \geq 3$. It is known that a quasihyperbolic geodesic between any two points in $E$ exists if the dimension of $E$ is finite, see [8, Lemma 1]. But this is not true in arbitrary spaces. A counterexample (due to Alestalo) has been given in [18, Section 3], see also [19, Section 2]. Hence the method of proof used in Theorem B is invalid either. By using a different method of proof, we obtain the following theorems and delay their proofs until a few necessary preliminaries have been developed. Moreover, our method of proof works also for the case $E = \mathbb{R}^2$.

**Theorem 1.8.** Let $E$ be a real normed vector space with $\dim(E) \geq 2$. Then a domain $D$ in $E$ is uniform if and only if it has the uniform domain decomposition property.

**Theorem 1.9.** Every quasiball in $\mathbb{R}^n$ has the quasiball decomposition property.

We see from the following example that the converse of Theorem 1.9 is not necessarily true.

**Example 1.10.** Let $e_1 = (1, 0, 0)$ denote the unit vector in the direction of $x_1$-axis and $D = B^3 \setminus L$ in $\mathbb{R}^3$, where $L = \{te_1 : \frac{1}{2} \leq t < 1\}$. Then $D$ has the quasiball decomposition property, but $D$ is not a quasiball.

**2. Proof of Theorem 1.8**

We start with some preliminary results. The proof of Theorem 1.8 is given in Subsection 2.24.

**Lemma 2.1.** For any $x_1, x_2 \in G \subset E$, if $\overline{B}(x_1, r_1) \cap \overline{B}(x_2, r_2) \neq \emptyset$, $\frac{1}{4}d_G(x_1) \leq r_1 \leq \frac{8}{5}d_G(x_1)$ and $\frac{1}{4}d_G(x_2) \leq r_2 \leq \frac{8}{5}d_G(x_2)$, then

$$\frac{1}{17}d_G(x_2) \leq d_G(x_1) \leq 17d_G(x_2) \quad \text{and} \quad \frac{1}{68}r_1 \leq r_2 \leq 68r_1.$$

**Proof.** For any $y \in \partial B(x_1, r_1) \cap \overline{B}(x_2, r_2)$, since

$$d_G(y) \geq d_G(x_2) - r_2, \quad d_G(x_1) \geq d_G(y) - r_1$$

and

$$d_G(y) \geq d_G(x_1) - r_1, \quad d_G(x_2) \geq d_G(y) - r_2,$$

we see that the lemma holds.

For any $z_1, z_2 \in D$, we assume that $\alpha \subset D$ is a rectifiable arc joining them with
Lemma 2.2. There exists a simply connected domain $D_{1,0} = \bigcup_{i=1}^{k_1} B_{1,i} \subset D$ such that

1. $z_1, z_0 \in D_{1,0}$;
2. For each $i \in \{1, \ldots, k_1\}$, $\frac{1}{7} d_D(x_{1,i}) \leq r_{1,i} \leq \frac{7}{8} d_D(x_{1,i})$;
3. If $k_1 \geq 3$, then for any $i, j \in \{1, \ldots, k_1\}$ with $|i - j| \geq 2$, we have $\text{dist}(B_{1,i}, B_{1,j}) \geq \frac{1}{32M} \max\{r_{1,i}, r_{1,j}\}$;
4. If $k_1 \geq 2$, then $r_{1,i} + r_{1,i+1} - |x_{1,i} - x_{1,i+1}| \geq \frac{1}{32M} \max\{r_{1,i}, r_{1,i+1}\}$ for each $i \in \{1, \ldots, k_1 - 1\}$,

where $B_{1,i} = B(x_{1,i}, r_{1,i})$, $x_{1,i} \in \gamma$, $x_{1,i} \notin B_{1,i-1}$ and $\text{dist}(B_{1,i}, B_{1,j})$ denotes the distance from $B_{1,i}$ to $B_{1,j}$.

Proof. Let $x_{1,1} = z_1$. Set $A_{1,1} = B(x_{1,1}, r_{1,1})$ with $r_{1,1} = \frac{1}{2} d_D(x_{1,1})$.

If $z_0 \in A_{1,1}$, then we let $B_{1,1} = A_{1,1}$, and the domain $D_{1,0} = B_{1,1}$ is the desired.

If $z_0 \notin A_{1,1}$, then we let $x_{1,2}$ be the last intersection point of $\gamma$ from $z_1$ to $z_0$ with $\partial A_{1,1}$. Set $A_{1,2} = B(x_{1,2}, r_{1,2})$ with $r_{1,2} = \frac{1}{2} d_D(x_{1,2})$.

If $z_0 \in A_{1,2}$ and $A_{1,1}$ is contained in $A_{1,2}$, then we let $B_{1,1} = A_{1,2}$, and the domain $D_{1,0} = B_{1,1}$ is the needed. If $z_0 \in A_{1,2}$ and $A_{1,1}$ is not contained in $A_{1,2}$, then we let $B_{1,1} = A_{1,1}$, $B_{1,2} = A_{1,2}$, and the domain $D_{1,0} = B_{1,1} \cup B_{1,2}$ is the desired.

