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DISCRETE HARDY SPACES RELATED TO POWERS OF THE POISSON KERNEL

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Abstract

Discrete Hardy spaces $H^1_\alpha(\partial T)$, related to powers $\alpha \geq 1/2$ of the Poisson kernels on boundaries ∂T of regular rooted trees, are studied. The spaces for $\alpha > 1/2$ coincide with the ordinary atomic Hardy space on ∂T, which in turn is strictly smaller than $H^1_{1/2}(\partial T)$. The Orlicz space $L \log\log L(\partial T)$ characterizes the positive and increasing functions in $H^1_{1/2}(\partial T)$, but there is no such characterization for general positive functions.

1. Introduction

Let f be an integrable function on the unit circle T. Recall that the Poisson extension of f gives a harmonic function, denoted Pf, in the unit disc. By taking the radial maximal function – that is, letting the function value at $e^{i\theta}$ be $\sup_{r\in[0,1[} |(Pf)(re^{i\theta})|$ – we get a new function defined on the unit circle. If this function is integrable, we say that f is in the Hardy space $H^1(\mathsf{T})$.

In [8], we studied the Hardy space $H^1_{\surd}(\mathsf{T})$ defined in a similar manner, but using instead the normalized square root of the Poisson kernel. Here we are interested in the corresponding space for powers of the Poisson kernel in the discrete setting, more precisely on rooted regular trees T of fixed degree $q + 1 \geq 3$. The boundary of the tree, denoted ∂T, consists of all infinite geodesic paths starting at the root. The measure on ∂T is always the standard harmonic measure ω, defined by (2) below.

Given $f \in L^1(\partial T)$, $\alpha \in \left[\frac{1}{2}, +\infty\right[$, and $x \in T$, we let

$$(P_\alpha f)(x) = \int_{\partial T} \bigl(q^{2N(x,\zeta)-|x|}\bigr)^\alpha f(\zeta)\,\mathrm{d}\omega(\zeta),$$

where $N(x, \zeta)$ is the number of edges shared by the geodesic from the root to x, having $|x|$ edges, and ζ. The normalized *α-Poisson extension of f* to T is defined by

$$(\mathscr{P}_\alpha f)(x) = \frac{(P_\alpha f)(x)}{(P_\alpha 1)(x)},$$

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for $x \in T$, and the corresponding radial maximal function is

$$(1) \qquad (\mathscr{P}^*_\alpha f)(\zeta) = \sup_{n \in \mathbb{N}} |(\mathscr{P}_\alpha f)(\zeta_n)|, \qquad \text{for all} \quad \zeta \in \partial T,$$

where ζ_n is the vertex along ζ which is n edges away from the root. A Hardy space is now defined just as in the continuous case.

DEFINITION 1.1. Let $\alpha \in \left[\frac{1}{2}, +\infty\right[$. The *$\alpha$-Hardy space*, denoted $H^1_\alpha(\partial T)$, is the subspace of $L^1(\partial T)$ consisting of functions $f \in L^1(\partial T)$ such that $\mathscr{P}^*_\alpha f \in L^1(\partial T)$. A norm on $H^1_\alpha(\partial T)$ is defined by $\|f\|_{H^1_\alpha(\partial T)} = \left\|\mathscr{P}^*_\alpha f\right\|_{L^1(\partial T)}$.

Since the space ∂T is a space of homogeneous type, an *atomic Hardy space* $H^1_{\mathrm{at}}(\partial T)$ is defined as in [1], see Section 3 below.

Probabilistically, the kernel $\left(q^{2N(x,\cdot)-|x|}\right)^\alpha/(P_\alpha 1)(x)$ is the exit distribution of a certain nearest neighbor random walk – the $\mathscr{P}_\alpha$-random walk – starting at x. This random walk was described by Picardello and Taibleson [5], and later also studied by the present author [9, papers 4 and 5]. If $\alpha = 1$ we regain the ordinary simple random walk, which at each step selects one of the $q + 1$ neighbours with equal probability. For $\alpha > \frac{1}{2}$, the $\mathscr{P}_\alpha$-random walk is of the type studied in [4], but – just as in the continuous setting – the case $\alpha = \frac{1}{2}$ is special.

Finally, C will denote a sufficiently large constant in $[1, +\infty[$, the value of which is uninteresting and can change even within the same line. Similarly, $c \in]0, 1]$ denotes a sufficiently small constant. If f, g are functions such that $cf \le g \le Cf$ at all points, then we write $f \sim g$. The indicator function of a set A is denoted 1_A, that is $1_A(x) = 1$ if $x \in A$ and 0 otherwise. A function f is said to be positive if it satisfies $f \ge 0$; that is, we do not require a strict inequality.

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2. Results

Our first result states that the atomic Hardy space is contained in all α-Hardy spaces, and that the reverse inclusion holds if $\alpha > \frac{1}{2}$. In particular, all α-Hardy spaces are the same for $\alpha > \frac{1}{2}$.

PROPOSITION 2.1. *Fix $\alpha \in \left[\frac{1}{2}, +\infty\right[$. Then $H^1_{\mathrm{at}}(\partial T) \subseteq H^1_\alpha(\partial T)$, and $\|f\|_{H^1_\alpha(\partial T)} \le C\,\|f\|_{H^1_{\mathrm{at}}(\partial T)}$, for all $f \in H^1_{\mathrm{at}}(\partial T)$. Furthermore, if $\alpha > \frac{1}{2}$, then $H^1_{\mathrm{at}}(\partial T) = H^1_\alpha(\partial T)$ and $\|f\|_{H^1_{\mathrm{at}}(\partial T)} \le C\,\|f\|_{H^1_\alpha(\partial T)}$, for all $f \in H^1_\alpha(\partial T)$.*

The positive functions in the atomic Hardy space admit a nice characterization, identical to that in the continuous setting [7], in terms of an Orlicz space.

PROPOSITION 2.2. *If* $f \in L^1(\partial T)$ *is positive, then* $f \in H^1_{\mathrm{at}}(\partial T)$ *if and only if* $f \in L \log L(\partial T)$.

Our next proposition will allow us to deduce the corresponding characterization of positive and increasing – as defined below – functions in $H^1_{1/2}(\partial T)$.

PROPOSITION 2.3. *Let* T *be a rooted* $(q+1)$*-regular tree with* $q \geq 2$*. The following hold for all positive functions* $f \in L^1(\partial T)$.

(i) $\sum_{n\geq 0} f_{E(\zeta_n)} \frac{q^{-n}}{n+1} \leq C \, \|f\|_{H^1_{1/2}(\partial T)}$, *for all* $\zeta \in \partial T$.

(ii) *If there exists* $\zeta \in \partial T$ *such that* $f_{E(\eta_k)} \leq f_{E(\eta_{N(\eta,\zeta)})}$, *for all* $\eta \in \partial T \setminus \{\zeta\}$ *and all* $k \in \mathsf{N}$, *then* $\|f\|_{H^1_{1/2}(\partial T)} \leq C \sum_{n\geq 0} f_{E(\zeta_n)} q^{-n}$.

