ON BOUNDED WEAK AND STRONG SOLUTIONS OF NON LINEAR DIFFERENTIAL EQUATIONS WITH AND WITHOUT DELAY IN BANACH SPACES

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Abstract
Assume that $E$ is a Banach space, $B_r = \{x \in E : \|x\| \leq r\}$ and $C([-d, 0], B_r)$ is the Banach space of continuous functions from $[-d, 0]$ into $B_r$. Consider $f : \mathbb{R}^+ \times E \to E; f^d : [0, T] \times C([-d, 0], B_r) \to E$; for each $t \in [0, T]$ the mapping $\theta_t \in C([-d, 0], B_r)$ is defined by $\theta_t x(s) = x(t + s), s \in [-d, 0]$ and let $A(t)$ be a linear operator from $E$ into itself. In this paper we give existence theorems for bounded weak and strong solutions of the nonlinear differential equation

\[(P) \quad \dot{x}(t) = A(t)x + f(t, x), \quad t \in \mathbb{R}^+,\]

and we prove that, with certain conditions, the differential equation with delay

\[(Q) \quad \dot{x}(t) = L(t)x(t) + f^d(t, \theta_t x), \quad \text{if} \quad t \in [0, T]\]

has at least one weak solution where $L(t)$ is a linear operator from $E$ into $E$. Moreover, under suitable assumptions, the problem $(Q)$ has a solution. Furthermore under a generalization of the compactness assumptions, we show that $(Q)$ has a solution too.

1. Introduction and preliminaries
In this paper the dual space of an infinite dimensional Banach space $E$ will be denoted by $E^*$ and the pairing between $E$ and $E^*$ is denoted by $\langle \rangle$. Denote by $E_w$ the Banach space $E$ endowed with the weak topology. We denote the closed unit sphere in $E$ by $B_1 = \{x \in E : \|x\| \leq 1\}$. Further, let $\mathcal{L}(\mathbb{R}^+, E)$ be the space of measurable functions $u : \mathbb{R}^+ \to E$, $\mathcal{L}(E)$ be the space of linear operators from $E$ into itself and $\lambda$ be the Lebesgue measure on $I = [0, T]$. Furthermore, let $C(I, E)$ be the space of all continuous functions from $I$ to $E$ with the usual supremum norm and $C_w(I, E)$ be the space of all weakly continuous functions from $I$ to $E$ endowed with the topology of weak uniform convergence. Let $C([-d, 0], E)$ be the Banach space of continuous functions from the closed interval $[-d, 0]$ ($d \geq 0$) into $E$ and $\mathcal{B}$ be the family of all bounded subsets of $E$.

Let $\mathcal{M} = \mathcal{M}(\mathbb{R}^+, E)$ be a Banach space of measurable functions $x : \mathbb{R}^+ \to E$ with $\|x\| \in \mathcal{M}(\mathbb{R}^+, \mathbb{R}), \|x\|_{\mathcal{M}} = \|\|x\|\|_{\mathcal{M}(\mathbb{R}^+, \mathbb{R})}$, where

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(1) \( \mathcal{M}(\mathbb{R}^+, \mathbb{R}) \subset \mathcal{L}(\mathbb{R}^+, \mathbb{R}) \),
(2) \( \mathcal{M}(\mathbb{R}^+, \mathbb{R}) \) contains all essentially bounded functions with compact support,
(3) if \( x \in \mathcal{M}(\mathbb{R}^+, \mathbb{R}) \), \( y : \mathbb{R}^+ \to \mathbb{R} \) is measurable with \( |y| \leq |x| \), then \( y \in \mathcal{M}(\mathbb{R}^+, \mathbb{R}) \) and \( \|y\|_{\mathcal{M}(\mathbb{R}^+, \mathbb{R})} \leq \|x\|_{\mathcal{M}(\mathbb{R}^+, \mathbb{R})} \),
(4) if \( x \in \mathcal{M}(\mathbb{R}^+, \mathbb{R}) \), \( x_n \in \mathcal{M}(\mathbb{R}^+, \mathbb{R}) \), \( |x_n| \leq |x| \) and \( \lim_{n \to \infty} x_n(t) = 0 \) a.e. on \( \mathbb{R}^+ \), then \( \lim_{n \to \infty} \|x_n\|_{\mathcal{M}(\mathbb{R}^+, \mathbb{R})} = 0 \).