If $z_0 \notin A_{1,2}$, then we let $x_{1,3}$ be the last intersection point of $\gamma$ from $z_1$ to $z_0$ with $\partial A_{1,2}$. Set $A_{1,3} = B(x_{1,3}, r_{1,3})$ with $r_{1,3} = \frac{1}{2} d_D(x_{1,3})$.

We continue this procedure until there is some $i \in \{1, \ldots, s - 2\}$ such that $\text{dist}(B_{1,i}, B_{1,i}) < \frac{1}{32M} \max\{r_{1,i}, r_{1,i+1}\}$. Obviously, $s \geq 3$.

Let $A_{1,t}$ be the first ball from $A_{1,1}$ to $A_{1,s-1}$ such that $\overline{A_{1,t}} \cap \overline{A_{1,s}} \neq \emptyset$. For the case $t = 1$ and $z_0 \in A_{1,s}$, if $A_{1,1}$ is contained in $B(x_{1,s}, \frac{3}{4} d_D(x_{1,s}))$, we take $D_{1,0} = B_{1,1} = B(x_{1,s}, \frac{3}{4} d_D(x_{1,s}))$. Otherwise, the similar reasoning as in Lemma 2.1 shows that we can let $D_{1,0} = B_{1,1} \cup B_{1,2}$, where $B_{1,1} = A_{1,1}$ and $B_{1,2} = B(x_{1,s}, \frac{3}{4} d_D(x_{1,s}))$. When $t = 1$ and $z_0 \notin A_{1,s}$ or $t \neq 1$, we have the following claim.
Claim 2.3. There are \( q \) balls \( C_{1,1} = B(y_{1,1}, p_{1,1}), \ldots, C_{1,q} = B(y_{1,q}, p_{1,q}) \) (possibly, \( q = 1 \)) in \( D \) such that

(a) \( \{y_{1,1}, \ldots, y_{1,q}\} \subset \{x_{1,1}, \ldots, x_{1,s}\} \);

(b) the conditions (2), (3) and (4) in the lemma are satisfied by the balls \( C_{1,1}, \ldots, C_{1,q} \).

The proof for the case \( t = 1 \) is obvious: If \( A_{1,1} \) is contained in \( B(x_{1,s}, \frac{3}{4}d_D(x_{1,s})) \), then we let \( C_{1,1} = B(x_{1,s}, \frac{3}{4}d_D(x_{1,s})) \) and so \( q = 1 \). Otherwise, we let \( C_{1,1} = A_{1,1}, C_{1,2} = B(x_{1,s}, \frac{3}{4}d_D(x_{1,s})) \). The similar reasoning as in Lemma 2.1 implies that \( C_{1,1} \) and \( C_{1,2} \) satisfy Conditions (2) and (4) in the lemma, and hence \( q = 2 \). For the remaining case \( t > 1 \), we divide the proof into two cases.

**Case 2.4.** \( r_{1,t} + r_{1,s} - |x_{1,t} - x_{1,s}| \geq \frac{1}{8M} r_{1,s} \).

We let \( C_{1,i} = A_{1,i} \) for each \( i \in \{1, \ldots, t\} \) and \( C_{1,t+1} = B(x_{1,s}, (1 - \frac{1}{16M})r_{1,s}) \). Since for each \( i \in \{1, \ldots, t\} \), \( r_{1,s} = \frac{1}{2}d_D(x_{1,s}) \geq \frac{1}{2\varepsilon} \ell(\alpha[z_{1}, x_{1,s}]) \geq \frac{1}{2\alpha}r_{1,i} \), we see that the balls \( C_{1,1}, C_{1,2}, \ldots, C_{1,t}, C_{1,t+1} \) satisfy the conditions (2) \( \sim (4) \) in the lemma. Hence \( q = t + 1 \).

**Case 2.5.** \( r_{1,t} + r_{1,s} - |x_{1,t} - x_{1,s}| < \frac{1}{8M} r_{1,s} \).

We consider the ball \( A'_{1,s} = B(x_{1,s}, \frac{7}{4}r_{1,s}) \). Let \( A_{1,s_1} \) be the first ball from \( A_{1,1} \) to \( A_{1,t} \), whose closure \( \overline{A}_{1,s_1} \) has nonempty intersection with \( \overline{A}_1 \). Denote \( d_{i} = \text{dist}(A_{1,i}, A_{1,s_1}) (s_1 \leq i \leq t) \). Clearly, \( d_t = 0 \). We divide the rest argument into two parts.

**Subcase 2.6.** \( d_{s_1} \leq \frac{5}{16} r_{1,s} \).

In this case, we take \( C_{1,i} = A_{1,i} (1 \leq i \leq s_1) \) and \( C_{1,s_1+1} = B(x_{1,s}, \frac{25}{16} r_{1,s}) \). Then the balls \( C_{1,1}, C_{1,2}, \ldots, C_{1,s_1}, C_{1,s_1+1} \) satisfy the conditions (2) \( \sim (4) \) in our lemma. This shows \( q = s_1 + 1 \).