(iii) *If* f *is increasing towards* $\zeta \in \partial T$, *then* $\|f\|_{H^1_{1/2}(\partial T)} \leq C \sum_{n\geq 0} f_{E(\zeta_n)} \cdot \frac{q^{-n}}{n+1}$.

The constant C *does not depend on* f *or* ζ.

We have the following characterization, similar to that in Proposition 2.2, of the positive and increasing functions in $H^1_{1/2}(\partial T)$. In particular, we deduce that $H^1_{1/2}(\partial T)$ is strictly larger than $H^1_{\mathrm{at}}(\partial T)$, and hence also strictly larger than $H^1_{\alpha}(\partial T)$ for all $\alpha > \frac{1}{2}$.

PROPOSITION 2.4. *If* $f \in L^1(\partial T)$ *is positive and increasing, then* $f \in H^1_{1/2}(\partial T)$ *if and only if* $f \in L \log\log L(\partial T)$.

Our main result is, roughly speaking, that neither implication in Proposition 2.4 holds for general positive functions, not even if we allow a larger or smaller Orlicz space.

THEOREM 2.5.

(i) *Let* $L^{\Phi}(\partial T)$ *be an Orlicz space. If* $\lim_{t\to+\infty} \Phi(t)/(t \log t) = 0$, *then there exists a positive function* $f \in L^{\Phi}(\partial T)$ *such that* $f \notin H^1_{1/2}(\partial T)$.

(ii) *If* $L^{\Phi}(\partial T)$ *is of type* Δ_2 *– that is, if* $\Phi(2t) \leq C\Phi(t)$ *for all sufficiently large* t *– and* $\lim_{t\to+\infty} \Phi(t)/t = +\infty$, *then there exists a positive function* $g \in H^1_{1/2}(\partial T)$ *such that* $g \notin L^{\Phi}(\partial T)$.

3. Preliminaries

Identifying each vertex with the geodesic path connecting it to the root, we let $N(u, v)$ denote the number of edges shared by $u, v \in T \cup \partial T$. The distance between $x, y \in T$ is defined as the number of edges on the geodesic between

x and y, and $|x|$ denotes the distance between x and the root. Recall that ζ_n denotes the vertex $x \in T$ with $|x| = n$ and $N(x, \zeta) = n$. Every vertex $x \in T$ determines an interval $E(x) \subseteq \partial T$ defined by

$$E(x) = \{\eta \in \partial T;\ N(x, \eta) = |x|\}.$$

This allows us to define a Borel probability measure ω on ∂T by letting

$$\omega(E(x)) = \begin{cases} 1, & \text{if } |x| = 0, \quad \text{and} \\ \dfrac{1}{(q+1)q^{|x|-1}}, & \text{if } |x| \geq 1, \end{cases} \tag{2}$$

where $q + 1$ is the degree of the tree. Unless otherwise stated, the measure on ∂T is always ω.

For trees, the atomic Hardy space [1] is defined as follows.

DEFINITION. The *atomic Hardy space* on ∂T, denoted $H^1_{\text{at}}(\partial T)$, is the subspace of $L^1(\partial T)$ consisting of functions $f \in L^1(\partial T)$ such that $f = \sum_j \lambda_j a_j$, where $\sum_j |\lambda_j| < +\infty$ and a_j are atoms. A norm on $\|\cdot\|_{H^1_{\text{at}}(\partial T)}$ is defined by $\|f\|_{H^1_{\text{at}}(\partial T)} = \inf \sum_j |\lambda_j|$, where the infimum is over all atomic decompositions $\sum_j \lambda_j a_j$ of f.

An atom [1], [4] is a function of the following type.

DEFINITION. A function $a \in L^1(\partial T)$ is said to be an *atom* if it is the constant 1, or if there exists an interval $E(x) \subseteq \partial T$ such that

(i) $\operatorname{supp} a \subseteq E(x)$,

(ii) $\|a\|_{L^{+\infty}(\partial T)} \leq \frac{1}{\omega(E(x))}$, and

(iii) $\int_{\partial T} a(\zeta)\, \mathrm{d}\omega(\zeta) = 0$.

Apart from the radial Poisson maximal function $\mathscr{P}^*_\alpha$, defined by (1) in the introduction, we will also make use of the Hardy-Littlewood maximal function M on $L^1(\partial T)$, defined by $(Mf)(\zeta) = \sup_{n \in \mathsf{N}} |f_{E(\zeta_n)}|$, where $f_E = \frac{1}{\omega(E)} \int_E f \,\mathrm{d}\omega$ is the mean value of f over $E \subseteq \partial T$.

As mentioned in the introduction, the kernel of the operator $\mathscr{P}_\alpha$ is the exit distribution of the $\mathscr{P}_\alpha$-random walk, but we will not make use of the explicit step probabilities of these random walks. However, to prove Proposition 2.1 we will use the fact [9, paper 4 or 5] that, for $\alpha > \frac{1}{2}$, the probability $p(x, y)$ of stepping from x to y satisfies

$$\delta \leq p(x, y) \leq \frac{1}{2} - \delta, \tag{3}$$

for all neighbors x and y, where $\delta > 0$ depends only on α. For $\alpha = \frac{1}{2}$, no $\delta > 0$ satisfies the upper bound for all x and y, but the following result will allow us to prove the inclusion $H^1_{\mathrm{at}}(\partial T) \subseteq H^1_{1/2}(\partial T)$ in Proposition 2.1.

PROPOSITION 3.1. *Let $q \geq 2$ be an integer. The probability that a $\mathscr{P}_{\frac{1}{2}}$-random walk on the $(q+1)$-regular rooted tree T ever reaches y when starting at x, denoted $F_{\frac{1}{2}}(x, y)$, is given by*

$$F_{\frac{1}{2}}(x, y) = \frac{(P_{\frac{1}{2}}1)(y)}{(P_{\frac{1}{2}}1)(x)} \cdot q^{-d(x,y)/2}.$$

The proof can be found in [9, paper 4 or 5], and the following closed formula for the spherical function $P_{\frac{1}{2}}1$ is well known [2].

LEMMA 3.2. *Let $q \geq 2$ be an integer. Then $(P_{\frac{1}{2}}1)(z) = (P_{\frac{1}{2}}1)(|z|) = q^{-|z|/2}\big(1 + \frac{q-1}{q+1} \cdot |z|\big)$, for all $z \in T$.*

We now define increasing functions on ∂T.

DEFINITION 3.3. A function $f \in L^1(\partial T)$ is said to be *increasing* – or, more specifically, *increasing towards* $\zeta \in \partial T$ – if there exists an increasing function $\tilde{f} \colon \mathsf{N} \to \mathsf{R}$ such that $f(\eta) = \tilde{f}\big(N(\eta, \zeta)\big)$, for all $\eta \in \partial T \setminus \{\zeta\}$.