Let \( \mathcal{M}' \) denote the associate space to \( \mathcal{M} \).[20]

**Definition 1.1.** The map \( \gamma : \mathcal{B} \to \mathbb{R}^+ \) is called a measure of strong (weak) noncompactness on \( \mathcal{B} \) if, for \( U, V \in \mathcal{B} \),

\( (M_1) \ U \subset V \rightarrow \gamma(U) \leq \gamma(V), \)
\( (M_2) \ \gamma(U \cup V) \leq \max(\gamma(U), \gamma(V)), \)
\( (M_3) \ \gamma(\text{conv} U) = \gamma(U), \)
\( (M_4) \ \gamma(U + V) \leq \gamma(U) + \gamma(V), \)
\( (M_5) \ \gamma(cU) = |c|\gamma(U), \ c \in \mathbb{R}, \)
\( (M_6) \ \gamma(U) = 0 \iff U \) is relatively strongly (weakly) compact in \( E \),
\( (M_7) \ \gamma(U \cup \{x\}) = \gamma(U), \ x \in E. \)

**Definition 1.2.** A function \( u : [a, b] \to E, (a, b) \in \mathbb{R}^2 \), is called:

(a) Pettis integrable if for any measurable subset \( D \) of \( [a, b] \) there is an element \( v_D \) in \( E \) such that \( \langle v_D, f \rangle = \int_D \langle u(s), f \rangle \ ds \), for all \( f \in E^* \), we write \( v_D = \int_D u(s) \ ds \),
(b) Bochner integrable if there exists a sequence of countable-valued functions \( \{u_n\} \) converging almost everywhere on \( [a, b] \) such that \( \lim_{n \to \infty} \int_a^b \|u_n(s) - u(s)\| \ ds = 0 \).

We note that every Bochner integrable function is Pettis integrable (see [14]).

**Definition 1.3.** The Hausdorff measure of weak noncompactness \( \beta : \mathcal{B} \to \mathbb{R}^+ \) and the Kuratowski measure of noncompactness \( \alpha : \mathcal{B} \to \mathbb{R}^+ \) are defined as follows: for each \( U \in \mathcal{B} \),

(i) \( \beta(U) = \inf \{ \varepsilon > 0 : \exists K = \text{weakly compact subset of } E, U \subseteq K + \varepsilon B_1 \}, \)
(ii) \( \alpha(U) = \inf \{ \varepsilon > 0 : U \) admits a finite cover of sets with diameter \( < \varepsilon \} \).

For more details of \( \beta \) and \( \alpha \) we refer the reader to [1], [8].
Definition 1.4. By a Kamke function we mean a function \( w : I \times \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) such that:

(i) \( w \) is a Carathéodory function,

(ii) for all \( t \in I \); \( w(t, 0) = 0 \),

(iii) for any \( c \in (0, b] \), \( u \equiv 0 \) is the only absolutely continuous function on \([0, c]\) which satisfies \( \dot{u}(t) \leq w(t, u(t)) \) a.e. on \([0, c]\) and such that \( u(0) = 0 \).

Definition 1.5. A continuous function \( x : [-d, T] \rightarrow E_w \) is called a weak solution of problem (P) if, for some \( \xi \in C([-d, 0], E) \),

\[
x = \xi \quad \text{on} \quad [-d, 0]
\]

and

\[
x(t) = G(t, 0)\xi(0) + \int_0^t G(t, s)f(s, x(s))\, ds \quad \text{for all} \quad t \in I.
\]

Lemma 1.6. Let \( F \) be a continuous mapping from a compact interval \( I \) to \( \mathcal{L}(E) \) and \( \mathcal{U} \) be a bounded subset of \( E \), then

\[
\gamma\left( \bigcup_{t \in I} F(t)\mathcal{U} \right) \leq \sup_{t \in I} \| F(t) \| \gamma(\mathcal{U}).
\]

Proof. \( \mathcal{U} \) is bounded, so \( \exists \ c > 0; \| \mathcal{U} \| = \sup\{\| u \| : u \in \mathcal{U} \} \leq c \). From the continuity of \( F \), for \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that if \( P = \{x_0, x_1, x_2, \ldots, x_n\} \) is a partition of \( I \), that is, \( a = x_0 < x_1 < x_2 < \cdots < x_n = b \) with \( \| P \| = \sup\{|x_{i+1} - x_i| : i = 0, 1, 2, \ldots, n - 1\} < \delta \), then \( \| F(x_{i+1}) - F(x_i) \| < \frac{\varepsilon}{c} \). Since \( B_1 \) is the closed unit ball in \( E \), there exists a weakly compact subset \( \mathcal{H} \) of \( E \) such that \( \mathcal{U} \subset \mathcal{H} + \frac{(\gamma(\mathcal{H}) + \varepsilon)}{\gamma(B_1)} B_1 \). But for each \( t \in I, t = [x_i, x_{i+1}], F(t)\mathcal{U} \subset \{F(t)u - F(t_{i+1})u : u \in \mathcal{U}\} + F(t_{i+1})\mathcal{U} \) and \( \| F(t) - F(t_{i+1}) \| \|\mathcal{U}\| < \frac{\varepsilon}{c} \cdot c = \varepsilon \). Hence \( \{F(t)u - F(t_{i+1})u : u \in \mathcal{U}\} \subset \varepsilon B_1 \) and \( F(t)\mathcal{U} \subset \varepsilon B_1 + F(t_{i+1})\mathcal{U} \). Therefore