**Subcase 2.7.** \( d_{s_1} > \frac{5}{16} r_{1,s} \).

Let \( \delta_1 = d_{s_1} \) and \( \delta_2 \) be the first \( d_{i} \) from \( d_{s_1} \) to \( d_t \) satisfying \( d_{i} < \delta_1 \). Clearly, \( \delta_1 > \delta_2 \). By repeating the procedure, we get \( \delta_1, \ldots, \delta_m \in \{d_{s_1}, \ldots, d_t\} \) such that

\[
\delta_1 > \delta_2 > \cdots > \delta_m = 0.
\]

Observe that \( \delta_1 > \frac{5}{16} r_{1,s} \) and hence \( m \geq 2 \). For each \( h \in \{1, \ldots, m - 1\} \), we denote \( A_{1,i_h} = B(x_{1,i_h}, r_{1,i_h}) \) the first ball from \( A_{1,1} \) to \( A_{1,t} \) with \( d_{i_h} = \delta_h \) and define \( \varepsilon_h = \delta_h - \delta_{h+1} \).

**Subclaim 2.8.** There must exist some \( j \in \{1, \ldots, m - 1\} \) such that \( \varepsilon_j > \frac{1}{8M} r_{1,s} \).
If \( m \leq M \), then the existence of \( j \in \{1, \ldots, m-1\} \) with \( \epsilon_j > \frac{1}{8M} r_{1,s} \) is obvious because otherwise,

\[
\frac{5}{16} r_{1,s} < \delta_1 - \delta_m \leq (m - 1) \frac{1}{8M} r_{1,s} < \frac{1}{8} r_{1,s},
\]

which is a contradiction.

We assume that \( m > M \). To prove the existence of \( j \), we suppose on the contrary that \( \epsilon_h \leq \frac{1}{8M} r_{1,s} \) for all \( h \in \{1, \ldots, m-1\} \). Note that

\[
\delta_{m-M} - \delta_m = \epsilon_{m-M} + \cdots + \epsilon_{m-1} \leq \frac{1}{8} r_{1,s}.
\]

Then for any \( h \in \{m-M, \ldots, m-1\} \), we have

(2.9) \[
\delta_h \leq \frac{1}{8} r_{1,s}.
\]

If there exists some \( h \in \{m-M, \ldots, m-1\} \) such that \( A_{1,ih} = B(x_{1,ih}, r_{1,ih}) \not\subset (A'_{1,s} \setminus A_{1,s}) \) then \( (A'_{1,s} \setminus A_{1,s}) \cap A_{1,ih} \) contains a ball, denoted by \( A_{0,ih} \), with radius \( r_{0,ih} = \frac{\frac{5}{16} r_{1,s} - \delta_h}{2} \geq \frac{5}{16} r_{1,s} \). Hence \( r_{1,ih} \geq \frac{5}{16} r_{1,s} \).

On the other hand, if \( A_{1,ih} = B(x_{1,ih}, r_{1,ih}) \subset (A'_{1,s} \setminus A_{1,s}) \) for some \( h \in \{m-M, \ldots, m-1\} \) then we see that \( r_{1,ih} \geq \frac{1}{8} r_{1,s} \). Otherwise,

\[
\frac{1}{8} r_{1,s} \geq r_{1,ih} \geq \frac{1}{3} d_D(x_{1,ih}) \geq \frac{1}{3} \left( \frac{3}{4} r_{1,s} - \delta_h - r_{1,ih} \right) \geq \frac{1}{6} r_{1,s},
\]

which obviously is a contradiction. Thus we have proved that for each \( h \in \{m-M, \ldots, m-1\} \),

(2.10) \[
r_{1,ih} > \frac{1}{8} r_{1,s}.
\]

It follows that

(2.11) \[
3cr_{1,s} \geq c d_D(x_{1,s}) \geq \ell(\gamma[z_1, x_{1,s}]) \geq \frac{M - 1}{8} r_{1,s},
\]

which is the desired contradiction since \( M = [2^{16c}] \). The proof of Subclaim 2.8 is complete.
We come back to the proof of Claim 2.3. Let $j$ be the least number in $\{1, \ldots, m-1\}$ satisfying Subclaim 2.8 and let $A''_{1,s} = B(x_1,r''_{1,s})$, where

$$r''_{1,s} = r_{1,s} + \delta_{j+1} + \frac{1}{16M}r_{1,s}.$$

Then for all $i < i_{j+1}$, $A''_{1,s} \cap A_{1,i} = \emptyset$. We take $C_{1,i} = A_{1,i}$ for each $i \in \{1, \ldots, i_{j+1}\}$ and $C_{1,i_{j+1}+1} = A''_{1,s}$. It follows from $r''_{1,s} \leq \frac{7}{4}r_{1,s}$ that the balls $C_{1,1}, \ldots, C_{1,i_{j+1}}, C_{1,i_{j+1}+1}$ satisfy the conditions (2), (3) and (4). Thus $q = i_{j+1} + 1$ in the case. The proof of Claim 2.3 is finished.