Propositions 2.2 and 2.4, as well as Theorem 2.5, make use of Orlicz spaces. Let $\Phi \colon [0, +\infty[$ be a convex and increasing function with $\Phi(0) = 0$ and $\lim_{x\to\infty} \Phi(x) = +\infty$. The Orlicz space $L^{\Phi}(\partial T)$ is the set of all measurable functions $f \colon \partial T \to \mathsf{R}$ for which there exists a constant $\gamma > 0$ such that $\int_{\partial T} \Phi\big(\gamma\, |f(\zeta)|\big)\, \mathrm{d}\omega(\zeta) < +\infty$. If $\Phi(2x)/\Phi(x)$ is bounded when x is large, then $L^{\Phi}(\partial T)$ is said to be of type Δ_2, and in this case it is enough to consider $\gamma = 1$ in the definition. We refer to [6] for proofs of these properties.

We also let $L \log L(\partial T)$ and $L \log\log L(\partial T)$ denote the Orlicz spaces with $\Phi(t) = t\log(1+t)$ and $\Phi(t) = t\log\big(1 + \log(1+t)\big)$, respectively. It is easy to see that both $L \log L(\partial T)$ and $L \log\log L(\partial T)$ are of type Δ_2.

4. Proofs

Proposition 2.1 is a special case of [4, Theorem 5] when $\alpha > \frac{1}{2}$, and the case $\alpha = \frac{1}{2}$ is also essentially contained in that theorem.

PROOF OF PROPOSITION 2.1. The result for $\alpha > \frac{1}{2}$ follows from [4, Theorem 5]. Indeed, in that case the operator $\mathscr{P}_\alpha$ is associated to a random walk – the $\mathscr{P}_\alpha$-random walk – which is of nearest neighbor type and satisfies (3); that is, the random walk satisfies conditions conditions (H1) and (H2) in [4]. When

$\alpha = \frac{1}{2}$, condition (3) no longer holds, but as we shall see, the same method of proof works.

In fact, by the proof of [4, Theorem 6] we see that $\|Mf\|_{L^1(\partial T)} \leq \|f\|_{H^1_{\mathrm{at}}(\partial T)}$, and by [4, Proposition 8] we have that

$$(\mathscr{P}_{\frac{1}{2}} f)(\zeta_m) = \sum_{k=0}^{m} f_{E(\zeta_k)}(1-\alpha_k) \prod_{i=k+1}^{m} \alpha_i, \qquad \text{all} \quad m \geq 1,$$

where $\alpha_0 = 0$ and $\alpha_i = F_{\frac{1}{2}}(\zeta_i, \zeta_{i-1})$. Hence, using Proposition 3.1 and Lemma 3.2, we get that

$$(1-\alpha_k) \prod_{i=k+1}^{m} \alpha_i = \left(1 - \frac{(P_{\frac{1}{2}}1)(k-1)}{(P_{\frac{1}{2}}1)(k)} \cdot q^{-\frac{1}{2}}\right) \cdot \frac{(P_{\frac{1}{2}}1)(k)}{(P_{\frac{1}{2}}1)(m)} \cdot q^{\frac{k-m}{2}} = \frac{1}{m + \frac{q+1}{q-1}},$$

so that

$$\left|(\mathscr{P}_{\frac{1}{2}} f)(\zeta_m)\right| \leq \frac{1}{m + \frac{q+1}{q-1}} \sum_{k=0}^{m} |f_{E(\zeta_k)}| \leq (Mf)(\zeta),$$

and finally $\left\|\mathscr{P}^*_{\frac{1}{2}} f\right\|_{L^1(\partial T)} \leq \|Mf\|_{L^1(\partial T)}$.

We now turn to proving our results for positive functions.

Proof of Proposition 2.2. The proof is identical to the continuous proof – see for instance [3, Theorems II.2.1 and II.2.4] – using the Calderón-Zygmund decomposition given by [4, Lemma 6] and the characterization of $H^1_{\mathrm{at}}(\partial T)$ in terms of the Hardy-Littlewood maximal operator [4, Theorem 5 (i)].

In the proof of Proposition 2.3, we will make use of the following approximation of $\mathscr{P}_{\frac{1}{2}} f$.

Lemma 4.1. *Assume that* $f \in L^1(\partial T)$ *is positive. Then* $(\mathscr{P}_{\frac{1}{2}} f)(\eta_n) \sim \frac{1}{n+1} \sum_{k=0}^{n} f_{E(\eta_k)}$, *for all* $\eta \in \partial T$ *and all* $n \in \mathsf{N}$, *with universal constants of comparison.*

Proof of Lemma 4.1. Let $q+1$ be the degree of the tree. By definition

$$(\mathscr{P}_{\frac{1}{2}} f)(x) = \frac{1}{(P_{\frac{1}{2}}1)(x)} \int_{\partial T} q^{N(x,\zeta) - \frac{|x|}{2}} f(\zeta)\, \mathrm{d}\omega(\zeta),$$

where, by Lemma 3.2, $(P_{\frac{1}{2}}1)(x) \sim (|x|+1) q^{-\frac{|x|}{2}}$, with universal constants of comparison. Hence

$$\begin{aligned}
&(n+1)\cdot(\mathscr{P}_{1/2}f)(\eta_n)\\
&\quad\sim\int_{\partial T} q^{N(\eta_n,\zeta)} f(\zeta)\,\mathrm{d}\omega(\zeta)\\
&\quad=\sum_{k=0}^{n}\int_{E(\eta_k)\setminus E(\eta_{k+1})} q^k\cdot f(\zeta)\,\mathrm{d}\omega(\zeta)+\int_{E(\eta_{n+1})} q^n\cdot f(\zeta)\,\mathrm{d}\omega(\zeta)\\
&\quad=\sum_{k=0}^{n} q^k\cdot\big(\omega(E(\eta_k))f_{E(\eta_k)}-\omega(E(\eta_{k+1}))f_{E(\eta_{k+1})}\big)\\
&\qquad\quad+q^n\cdot\omega(E(\eta_{n+1}))f_{E(\eta_{n+1})}\\
&\quad=f_{E(\eta_0)}+(1-q^{-1})\sum_{k=1}^{n} q^k\cdot\omega(E(\eta_k))\cdot f_{E(\eta_k)},
\end{aligned}$$

which proves the lemma, since $\omega(E(\eta_k)) = q^{1-k}/(q+1)$, for $k \geq 1$.

PROOF OF PROPOSITION 2.3. *Part* (i). Let $\zeta \in \partial T$. We have that

$$\begin{aligned}
\big\|\mathscr{P}^*_{\frac12} f\big\|_{L^1(\partial T)} &\geq \int_{\eta\in\partial T} (\mathscr{P}_{\frac12} f)(\eta_{N(\zeta,\eta)})\,\mathrm{d}\omega(\eta)\\
&= \sum_{n\geq 0}\int_{E(\zeta_n)\setminus E(\zeta_{n+1})} (\mathscr{P}_{\frac12} f)(\zeta_n)\,\mathrm{d}\omega(\eta)\\
&= \sum_{n\geq 0}(\mathscr{P}_{\frac12} f)(\zeta_n)\cdot\omega(E(\zeta_n)\setminus E(\zeta_{n+1}))\\
&\geq c\sum_{n\geq 0}\frac{q^{-n}}{n+1}\sum_{k=0}^{n} f_{E(\zeta_k)}\\
&\geq c\sum_{n\geq 0}\frac{q^{-n}}{n+1} f_{E(\zeta_n)},
\end{aligned}$$

where the second inequality follows from Lemma 4.1 and (2), completing the proof of part (i).