\[
\bigcup_{t \in I} F(t)\mathcal{U} = \bigcup_{t=0}^{n-1} \bigcup_{t \in I} F(t)\mathcal{U} \subset \varepsilon B_1 + \bigcup_{t=0}^{n-1} F(t_{i+1})\mathcal{U} \]

\[
\subset \varepsilon B_1 + \bigcup_{t=0}^{n-1} \bigcup_{t \in I} F(t)\mathcal{U} \subset \varepsilon B_1 + \bigcup_{t=0}^{n-1} (\varepsilon B_1 + F(t_{i+1})\mathcal{U})
\]
\[ \subseteq 2\varepsilon B_1 + \bigcup_{t=0}^{n-1} \mathcal{F}(t_{i+1})(\mathcal{H} + \frac{(\gamma(U) + \varepsilon)}{\gamma(B_1)} B_1) \]

\[ \subseteq 2\varepsilon B_1 + \bigcup_{t=0}^{n-1} \mathcal{F}(t_{i+1})\mathcal{H} + \bigcup_{t=0}^{n-1} \mathcal{F}(t_{i+1})\frac{(\gamma(U) + \varepsilon)}{\gamma(B_1)} B_1 \]

\[ \subseteq 2\varepsilon B_1 + \bigcup_{t=0}^{n-1} \mathcal{F}(t_{i+1})\mathcal{H} + \sup_{t \in I} \| \mathcal{F}(t) \| \frac{(\gamma(U) + \varepsilon)}{\gamma(B_1)} B_1. \]

Moreover

\[ \gamma \left( \bigcup_{t \in I} \mathcal{F}(t)U \right) \leq \gamma (2\varepsilon B_1 + \bigcup_{t=0}^{n-1} \mathcal{F}(t_{i+1})\mathcal{H} + \sup_{t \in I} \| \mathcal{F}(t) \| \frac{(\gamma(U) + \varepsilon)}{\gamma(B_1)} B_1) \]

\[ \leq 2\varepsilon \gamma(B_1) + \gamma \left( \sup_{t \in I} \| \mathcal{F}(t) \| \frac{(\gamma(U) + \varepsilon)}{\gamma(B_1)} B_1 \right) \]

\[ \leq 2\varepsilon \gamma(B_1) + \sup_{t \in I} \| \mathcal{F}(t) \| \frac{(\gamma(U) + \varepsilon)}{\gamma(B_1)} \gamma(B_1) \]

\[ \leq 2\varepsilon \gamma(B_1) + \sup_{t \in I} \| \mathcal{F}(t) \| (\gamma(U) + \varepsilon) \]

where \( \bigcup_{t=0}^{n-1} \mathcal{F}(t_{i+1})\mathcal{H} \) is weakly compact. Since \( \varepsilon \) is arbitrary the result follows.

**Lemma 1.7 ([3]).** Let \( Y \) and \( E \) be two Banach spaces, \( P_{fc}(Y) \) be the set of all closed and convex subsets of \( Y \) and \( F : E \to P_{fc}(Y) \) be weakly sequentially upper hemicontinuous. Further let \( (x_n)_{n \in \mathbb{N}} \subset C(I, E), x_n(t) \to x_0(t) \) weakly a.e. on \( I \) and \( (y_n)_{n \in \mathbb{N} \cup \{0\}} \subset L^1(I, E), y_n \to y_0 \) weakly. Suppose that there exists \( a \in L^1(I, \mathbb{R}) \) such that \( \|F(x)\| \leq a(t) \) for all \( x \in C(I, E) \) and \( y_n(t) \in F(x_n(t)) \) a.e. on \( I \). Then \( y_0(t) \in F(x_0(t)) \) a.e. on \( I \).

**Lemma 1.8 ([18], [1]).** If \( \gamma : \mathbb{R} \to \mathbb{R}^+ \) satisfies conditions (M2), (M4) and (M6) then, for any nonempty \( U \in \mathbb{B} \),

\[ \gamma(U) \leq \gamma(B_1)\alpha(U) \leq 2\gamma(B_1)\beta(U). \]

**Lemma 1.9 ([21], [17]).** If \( \gamma \) is a measure of weak (strong) noncompactness and \( A \subset C_w(I, E) \) is a family of strongly equicontinuous functions, then

\[ \gamma(A(I)) = \sup_{t \in I} \| \gamma(A) \| : t \in I \].
If for each \( t \in \mathbb{R}^+ \), \( A(t) \in \mathcal{L}(E) \) and \( \dot{x}(t) \) denotes the weak derivative of \( x \) at \( t \), then we consider the differential equation

\[
\dot{x}(t) = A(t)x(t).
\]

Let \( E \) be the direct sum of \( E_0 \) and \( E_1 \), where

\[
E_0 = \{ x_0 \in E : \exists \text{ a bounded weak solution } x \text{ of (1) and } x(0) = x_0 \}
\]
is closed and has a closed complement \( E_1 \).