We continue the proof of our lemma.

If $z_0 \in C_{1,q}$, then by letting $B_{1,i} = C_{1,i}$ for each $i \in \{1, \ldots, q\}$, we see that the domain $D_{1,0} = \bigcup_{i=1}^{q} B_{1,i}$ is the desired.

If $z_0 \notin C_{1,q}$, then we let $x_1,q+1$ be the last intersection point of $\gamma$ from $z_1$ to $z_0$ with $\partial C_{1,q}$. Set $C_{1,q+1} = B(x_1,q+1,r_{1,q+1})$ with $r_{1,q+1} = \frac{1}{2}d_D(x_1,q+1)$.

By repeating the procedure as above, we will get a set of points $\{x_{1,i}\}_{i=1}^{k_1}$ on $\gamma$ and a set of balls $\{C_{1,i} = B(x_{1,i},r_{1,i})\}_{i=1}^{k_1}$ in $D$ such that Conditions (2), (3) and (4) are satisfied and $z_0$ is contained in $C_{1,k_1}$. By letting $B_{1,i} = C_{1,i}$ for each $i \in \{1, \ldots, k_1\}$, we know that $D_{1,0} = \bigcup_{i=1}^{k_1} B_{1,i}$ is the needed domain. Hence we see that Lemma 2.2 holds.

By a similar argument as in the proof of Lemma 2.2, we get

**Corollary 2.12.** There exists a simply connected domain $D_{2,0} = \bigcup_{u=1}^{k_2} B_{2,u} \subset D$ such that

1. $z_2, z_0 \in D_{2,0}$;
2. For each $u \in \{1, \ldots, k_2\}$, $\frac{1}{2}d_D(x_{2,u}) \leq r_{2,u} \leq \frac{7}{8}d_D(x_{2,u})$;
3. If $k_2 \geq 3$, then for any $u, v \in \{1, \ldots, k_2\}$ with $|u-v| \geq 2$, we have $\text{dist}(B_{2,u}, B_{2,v}) \geq \frac{1}{32M} \max\{r_{2,u}, r_{2,v}\}$;
4. If $k_2 \geq 2$, then $r_{2,u} + r_{2,u+1} - |x_{2,u} - x_{2,u+1}| \geq \frac{1}{32M} \max\{r_{2,u}, r_{2,u+1}\}$ for each $u \in \{1, \ldots, k_2 - 1\}$,

where $B_{2,u} = B(x_{2,u}, r_{2,u})$, $x_{2,u} \in \beta$ and $x_{2,u} \notin B_{2,u-1}$.

**Lemma 2.13.** $d_D(x_{2,k_2}) \geq \frac{1}{2\epsilon} \ell(\beta)$.

**Proof.** If $|z_0 - x_{2,k_2}| \leq \frac{1}{2}d_D(z_0)$, then $d_D(x_{2,k_2}) \geq d_D(z_0) - |z_0 - x_{2,k_2}| \geq \frac{1}{2}d_D(z_0)$. If $|z_0 - x_{2,k_2}| > \frac{1}{2}d_D(z_0)$, then $d_D(x_{2,k_2}) \geq r_{2,k_2} \geq \frac{1}{2}d_D(z_0)$. From the inequality $\ell(\beta) \leq c d_D(z_0)$, our lemma follows.
Lemma 2.14. There exists a simply connected domain $D_1 = \bigcup_{i=1}^{k} B_i \subset D$ such that

1. $z_1, z_2 \in D_1$;
2. For each $i \in \{1, \ldots, k\}$, \(\frac{1}{12} d_D(x_i) \leq r_i \leq d_D(x_i)\);
3. If $k \geq 3$, then for any $i, j \in \{1, \ldots, k\}$ with $|i - j| \geq 2$, we have $\text{dist}(B_i, B_j) \geq \frac{1}{64M} \max \{r_i, r_j\}$;
4. If $k \geq 2$, then $r_i + r_{i+1} - |x_i - x_{i+1}| \geq \frac{1}{64M} \max \{r_i, r_{i+1}\}$ for each $i \in \{1, \ldots, k - 1\}$,

where $B_i = B(x_i, r_i)$, $x_i \in \alpha$ and $x_i \notin B_{i-1}$.

Proof. We divide the proof into two cases.

Case 2.15. For any $i \in \{1, \ldots, k\}$ and $u \in \{1, \ldots, k - 1\}$, we have $r_{i, i} + r_{2, u} - |x_{i, i} - x_{2, u}| \leq \frac{1}{64M} \max \{r_{1, i}, r_{2, u}\}$.

For each $i \in \{1, \ldots, k - 1\}$, we let $A_{1,i} = B(x_{1,i}, R_{1,i})$ with $R_{1,i} = (1 - \frac{1}{64M})r_{1,i}$ and for each $u \in \{1, \ldots, k - 1\}$, let $A_{2,u} = B(x_{2,u}, R_{2,u})$ with $R_{2,u} = (1 - \frac{1}{64M})r_{2,u}$. Let $A_{1,k_1} = B(x_{1,k_1}, r_{1,k_1})$. By Lemma 2.2 and Corollary 2.12, we have

Claim 2.16.