Part (ii). Using Lemma 4.1, the condition on f, and (2), we see that for all $\eta \in \partial T \setminus \{\zeta\}$ and all $n \in \mathsf{N}$,

$$(\mathscr{P}_{\frac12} f)(\eta_n) \sim \frac{1}{n+1}\sum_{k=0}^{n} f_{E(\eta_k)} \leq f_{E(\eta_{N(\eta,\zeta)})},$$

so that $(\mathscr{P}^*_{\frac{1}{2}} f)(\eta) \leq C f_{E(\zeta_{N(\eta,\zeta)})}$, and hence

$$\begin{aligned}\left\|\mathscr{P}^*_{\frac{1}{2}} f\right\|_{L^1(\partial T)} &\leq C \int_{\partial T} f_{E(\zeta_{N(\eta,\zeta)})}\,\mathrm{d}\omega(\eta)\\ &= C \sum_{n\geq 0} \int_{E(\zeta_n)\setminus E(\zeta_{n+1})} f_{E(\zeta_n)}\,\mathrm{d}\omega(\eta)\\ &= C \sum_{n\geq 0} f_{E(\zeta_n)}\omega(E(\zeta_n)\setminus E(\zeta_{n+1})) \sim \sum_{n\geq 0} f_{E(\zeta_n)} q^{-n},\end{aligned}$$

and (ii) follows.

Part (iii). Fix $\eta \in \partial T$ and $n \in \mathsf{N}$. Let $\zeta \in \partial T$ be the point towards which f is increasing. By definition and Lemma 3.2,

$$(\mathscr{P}_{\frac{1}{2}} f)(\eta_n) \sim \frac{1}{n+1} \int_{\partial T} q^{N(\eta_n,\nu)} f(\nu)\,\mathrm{d}\omega(\nu),$$

and we consider first the integration over $\nu \in \partial T$ with $N(\nu,\zeta) > N(\eta,\zeta)$. Observing that $N(\eta,\nu) = N(\eta,\zeta)$ in this case, we get that

$$\begin{aligned}&\frac{1}{n+1} \int_{\nu:N(\nu,\zeta)>N(\eta,\zeta)} q^{N(\eta_n,\nu)} f(\nu)\,\mathrm{d}\omega(\nu)\\ &\quad= \frac{q^{\min(n,N(\eta,\zeta))}}{n+1} \cdot \omega(E(\zeta_{N(\eta,\zeta)+1})) \cdot f_{E(\zeta_{N(\eta,\zeta)+1})}\\ &\quad\leq \frac{C q^{N(\eta,\zeta)}}{N(\eta,\zeta)+1} \cdot q^{-N(\eta,\zeta)} \cdot f_{E(\zeta_{N(\eta,\zeta)+1})}\\ &\quad= \frac{C}{N(\eta,\zeta)+1} \cdot f_{E(\zeta_{N(\eta,\zeta)+1})},\end{aligned}$$

where the last inequality uses the fact that $t \mapsto q^t/(t+1)$ is increasing for, say, $t \geq 1$. If instead $N(\nu,\zeta) \leq N(\eta,\zeta)$, then $f(\nu) \leq f(\eta)$, so that

$$\begin{aligned}\frac{1}{n+1} \int_{\nu:N(\nu,\zeta)\leq N(\eta,\zeta)} q^{N(\eta_n,\nu)} f(\nu)\,\mathrm{d}\omega(\nu) &\leq \frac{f(\eta)}{n+1} \int_{\partial T} q^{N(\eta_n,\nu)}\,\mathrm{d}\omega(\nu)\\ &\leq C f(\eta),\end{aligned}$$

and hence

$$(\mathscr{P}^*_{\frac{1}{2}} f)(\eta) \leq \frac{C}{N(\eta,\zeta)+1} \cdot f_{E(\zeta_{N(\eta,\zeta)+1})} + C f(\eta).$$

Integrating over ∂T we get that

$$\begin{aligned}\left\|\mathscr{P}^*_{\frac{1}{2}} f\right\|_{L^1(\partial T)} &\leq C \sum_{k\geq 0} \int_{E(\zeta_k)\setminus E(\zeta_{k+1})} \frac{f_{E(\zeta_{k+1})}}{k+1}\,\mathrm{d}\omega(\eta) + C\,\|f\|_{L^1(\partial T)} \\ &= C \sum_{k\geq 0} \omega(E(\zeta_k)\setminus E(\zeta_{k+1})) \cdot \frac{f_{E(\zeta_{k+1})}}{k+1} + C\,\|f\|_{L^1(\partial T)} \\ &\leq C \sum_{k\geq 0} \frac{q^{-k}}{k+1} \cdot f_{E(\zeta_k)},\end{aligned}$$

since $\|f\|_{L^1(\partial T)} = f_{E(\zeta_0)}$, completing the proof of part (iii).

Remark. Let $f \in L^1(\partial T)$ be a positive function, increasing towards $\zeta \in \partial T$. We claim that for every $\eta \in \partial T$ and every $k \in \mathsf{N}$,

$$f_{E(\eta_k)} \leq f_{E(\eta_{N(\eta,\zeta)})},$$

that is, f satisfies condition (ii) in Proposition 2.3. This immediately follows if $k > N(\eta,\zeta)$, since in that case $f(\nu) \leq f(\eta) \leq f(\tilde{\nu})$, for all $\nu \in E(\eta_k)$ and all $\tilde{\nu} \in E(\eta_{N(\eta,\zeta)}) \setminus \{\zeta\}$. If instead $k \leq N(\eta,\zeta)$, then

$$\begin{aligned}\int_{E(\eta_k)} f\,\mathrm{d}\omega &= \int_{E(\eta_{N(\eta,\zeta)})} f\,\mathrm{d}\omega + \int_{E(\eta_k)\setminus E(\eta_{N(\eta,\zeta)})} f\,\mathrm{d}\omega \\ &\leq \int_{E(\eta_{N(\eta,\zeta)})} f\,\mathrm{d}\omega + \omega(E(\eta_k)\setminus E(\eta_{N(\eta,\zeta)})) \cdot f_{E(\eta_{N(\eta,\zeta)})} \\ &= \omega(E(\eta_k)) \cdot f_{E(\eta_{N(\eta,\zeta)})},\end{aligned}$$

where the inequality follows since $f(\nu) \leq f_{E(\eta_{N(\eta,\zeta)})}$ if $\nu \in E(\eta_k)\setminus E(\eta_{N(\eta,\zeta)})$, in fact $f(\nu) \leq f(\tilde{\nu})$ for all $\tilde{\nu} \in E(\eta_{N(\eta,\zeta)}) \setminus \{\zeta\}$.

Proving the $L\log\log L(\partial T)$-characterization in Proposition 2.4 is now simply a matter of estimating the sums in (i) and (iii) of Proposition 2.3. The first part of that calculation is the following lemma, which does not require the function to be increasing.

Lemma 4.2. *Let T be a rooted $(q+1)$-regular tree with $q \geq 2$, and let $\zeta \in \partial T$. If $\tilde{f}\colon \mathsf{N} \to \mathsf{R}$ is a any function satisfying $\sum_{k\geq 0} |\tilde{f}(k)|q^{-k} < +\infty$, then $f(\eta) = \tilde{f}(N(\eta,\zeta))$, $\eta \in \partial T\setminus\{\zeta\}$, defines an integrable function $\partial T \to \mathsf{R}$ with*

$$\int_{\partial T} f\,\mathrm{d}\omega = \frac{q}{q+1}\tilde{f}(0) + \frac{q-1}{q+1}\sum_{k\geq 1} \tilde{f}(k)q^{-k}. \tag{4}$$

If in addition $\tilde{f} \geq 0$, *then*

$$\sum_{k\geq 0} f_{E(\eta_k)} \frac{q^{-k}}{k+1} \sim \sum_{k\geq 0} \tilde{f}(k) q^{-k} \log(k+2), \tag{5}$$

for all $\eta \in \partial T$, *with universal constants of comparison.*

Proof of Lemma 4.2. Obviously f is measurable. If $f \geq 0$, then

$$\int_{\partial T} f \,\mathrm{d}\omega = \sum_{k\geq 0} \int_{E(\eta_k)\setminus E(\eta_{k+1})} f \,\mathrm{d}\omega = \sum_{k\geq 0} \tilde{f}(k)\omega(E(\eta_k) \setminus E(\eta_{k+1})), \tag{6}$$

and applied to $|f|$, this shows – using (2) – that f is integrable, since $\sum_{k\geq 0} |\tilde{f}(k)| q^{-k} < +\infty$. Hence, (6) holds for general f and (4) immediately follows.

Now assume that $\tilde{f} \geq 0$. Letting $n \in \mathsf{N}$ and applying (4) to the function $k \mapsto \tilde{f}(k) 1_{[n,+\infty[}(k)$, we get that

$$\int_{E(\zeta_n)} f \,\mathrm{d}\omega \sim \sum_{k\geq n} \tilde{f}(k) q^{-k},$$

and hence $f_{E(\zeta_n)} \sim q^n \sum_{k\geq n} \tilde{f}(k) q^{-k}$, both estimates with universal constants of comparison. Consequently

$$\sum_{n\geq 0} f_{E(\eta_n)} \frac{q^{-n}}{n+1} \sim \sum_{k\geq 0} \tilde{f}(k) q^{-k} \sum_{n=0}^{k} \frac{1}{n+1} \sim \sum_{k\geq 0} \tilde{f}(k) q^{-k} \log(k+2),$$

with universal constants of comparison.

Proof of Proposition 2.4. Since $f \geq 0$ is increasing, there exists $\zeta \in \partial T$ and an increasing function $\tilde{f} \colon \mathsf{N} \cup \{+\infty\} \to [0, +\infty]$ such that $f(\eta) = \tilde{f}(N(\eta, \zeta))$, for all $\eta \in \partial T$.

Assume that $f \in H^1_{1/2}(\partial T)$. Without loss of generality, we may also assume that $\|f\|_{L^1(\partial T)} \leq 1$. First observe that $f(\eta) \leq q^{N(\eta,\zeta)+2}$, since by Lemma 4.2,

$$1 \geq \|f\|_{L^1(\partial T))} \geq \frac{q-1}{q+1} \sum_{k\geq 0} \tilde{f}(k) q^{-k} \geq \frac{1}{3} \tilde{f}(n) q^{-n},$$

and hence $\tilde{f}(n) \leq 3q^n \leq q^{n+2}$, Consequently

$$\begin{aligned} \log\bigl(1 + \log(1 + f(\eta))\bigr) &\leq \log(1 + \log(1 + q^{N(\eta,\zeta)+2})) \\ &\leq C \log\log(q^{N(\eta,\zeta)+2}) \\ &\leq C \log(N(\eta, \zeta) + 2), \end{aligned}$$

with C depending on q. By (4) and (5) in Lemma 4.2, we then get that

$$\begin{aligned}
&\int_{\partial T} f(\eta)\log\bigl(1+\log\bigl(1+f(\eta)\bigr)\bigr)\,\mathrm{d}\omega(\eta)\\
&\qquad\le C\int_{\partial T} f(\eta)\log\bigl(N(\eta,\zeta)+2\bigr)\,\mathrm{d}\omega(\eta)\\
&\qquad\le C\sum_{k\ge 0}\tilde{f}(k)\log(k+2)\cdot q^{-k}\\
&\qquad\le C\sum_{k\ge 0} f_{E(\zeta_k)}\frac{q^{-k}}{k+1},
\end{aligned}$$

and this last sum is finite by Proposition 2.3 (i).

Now assume instead that $f\in L\log\log L(\partial T)$ and let $g(\eta)=f(\eta)+q^{\frac{N(\eta,\zeta)+5}{2}}$ for $\eta\in\partial T\setminus\{\zeta\}$. Using (4) in Lemma 4.2 we see that the last term, as a function of η, is in $L\log\log L(\partial T)$, and hence so is g. Defining a function $\tilde{g}\colon \mathsf{N}\to[0,+\infty[$ by $\tilde{g}(k)=g(\zeta_k)$, we see that $g(\eta)=\tilde{g}\bigl(N(\eta,\zeta)\bigr)$, for all $\eta\in\partial T\setminus\{\zeta\}$. Furthermore, $\log\tilde{g}(k)\ge\frac{k+5}{4}$, and using (5) in Lemma 4.2 we get that

$$\begin{aligned}
\sum_{k\ge 0} f_{E(\zeta_k)}\frac{q^{-k}}{k+1} &\le C\sum_{k\ge 0}\tilde{f}(k)q^{-k}\log(k+2)\\
&\le C\sum_{k\ge 0}\tilde{g}(k)q^{-k}\log(\log\tilde{g}(k))\\
&\le C\int_{\partial T} g(\eta)\log(1+\log(1+g(\eta)))\,\mathrm{d}\omega(\eta),
\end{aligned}$$

where the last inequality follows from (4) in Lemma 4.2. Since $g\in L\log\log L(\partial T)$ this proves that $\sum_{k\ge 0} f_{E(\zeta_k)}\frac{q^{-k}}{k+1}<+\infty$, and hence $f\in H^1_{1/2}(\partial T)$ by Proposition 2.3 (iii).

Just as in the continuous case [8], we prove Theorem 2.5 by summing simpler functions with disjoint support. The functions we will use are given by the following two lemmas.

LEMMA 4.3. *Let T be a rooted $(q+1)$-regular tree with $q\ge 2$. Furthermore, let $n\ge 1$, $N\ge 0$ be integers, and let $x\in T$. If $q^N\ge n^2$ and $|x|\le n$, then there exists a subset M of the interval $E(x)$ such that $\omega(M)=\frac{q}{q+1}\cdot q^{-n-N}$ and*

$$\int_{E(x)}(\mathscr{P}^*_{\frac{1}{2}}1_M)(\eta)\,\mathrm{d}\omega(\eta)\ge cNq^{-n-N},$$

where c does not depend on n, N or x.