1. For any $i \in \{1, \ldots, k\}$, we have \(\frac{1}{4} d_D(x_{1,i}) \leq R_{1,i} \leq \frac{7}{8} d_D(x_{1,i})\), and for each $u \in \{1, \ldots, k - 1\}$, we have \(\frac{1}{4} d_D(x_{2,u}) \leq R_{2,u} \leq \frac{7}{8} d_D(x_{2,u})\);
2. If $k_1 \geq 3$, then for any $i, j \in \{1, \ldots, k\}$ with $|i - j| \geq 2$, we have $\text{dist}(A_{1,i}, A_{1,j}) \geq \frac{1}{32M} \max \{r_{1,i}, r_{1,j}\}$;
3. If $k_2 \geq 3$, then for any $u, v \in \{1, \ldots, k\}$ with $|u - v| \geq 2$, we have $\text{dist}(A_{2,u}, A_{2,v}) \geq \frac{1}{32M} \max \{r_{2,u}, r_{2,v}\}$;
4. For any $i \in \{1, \ldots, k\}$ and $u \in \{1, \ldots, k - 1\}$, we have $\text{dist}(A_{1,i}, A_{2,u}) \geq \frac{1}{32M} \max \{r_{1,i}, r_{2,u}\}$.

If $B(x_{2,k_2}, (1 + \frac{1}{64M})r_{2,k_2}) \cap \bigcup_{i=1}^{k_1-1} \overline{B}_{1,i} = \emptyset$, then we let $A_{2,k_2} = B(x_{2,k_2}, (1 + \frac{1}{128M^2})r_{2,k_2})$. It follows from Corollary 2.12 and Lemma 2.13 that the balls $A_{1,1}, \ldots, A_{1,k_1-1}, A_{1,k_1}$ and $A_{2,1}, \ldots, A_{2,k_2}$ satisfy the conditions (1) ~ (4) in the lemma, where $k = k_1 + k_2$.

In the following, we assume that $\overline{B}(x_{2,k_2}, (1 + \frac{1}{64M})r_{2,k_2}) \cap \bigcup_{i=1}^{k_1-1} \overline{B}_{1,i} \neq \emptyset$.

We let $A_{1,q}$ be the first ball from $A_{1,1}$ to $A_{1,k_1-1}$ such that the closure $\overline{A}_{1,q}$ has nonempty intersection with $\overline{B}(x_{2,k_2}, (1 + \frac{1}{64M})r_{2,k_2})$.

Let $R_{2,k_2} = (1 + \frac{1}{64M})r_{2,k_2}$. We choose $B_i = A_{1,i}$ $(1 \leq i \leq q)$, $B_{q+1} = B(x_{2,k_2}, (1 + \frac{7}{512M^2})r_{2,k_2})$, $B_{q+2} = A_{2,k_2-1}$, ..., $B_k = A_{2,1}$ whenever

$R_{2,k_2} + R_{1,q} - |x_{2,k_2} - x_{1,q}| \geq \frac{1}{256M^2} R_{2,k_2}^2$. 


Then Corollary 2.12 and Lemma 2.13 show that the balls $B_1, B_2, \ldots, B_k$ satisfy the conditions (1) $\sim$ (4) in our lemma, where $k = q + k_2$.

On the other hand, in the case of

$$R_{2,k_2}^{'} + R_{1,q} - |x_{2,k_2} - x_{1,q}| < \frac{1}{256M^2} R_{2,k_2}^{'},$$

we consider the ball $B_{2,k_2}'' = B(x_{2,k_2}, R_{2,k_2}'')$ with $R_{2,k_2}'' = (1 + \frac{1}{128M^2}) r_{2,k_2}$.

Obviously, $A_{1,k_1} \cap B_{2,k_2}'' \neq \emptyset$. Let $A_{1,q_1}$ be the first ball from $A_{1,q}$ to $A_{1,k_1}$ such that the closure $\overline{A_{1,q_1}}$ has nonempty intersection with $\overline{B(x_{2,k_2}, (1 + \frac{1}{128M^2}) r_{2,k_2})}$. For each $i \in \{q, \ldots, q_1\}$, we denote $\text{dist}(A_{1,i}, B_{2,k_2}'')$ by $d_i$. Clearly, $d_{q_1} = 0$ and $d_q > \frac{1}{512M^2} r_{2,k_2}$.