LEMMA 4.4. *Let T be a rooted $(q+1)$-regular tree with $q \geq 2$. Furthermore, let $x \in T$ and let $N \geq 1$ be an integer. There exists a subset M of the interval $E(x)$ with measure $\omega(M) = \frac{q}{q+1} \cdot q^{-|x|-q^N}$ and whose indicator function satisfies $\|\mathscr{P}^*_{\frac{1}{2}} 1_M\|_{L^1(\partial T)} \leq C q^{-q^N}$, where C is a universal constant.*

PROOF OF LEMMA 4.3. Without loss of generality we may assume that $N \geq 4$ and that $|x| = n$. For each $m \geq 0$ we denote by $x_1^m, \ldots, x_{q^m}^m$ the elements in T with $N(x, x_i^m) = |x|$ that are at a distance m from x, numbered so that $N(x_i^m, x_{q\cdot(i-1)+j}^{m+1}) = |x| + m$, for $i = 1, \ldots, q^m$ and $j = 1, \ldots, q$ (figure 1). Then, of course, the intervals $E(x_1^m), \ldots, E(x_{q^m}^m)$ partition $E(x)$.

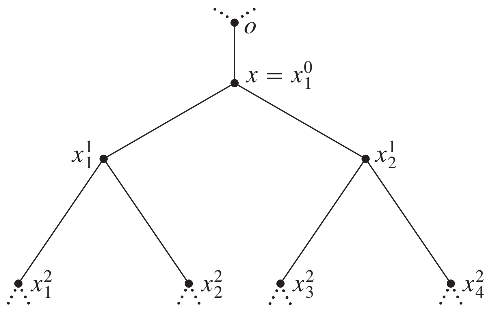

FIGURE 1. The vertices $x_1^m, \ldots, x_{q^m}^m$ are drawn for $m = 0, 1, 2$ and a vertex $x \in T$ with $|x| = 1$. In the figure, $q = 2$ and the root is labeled o.

Now let $K_0 = E(x)$ and, for $\mu \geq 1$,

$$K_\mu = \bigcup_{i=1}^{q^{q^\mu - 1}} E(x_{q\cdot i}^{q^\mu}) \cap K_{\mu-1}, \qquad I_\mu = K_{\mu-1} \setminus K_\mu.$$

The set K_μ consists of N_μ – whose exact value we will not need – pairwise disjoint intervals $K_{\mu,1}, \ldots, K_{\mu,N_\mu}$. Note that $\omega(K_\mu) = \frac{1}{q^\mu}\omega(K_0) = \frac{q}{q+1} \cdot q^{-n-\mu}$.

Choose $M = K_N$ – which is a set with $\omega(M) = \frac{q}{q+1} \cdot q^{-n-N}$ – and pick an integer μ such that $\frac{N}{2} \leq \mu \leq N$. Suppose that $\eta \in I_\mu$ and let k be such that $K_{\mu-1,k} \supseteq I_\mu$. Then, by definition and Lemma 3.2,

$$\begin{aligned}
(\mathscr{P}^*_{\frac{1}{2}} 1_M)(\eta) &\geq (\mathscr{P}_{\frac{1}{2}} 1_M)(\eta_{q^\mu+n-1}) \\
&\geq \frac{c}{q^\mu + n} \int_{K_{\mu-1,k}} q^{N(\eta_{q^\mu+n-1},\zeta)} 1_M(\zeta)\, \mathrm{d}\omega(\zeta) \\
&= \frac{c}{q^\mu + n} \sum_i \int_{K_{\mu,i}} q^{N(\eta_{q^\mu+n-1},\zeta)} 1_M(\zeta)\, \mathrm{d}\omega(\zeta)
\end{aligned}$$

where the sum is over those i for which $K_{\mu,i} \subseteq K_{\mu-1,k}$. Note that if $\zeta, \zeta' \in E(x_j^{q^\mu-1})$, for some j, then $N(\zeta,\zeta') \geq q^\mu + n - 1$, since $|x| = n$. In particular, this implies that $q^{N(\eta_{q^\mu+n-1},\zeta)}$ is constant for $\zeta \in K_{\mu,i}$, allowing us to replace 1_M by its average value $q^{\mu-N}$ in the integral above, so that

$$(\mathscr{P}^*_{\frac{1}{2}} 1_M)(\eta) \geq \frac{cq^{\mu-N}}{q^\mu+n} \sum_i \int_{K_{\mu,i}} q^{N(\eta_{q^\mu+n-1},\zeta)} \, \mathrm{d}\omega(\zeta)$$
$$= \frac{cq^{\mu-N}}{q^\mu+n} \cdot \frac{1}{q} \int_{K_{\mu-1,k}} q^{N(\eta_{q^\mu+n-1},\zeta)} \, \mathrm{d}\omega(\zeta),$$

where the last equality follows by extending the integration from $K_{\mu,i}$ to include the neighboring intervals in I_μ. Now, since $\eta \in I_\mu \subseteq K_{\mu-1,k}$ by assumption,

$$\int_{K_{\mu-1,k}} q^{N(\eta_{q^\mu+n-1},\zeta)} \, \mathrm{d}\omega(\zeta) \geq \sum_{i=q^{\mu-1}+n}^{q^\mu+n-1} q^i \cdot \omega(E(\eta_i) \setminus E(\eta_{i+1})) \geq cq^\mu,$$

so that

$$(\mathscr{P}^*_{\frac{1}{2}} 1_M)(\eta) \geq \frac{cq^{\mu-N}}{q^\mu+n} \cdot q^\mu,$$

and hence

$$\int_{I_\mu} (\mathscr{P}^*_{\frac{1}{2}} 1_M)(\eta) \, \mathrm{d}\omega(\eta) \geq \frac{cq^{\mu-N}}{q^\mu+n} \cdot q^\mu \cdot \omega(I_\mu) \geq cq^{-n-N},$$

where the last inequality follows since $n \leq q^{\frac{N}{2}} \leq q^\mu$. Finally, summing over integers μ such that $\frac{N}{2} \leq \mu \leq N$ we get that

$$\int_{E(x)} (\mathscr{P}^*_{\frac{1}{2}} 1_M)(\eta) \, \mathrm{d}\omega(\eta) \geq cNq^{-n-N},$$

since the intervals $(I_\mu)_\mu$, all subsets of $E(x)$, are pairwise disjoint.

As a warm-up for the proof of Lemma 4.4, we give the following estimate of $\|\mathscr{P}^*_{\frac{1}{2}} 1_{E(y)}\|_{L^1(\partial T)}$, where $E(y)$ is an interval.

PROPOSITION 4.5. *Let T be a rooted $(q+1)$-regular tree with $q \geq 2$. There exists a universal constant C such that*

$$\|\mathscr{P}^*_{\frac{1}{2}} 1_{E(y)}\|_{L^1(\partial T)} \leq C(|y|+1)q^{-|y|},$$

for all $y \in T$.

PROOF OF PROPOSITION 4.5. By Lemma 4.1, there exists a universal constant $C < +\infty$ such that for all $y \in T$, $\eta \in \partial T$, and $n \in \mathsf{N}$,

$$(\mathscr{P}_{\frac{1}{2}} 1_{E(y)})(\eta_n) \leq \frac{C}{n+1} \sum_{k=0}^{n} \frac{\omega(E(y) \cap E(\eta_k))}{\omega(E(\eta_k))},$$

and hence

$$(\mathscr{P}^*_{\frac{1}{2}} 1_{E(y)})(\eta) \leq C \sup_{k \geq 0} \frac{\omega(E(y) \cap E(\eta_k))}{\omega(E(\eta_k))}.$$

Since $\omega(E(y) \cap E(\eta_k)) = \min(\omega(E(y)), \omega(E(\eta_k)))$ if $N(y, \eta) \geq \min(|y|,k)$, and zero otherwise, we get that

$$\begin{aligned}
c(\mathscr{P}^*_{\frac{1}{2}} 1_{E(y)})(\eta) &\leq \max_{0 \leq k \leq |y|} \frac{\omega(E(y) \cap E(\eta_k))}{\omega(E(\eta_k))} + \sup_{k \geq |y|} \frac{\omega(E(y) \cap E(\eta_k))}{\omega(E(\eta_k))} \\
&= \max_{0 \leq k \leq \min(N(y,\eta),|y|)} \frac{\omega(E(y))}{\omega(E(\eta_k))} + \sup_{k \geq |y|} \frac{\omega(E(\eta_k))}{\omega(E(\eta_k))} 1_{E(y)}(\eta) \\
&\leq \frac{\omega(E(y))}{\omega(E(\eta_{N(y,\eta)}))} + 1_{E(y)}(\eta) \leq C q^{N(y,\eta)-|y|} + 1_{E(y)}(\eta).
\end{aligned}$$

Integrating over ∂T, we get that

$$\begin{aligned}
\|\mathscr{P}^*_{\frac{1}{2}} 1_{E(y)}\|_{L^1(\partial T)} &\leq C \int_{\partial T} q^{N(y,\eta)-|y|} \, \mathrm{d}\omega(\eta) + C\omega(E(y)) \\
&= C q^{-\frac{|y|}{2}} (P_{\frac{1}{2}} 1)(y) + C\omega(E(y)) \\
&\leq C q^{-\frac{|y|}{2}} \cdot q^{-\frac{|y|}{2}} (|y|+1) + C\omega(E(y)) \\
&\leq C(|y|+1) q^{-|y|},
\end{aligned}$$

where the estimate of $(P_{\frac{1}{2}} 1)(y)$ follows from Lemma 3.2.

Picking $y \in T$ with $|y| = |x| + q^N$ and $N(x, y) = |x|$, Proposition 4.5 immediately implies a weaker version of Lemma 4.4. As we shall see, the original claim follows if we instead pick q^{q^N} suitable vertices $y \in T$ with $|y| = |x| + 2q^N$ and $N(x, y) = |x|$.

PROOF OF LEMMA 4.4. Pick vertices y^i, $i = 1, \ldots, q^{q^N}$, with $N(y^i, x) = |x|$, $\left|y^i\right| = |x| + 2q^N$, and $N(y^i, y^j) < |x| + q^N$, for all $i \neq j$. Let $M = \bigcup_{i=1}^{q^{q^N}} E(y^i)$, which is a subset of $E(x)$ with $\omega(M) = \frac{q}{q+1} \cdot q^{-|x|-q^N}$. For notational convenience, let $f = 1_M$.

Lemma 4.1 implies that

$$c(\mathscr{P}^*_{\frac{1}{2}} f)(\eta) \leq \sup_{0\leq n<|x|+2q^N} (\mathscr{P}_{\frac{1}{2}} f)(\eta_n) + \sup_{n\geq|x|+2q^N} f_{E(\eta_n)}, \tag{7}$$

where $c > 0$ is a universal constant. To bound the last term, note that if $n \geq |x| + 2q^N$, then $f_{E(\eta_n)} = 1$ if $\eta \in M$ and zero otherwise, so that $\sup_{n\geq|x|+2q^N} f_{E(\eta_n)} = f(\eta)$, since $f = 1_M$.

Now consider $(\mathscr{P}_{\frac{1}{2}} f)(\eta_n)$ for $0 \leq n < |x| + 2q^N$. By definition of $\mathscr{P}_{\frac{1}{2}} f$, using Lemma 3.2, we see that

$$\begin{aligned}
(\mathscr{P}_{\frac{1}{2}} f)(\eta_n) &\leq \frac{C}{n+1}\int_{\partial T} q^{N(\eta_n,\zeta)} f(\zeta)\,\mathrm{d}\omega(\zeta) \\
&= \frac{C}{n+1}\sum_{i=1}^{q^{q^N}}\int_{E(y^i)} q^{N(\eta_n,\zeta)}\mathrm{d}\omega(\zeta) \\
&\leq \frac{Cq^{-|x|-2q^N}}{n+1}\sum_{i=1}^{q^{q^N}} q^{N(\eta_n,y^i)},
\end{aligned}$$

since $N(\eta_n, \zeta) = N(\eta_n, y^i)$ for all $\zeta \in E(y^i)$. If $n \leq |x|$, or if $\eta \notin E(x)$, then $N(\eta_n, y^i) \leq |x|$, for all i, and hence

$$(\mathscr{P}_{\frac{1}{2}} f)(\eta_n) \leq \frac{Cq^{-|x|-2q^N}}{n+1}\cdot q^{q^N}\cdot q^{|x|} \leq Cq^{-q^N}$$

in this case.

It remains to consider $\eta \in E(x)$ and n with $|x| < n < |x|+2q^N$. Note that $N(\eta, y^i) = |x|+m$ for $q^{q^N}\cdot q^{-m}\cdot(1-q^{-1})$ values of i, if $m = 0, \ldots, q^N-1$, and that $N(\eta, y^i) \geq |x| + q^N$ for exactly one value of i. Since

$$\begin{aligned}
(\mathscr{P}_{\frac{1}{2}} f)(\eta_n) \leq{}& \frac{Cq^{-|x|-2q^N}}{n+1}\sum_{m=0}^{q^N-1} q^{|x|+m}\cdot q^{q^N}\cdot q^{-m}\cdot(1-q^{-1}) \\
&+ \frac{Cq^{-|x|-2q^N}}{n+1}\sum_{i=1}^{q^{q^N}}\sum_{k=|x|+q^N}^{|x|+2q^N} q^k\cdot 1_{E((y^i)_k)}(\eta),
\end{aligned}$$

where the last sum accounts for the case when $N(\eta, y^i) \geq |x| + q^N$, we get that when $n \geq |x| + q^N$,

$$(\mathscr{P}_{\frac{1}{2}} f)(\eta_n) \leq Cq^{-q^N} + \frac{Cq^{-|x|-2q^N}}{|x|+q^N}\sum_{i=1}^{q^{q^N}}\sum_{k=|x|+q^N}^{|x|+2q^N} q^k\cdot 1_{E((y^i)_k)}(\eta).$$