Let $\eta_1 = d_q$ and $\eta_2$ be the first $d_i$ from $d_q$ to $d_{q_1}$ satisfying $d_i < \eta_1$. Clearly, $\eta_1 > \eta_2$. By repeating the procedure, we get $\eta_1, \ldots, \eta_{m_1} \in \{d_q, \ldots, d_{q_1}\}$ such that

$$\eta_1 > \eta_2 > \cdots > \eta_{m_1} = 0.$$ 

Observe that $\eta_1 > \frac{1}{512M^2} r_{2,k_2}$ and $m_1 \geq 2$. For each $i \in \{1, \ldots, m_1 - 1\}$, we denote the first ball from $A_{1,q}$ to $A_{1,\eta_i}$ with $d_{\eta_i} = \eta_i$ by $A_{1,\eta_i}$, i.e. $B(x_{1,\eta_i}, R_{1,\eta_i})$, and define $\epsilon_i = \eta_i - \eta_{i+1}$.

Replacing $\frac{\sqrt{2}}{16} r_{1,s}$ by $\frac{1}{512M^2} r_{2,k_2}$ and $M$ by $M^4$, the similar reasoning as in the proof of Subclaim 2.8 shows

**Claim 2.17.** There must exist some $j \in \{1, \ldots, m_1 - 1\}$ such that $\epsilon_j > \frac{1}{256M^2} r_{2,k_2}$.

We now consider the ball $C_{2,k_2}'' = B(x_{2,k_2}, r_{2,k_2}'')$, where

$$r_{2,k_2}'' = R_{2,k_2}'' + \eta_{j+1} + \frac{1}{512M^2} r_{2,k_2}.$$ 

By Claim 2.17, we see that $C_{2,k_2}'' \cap A_{1,i} = \emptyset$ for all $i < j+1$. We take $B_i = A_{1,i}$ for each $i \in \{1, \ldots, i+1\}$, $B_{i+1} = C_{2,k_2}''$, $B_{i+2} = A_{2,k_2-1}, \ldots, B_k = A_{1,1}$. Then Lemma 2.13 yields that the balls $B_1, \ldots, B_{i+1}, B_{i+2}, \ldots, B_k$ satisfy the conditions (1) $\sim$ (4) in the lemma, where $k = i+1 + k_2$.

**Case 2.18.** There exist $i \in \{1, \ldots, k_1\}$ and $u \in \{1, \ldots, k_2 - 1\}$ such that

$$r_{1,i} + r_{2,u} - |x_{1,i} - x_{2,u}| > \frac{1}{64M^2} \max\{r_{1,i}, r_{2,u}\}.$$ 

Let $B_{2,s}$ be the first ball from $B_{2,1}$ to $B_{2,k_2-1}$ such that there exists some $i \in \{1, \ldots, k_1\}$ satisfying $r_{1,i} + r_{2,s} - |x_{1,i} - x_{2,s}| > \frac{1}{64M^2} \max\{r_{1,i}, r_{2,s}\}$.

Let $B_{1,t}$ be the first ball from $B_{1,1}$ to $B_{1,k_1}$ satisfying $r_{1,t} + r_{2,s} - |x_{1,t} - x_{2,s}| > \frac{1}{64M^2} \max\{r_{1,t}, r_{2,s}\}$.
For any $i \in \{1, \ldots, t - 1\}$, we let $C_{1,i} = B(x_{1,i}, (1 - \frac{1}{64 M^8}) r_{1,i})$ and $C_{1,t} = B(x_{1,t}, (1 - \frac{1}{M^8}) r_{1,t})$, and for any $u \in \{1, \ldots, s - 1\}$, let $C_{2,u} = B(x_{2,u}, (1 - \frac{1}{64 M^8}) r_{2,u})$ and $C_{2,s} = B(x_{2,s}, (1 - \frac{1}{M^8}) r_{2,s})$. By letting $B_1 = C_{1,1}, \ldots, B_{t-1} = C_{1,t-1}, B_t = C_{1,t}, B_{t+1} = C_{2,s}, B_{t+2} = C_{2,s-1}, \ldots$ and $B_k = C_{2,1}$, we conclude from Lemma 2.1 that the balls $B_1, \ldots, B_t, B_{t+1}, \ldots, B_k$ satisfy the conditions (1) $\sim$ (4) in the lemma, where $k = t + s$.

The following two lemmas are also needed in the proof of Theorem 1.8.

**Lemma 2.19.** For any $i, j \in \{1, \ldots, k\}$ with $j \geq i + 2$, we have $\ell(\alpha[i, x_j]) \leq 36c^2 |x_i - x_j|$.

**Proof.** If $\{x_i, x_j\} \subset \gamma$ (resp. $\beta$), by the assumption $j \geq i + 2$ and Lemma 2.14, we get

$$
\ell(\alpha[i, x_j]) \leq c d_D(x_j) \leq 12cr_j \leq 12c|x_i - x_j|.
$$

For the rest case, without loss of generality, we may assume that $x_i \in \gamma$ and $x_j \in \beta$.

If $\max(|z_1 - x_i|, |z_2 - x_j|) \leq \frac{1}{3} |z_1 - z_2|$, then

$$
|z_1 - z_2| - |z_1 - x_i| - |z_2 - x_j| \geq \frac{1}{3} |z_1 - z_2|.
$$

Hence

$$
\ell(\alpha[i, x_j]) \leq \ell(\alpha) \leq c|z_1 - z_2| \leq 3c|x_i - x_j|.
$$

If $\max(|z_1 - x_i|, |z_2 - x_j|) \leq \frac{1}{3} |z_1 - z_2|$, we may assume that $\max(|z_1 - x_i|, |z_2 - x_j|) = |z_1 - x_i|$. Then by the assumption $j \geq i + 2$ and Lemma 2.14 we get

$$
\ell(\alpha[i, x_j]) \leq \ell(\alpha) \leq c|z_1 - z_2| \leq 3c|z_1 - x_i| \leq 36c^2 r_i \leq 36c^2 |x_i - x_j|.
$$

We conclude from (2.20) $\sim$ (2.22) that Lemma 2.19 holds.

**Lemma 2.23.** For any $w_1 \neq w_2 \in D$ and $r_1 \geq r_2 > 0$, we let $w_1 \in D \setminus B(w_2, r_2)$,

$$
r_1 + r_2 - |w_1 - w_2| \geq \frac{1}{64 M^8} r_2
$$

and $Q = B(w_1, r_1) \cup B(w_2, r_2)$. Then $Q$ is $2^{11} M^8$-uniform.

Before the proof of Lemma 2.23, we introduce the following lemma.
**Lemma C ([12, Theorem 1.2]).** Suppose that \( D_1 \) and \( D_2 \) are convex domains in \( E \), where \( D_1 \) is bounded and \( D_2 \) is \( c \)-uniform for some \( c > 1 \), and that there exist \( z_0 \in D_1 \cap D_2 \) and \( r > 0 \) such that \( B(z_0, r) \subset D_1 \cap D_2 \). If there exist constants \( R_1 > 0 \) and \( c_0 > 1 \) such that \( R_1 \leq c_0 r \) and \( D_1 \subset B(z_0, R_1) \), then \( D_1 \cup D_2 \) is a \( c' \)-uniform domain with \( c' = (c + 1)(2c_0 + 1) + c \).

**Proof of Lemma 2.23.** Obviously, there exists \( z_0 \in B(w_2, r_2) \cap B(w_1, r_1) \) such that the ball \( B(z_0, r) \) is contained in the intersection \( B(w_2, r_2) \cap B(w_1, r_1) \), where \( r = \frac{1}{128M^8} r_2 \). Hence \( B(w_2, r_2) \subset B(z_0, 256M^8 r) \). It follows from [20] that each ball in \( E \) is \( 2 \)-uniform. Then Lemma C implies that \( Q \) is \( 2^{11} M^8 \)-uniform.

2.24 Proof of Theorem 1.8. It suffices to prove the necessity since the sufficiency is obvious.

Assume that \( D \) is a \( c \)-uniform domain. Then for every pair of points \( z_1, z_2 \in D \), there is a rectifiable arc \( \alpha \subset D \) joining them with

\[
\ell(\alpha[z_1, z_2]) \leq c |z_1 - z_2| \quad \text{and} \quad \min_{j = 1, 2} \ell(\alpha[z_j, z]) \leq c d_D(z)
\]

for all \( z \in \alpha \).

It follows from Lemma 2.14 that there exists a domain \( D_1 \) which is simply connected satisfying Items (1) \( \sim \) (4) in Lemma 2.14. Let \( c_1 = \frac{1}{64M^8} \). We come to prove that \( D_1 \) is a \( c_2 \)-uniform domain, where \( c_2 = 72c^2(\frac{2}{c_1} + 1) \).

For any \( y_1, y_2 \in D_1 \), there must exist \( i, j \in \{1, \ldots, k\} \) such that \( y_1 \in B(x_i, r_i) \) and \( y_2 \in B(x_j, r_j) \).

If \( |j - i| \leq 1 \), then it follows from Lemma 2.23 and the fact \( r_i + r_{i+1} - |x_i - x_{i+1}| \geq c_1 \max\{r_i, r_{i+1}\} \) (see Lemma 2.14 (4)) that there exists a rectifiable curve \( \alpha_1 \) joining \( y_1 \) and \( y_2 \) in \( B(x_i, r_i) \cup B(x_{i+1}, r_{i+1}) \) such that

\[
\ell(\alpha_1) \leq 2^{11} M^8 |y_1 - y_2|
\]

and

\[
\min_{i = 1, 2} \ell(\alpha_1[y, y]) \leq 2^{11} M^8 d_{D_1}(y)
\]

for all \( y \in \alpha_1 \).