If instead $n < |x| + q^N$, then a similar calculation shows that

$$
\begin{aligned}
(\mathscr{P}_{\frac{1}{2}} f)(\eta_n) &\leq \frac{Cq^{-|x|-2q^N}}{n+1} \cdot q^n \cdot q^{q^N} \cdot q^{-(n-|x|)} \\
&\quad + \frac{Cq^{-|x|-2q^N}}{n+1} \sum_{k=|x|}^{n-1} q^k \cdot q^{q^N} \cdot q^{-(k-|x|)} \cdot (1-q^{-1}) \\
&\leq Cq^{-q^N}.
\end{aligned}
$$

By (7), all the estimates together yield

$$
(\mathscr{P}^*_{\frac{1}{2}} f)(\eta) \leq Cq^{-q^N} + \frac{Cq^{-|x|-2q^N}}{|x|+q^N} \sum_{i=1}^{q^{q^N}} \sum_{k=|x|+q^N}^{|x|+2q^N} q^k \cdot 1_{E((y^i)_k)}(\eta),
$$

and hence $\|\mathscr{P}^*_{\frac{1}{2}} f\|_{L^1(\partial T)} \leq Cq^{-q^N}$, which finishes the proof of Lemma 4.4.

PROOF OF THEOREM 2.5. *Part* (i). Let ψ be a continuous function such that $\psi(t)q^{-2\psi(t)} = t$, for all sufficiently small $t > 0$, and $\psi(t) \to +\infty$ as $t \to 0^+$. Since $\lim_{t\to+\infty} \frac{\Phi(t)}{t \log t} = 0$, we may to each $k \in \mathsf{N}^+$ pick $\alpha_k > 1$ such that $\frac{\Phi(\alpha_k)}{\alpha_k \log \alpha_k} \leq \frac{1}{k}$. Let $n_k = \psi\big(\frac{1}{\alpha_k k}\big)$, which we may assume – by choosing α_k sufficiently large – is an integer such that $n_k \geq k$. Since $\sum_{k\geq 1} \frac{q}{q+1} q^{-n_k} < 1$, we may pick pairwise disjoint intervals $E(x_k) \subseteq \partial T$ with $|x_k| = n_k$.

Noting that $q^{n_k} \geq n_k^2$ for $k \geq 4$, we may apply Lemma 4.3 with $n = N = n_k$ and $x = x_k$ to get sets $M_k \subseteq E(x_k)$ with $\omega(M_k) = \frac{q}{q+1} \cdot q^{-2n_k}$ and $\int_{E(x_k)} (\mathscr{P}^*_{\frac{1}{2}} 1_{M_k})(\eta)\, \mathrm{d}\omega(\eta) \geq c n_k q^{-2n_k}$. Then $f = \sum_{k\geq 4} \alpha_k 1_{M_k}$ is a measurable function satisfying

$$
\int_{\partial T} \Phi(f(\eta))\, \mathrm{d}\omega(\eta) = \sum_{k\geq 4} \Phi(\alpha_k)\omega(M_k) \leq \sum_{k\geq 4} \frac{\alpha_k \log \alpha_k}{k} \cdot q^{-2n_k},
$$

and since $\alpha_k = \frac{q^{2n_k}}{n_k k}$ – using the defining property of ψ – we get that

$$
\int_{\partial T} \Phi(f(\eta))\, \mathrm{d}\omega(\eta) \leq \sum_{k\geq 4} \frac{q^{2n_k}}{n_k k} \cdot \frac{2n_k \log q}{k} \cdot q^{-2n_k} < +\infty,
$$

so that $f \in L^\Phi(\partial T)$.

On the other hand,

$$\int_{\partial T} (\mathscr{P}^*_{\frac{1}{2}} f)(\eta)\, \mathrm{d}\omega(\eta) \geq \sum_{k\geq 4} \int_{E(x_k)} (\mathscr{P}^*_{\frac{1}{2}} (\alpha_k 1_{M_k}))(\eta)\, \mathrm{d}\omega(\eta)$$
$$\geq c \sum_{k\geq 4} \alpha_k n_k q^{-2n_k} = c \sum_{k\geq 4} \frac{1}{k} = +\infty,$$

and hence $f \notin H^1_{\frac{1}{2}}(\partial T)$, completing the proof of part (i) of Theorem 2.5.

Part (ii). Let $k_0 = q^{q/2}$. Since $\lim_{t\to+\infty} \Phi(t)/t = +\infty$ by assumption, we may to each integer $k \geq k_0$ pick $\alpha_k \geq 1$ such that $\Phi(\alpha_k)/\alpha_k \geq k^3$. Also assume, by choosing α_k sufficiently large, that the quantity $N_k = \log_q \log_q (k^2 \alpha_k)$ is an integer.

Let $l_k = \lceil 2 \log_q k \rceil$, where $\lceil s \rceil$ denotes the smallest integer such that $\lceil s \rceil \geq s$. Since $\sum_{k\geq k_0} \frac{q}{q+1} \cdot q^{-l_k} < 1$, we may pick pairwise disjoint intervals $(E(x_k))_{k\geq k_0}$, where $x_k \in T$ has $|x_k| = l_k$.

By Lemma 4.4, with $x = x_k$ and $N = N_k$, we get sets $(M_k)_{k\geq k_0}$ satisfying $M_k \subseteq E(x_k)$, $\omega(M_k) = \frac{q}{q+1} \cdot q^{-l_k - q^{N_k}}$, and $\|\mathscr{P}^*_{\frac{1}{2}} 1_{M_k}\|_{L^1(\partial T)} \leq C q^{-q^{N_k}}$, for a universal constant $C < +\infty$.

Let $g = \sum_{k\geq k_0} \alpha_k 1_{M_k}$, which is a function in $L^1(\partial T)$, owing to the fact that $\alpha_k \omega(M_k) \leq \frac{1}{k^4}$. Noting that

$$\|\mathscr{P}^*_{\frac{1}{2}} g\|_{L^1(\partial T)} \leq \sum_{k\geq k_0} \alpha_k \|\mathscr{P}^*_{\frac{1}{2}} 1_{M_k}\|_{L^1(\partial T)} \leq C \sum_{k\geq k_0} \alpha_k \cdot q^{-q^{N_k}}$$
$$= C \sum_{k\geq k_0} \frac{1}{k^2} < +\infty$$

we get that $g \in H^1_{1/2}(\partial T)$. However, $g \notin L^\Phi(\partial T)$, since

$$\int_{\partial T} \Phi(g(\eta))\, \mathrm{d}\omega(\eta) = \sum_{k\geq k_0} \Phi(\alpha_k)\omega(M_k) \geq \sum_{k\geq k_0} \alpha_k k^3 \cdot \frac{q}{q+1} \cdot q^{-l_k - q^{N_k}}$$
$$\geq \sum_{k\geq k_0} \frac{1}{q+1} \cdot \frac{1}{k} = +\infty,$$

implying that $g \notin L^\Phi(\partial T)$, as the space is of type Δ_2.

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