The remaining case we need to consider is: There are \( i, j \in \{1, \ldots, k\} \) such that \( j - i \geq 2 \), \( y_1 \in B_i \), \( y_2 \in B_j \) and \( \{y_1, y_2\} \) is not contained in \( B_i \cup B_{i+1} \) for any \( t \in \{i, \ldots, j - 1\} \). It suffices to prove the case: \( y_1 \notin [x_i, x_{i+1}] \) and \( y_2 \notin [x_{j-1}, x_j] \) since the discussions for other cases are similar. Set

\[
\alpha_2 = [y_1, x_i] \cup [x_i, x_{i+1}] \cup \ldots \cup [x_{j-1}, x_j] \cup [x_j, y_2].
\]
By Items (2) and (3) in Lemma 2.14 and Lemma 2.19, we have

\begin{equation}
\ell(\alpha_2) \leq |y_1 - x_i| + |x_j - y_2| + \ell(\alpha[x_i, x_j]) \\
\leq 2 \ell(\alpha[x_i, x_j]) \\
\leq 72c^2 |x_j - x_i| \\
= 72c^2(r_i + r_j + \text{dist}(B_i, B_j))
\end{equation}

\begin{equation}
\leq 72c^2 \left( \frac{2}{c_1} + 1 \right) |y_1 - y_2|,
\end{equation}

since \(|y_1 - y_2| \geq \text{dist}(B_i, B_j)|.

For any \(y \in \alpha_2\), if \(y \in [y_1, x_i]\) or \([x_j, y_2]\), then we easily have that

\begin{equation}
\min_{j=1,2} \ell(\alpha_2[y_j, y]) \leq d_D(y).
\end{equation}

For the case \(y \in [x_i, x_{i+1}] \cup \ldots \cup [x_{j-1}, x_j]\), obviously, there exists some \(m \in \{i, \ldots, j - 1\}\) such that \(y \in [x_m, x_{m+1}]\). Without loss of generality, we may assume that \(\min \{\ell(\alpha[z_1, x_m]), \ell(\alpha[x_m, z_2])\} = \ell(\alpha[z_1, x_m]).\) The proof for the case \(\min \{\ell(\alpha[z_1, x_m]), \ell(\alpha[x_m, z_2])\} = \ell(\alpha[z_2, x_m])\) follows from the similar reasoning.

It follows from Lemma 2.14 (2) that

\[\ell(\alpha[z_1, x_m]) \leq 12c d_D(x_m),\]

which in turn yields that

\begin{equation}
\ell(\alpha_2[y_1, x_m]) \leq |y_1 - x_i| + \ell(\alpha[x_i, x_m]) \\
\leq 24c d_D(x_m).
\end{equation}

If \(\min \{\ell(\alpha[z_1, x_{m+1}]), \ell(\alpha[x_{m+1}, z_2])\} = \ell(\alpha[z_1, x_{m+1}])\), then (2.29) yields that

\begin{equation}
\min_{s=1,2} \ell(\alpha_2[y_s, y]) \leq \ell(\alpha_2[y_1, y]) \\
\leq 24c d_D(x_m) + |y - x_m| \\
\leq (24c + 1) d_D(x_m) + d_D(y) \\
\leq \frac{2}{c_1} \left( 24c + \frac{c_1}{2} + 1 \right) d_D(y).
\end{equation}

Now we assume that \(\min \{\ell(\alpha[z_1, x_{m+1}]), \ell(\alpha[x_{m+1}, z_2])\} = \ell(\alpha[z_2, x_{m+1}]).\) Then Lemma 2.14 (2) implies that \(\ell(\alpha[z_2, x_{m+1}]) \leq 12c d_D(x_{m+1}).\) Hence

\begin{equation}
\ell(\alpha_2[y_2, x_{m+1}]) \leq 24c d_D(x_{m+1}).
\end{equation}
We infer from (2.31) that

\begin{equation}
\begin{aligned}
\min_{s=1,2} \ell(\alpha_2[y_s, y]) &\leq \ell(\alpha_2[y_2, y]) \\
&\leq 24 c d_{D_1}(x_{m+1}) + |y - x_{m+1}|
\end{aligned}
\end{equation}

\leq (24 + 1) d_{D_1}(x_{m+1}) + d_{D_1}(y)

\leq \frac{2}{c_1} \left( 24 c + \frac{c_1}{2} + 1 \right) d_{D_1}(y).

Thus the inequalities (2.25) \sim (2.28), (2.30) and (2.32) show that $D_1$ is a $c_2$-uniform domain. The proof of Theorem 1.8 is complete.

3. Proofs of Theorem 1.9 and Example 1.10

3.1 Proof of Theorem 1.9. Let $f: D \to \mathbb{B}^n$ be a quasiconformal map of $\mathbb{R}^n$. For any $z_1, z_2 \in D$, there exists a closed ball $\overline{B}_1^n \subset \mathbb{B}^n$ such that $f(z_1), f(z_2) \in \overline{B}_1^n$. Then $f^{-1}(B_1^n)$ is a quasiball. This shows that $D$ has the quasiball decomposition property.

3.2 Proof of Example 1.10. A result of Väisälä [17, Theorem 17.22] implies that $D$ is not a quasiball.

For any $z_1, z_2 \in D$, let $P$ be the plane determined by $z_1$ and $L$. Then $P$ divides $\mathbb{B}^3$ into two parts which are denoted by $B_1^3$ and $B_2^3$, respectively. We may assume that $z_1, z_2 \in \overline{B}_1^3$. Since $B_1^3$ is a bounded convex domain, the result in [22] shows that $B_1^3$ is a quasiball. This implies that $D$ has the quasiball decomposition property.

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