Let \( G \in C(\mathbb{R}^+ \times \mathbb{R}^+, E) \) be the Green function corresponding to (1):

\[
G(t, s) = \begin{cases} 
S(t)P S^{-1}(s) & \text{if } 0 \leq s \leq t \\
-S(t)(\text{id} - P)S^{-1}(s) & \text{if } 0 \leq t \leq s,
\end{cases}
\]

where \( S : \mathbb{R}^+ \to \mathcal{L}(E) \) is a solution of the differential equation

\[
\dot{S}(t) = A(t)S(t), \quad S(0) = \text{id},
\]

and \( P \) is the projection of \( E \) onto \( E_0 \); hence \( P(E_1) = \{0\} \).

2. Existence results for problem (P)

In this section we shall consider the nonlinear differential equation

\[
(P) \quad \dot{x}(t) = A(t)x(t) + f(t, x(t)), \quad t \in \mathbb{R}^+.
\]

This problem was studied by many authors (see, for instance, [5], [19], [6], [16], [10]). The next theorem is a generalization of Theorem 8 in [13]. Moreover we use here a general weak noncompactness measure, in contrast with the Hausdorff noncompactness measure used in [10]; hence, the result below is at the same time a generalization of Theorem 5 in [10].

**Theorem 2.1.** Let \( A : \mathbb{R}^+ \to \mathcal{L}(E) \) be strongly measurable and Bochner integrable on every subinterval \( I \) of \( \mathbb{R}^+ \). Let \( \gamma \) be a weak measure of noncompactness, for each \( t \in \mathbb{R}^+ \), let \( G(t, .) \in \mathcal{M} \) with \( \|G(t, .)\|_{\mathcal{M}} \leq c \) where \( c > 0 \). Let \( f \) be a continuous function from \( \mathbb{R}^+ \times E \) to \( E \) and \( m : \mathbb{R}^+ \to \mathbb{R}^+ \) belongs to \( \mathcal{M} \) such that \( \|f(t, x)\| \leq m(t) \) for every \( (t, x) \in \mathbb{R}^+ \times B_r \). Assume that \( c\|m\|_{\mathcal{M}} < r \) and for each \( T, \varepsilon > 0 \) there exists a closed subset \( I_\varepsilon \) of \( I \) with \( \lambda(I - I_\varepsilon) < \varepsilon \) such that for any nonempty bounded subset \( U \) of \( E \) one has \( \beta(f(J \times U)) \leq \sup_{t \in J} w(t, \beta(U)) \), for any compact subset \( J \) of \( I_\varepsilon \). Then, for each \( x_0 \in E_0 \) such that \( \|x_0\| \leq \frac{r - c\|m\|_{\mathcal{M}}}{\|G(t, 0)\|} \), there exists a bounded weak solution of (P).
Proof. Let \( S = \left\{ x \in C_w(R^+ , E) : \| x(t) - x(\tau) \| \leq r \int_{\tau}^t |A(s)| \, ds + \int_{\tau}^t m(s) \, ds, 0 \leq t \leq \tau \right\} \).

From (2) and by results from [20] there exists a positive number \( d \) such that \( \| G(t, 0) \| \leq d \). Let \( x_0 \in E \) with \( \| x_0 \| \leq \frac{r - c \| m \|}{d} \). Then \( G(t, 0)x_0 \) is a solution of (1) and \( \| G(t, 0)x_0 \| \leq d \| x_0 \| \leq r - c \| m \| \).

If \( \phi \) is defined by
\[
\phi(x)(t) = G(t, 0)x_0 + \int_0^\infty G(t, s)f(s, x(s)) \, ds \quad \text{for} \quad t \in R^+ \text{ and } x \in S,
\]
then
\[
\| \phi(x)(t) \| \leq d \| x_0 \| + c \| m \| \leq r.
\]

Since \( y = \phi(x) \) is a weak solution of the equation \( \dot{y}(t) = A(t)y(t) + f(t, x(t)) \), we have
\[
\| \phi(x)(t) - \phi(x)(\tau) \| \leq \int_{\tau}^t \| A(s)\phi(x)(s) + f(s, x(s)) \| \, ds
\]
\[
\leq r \int_{\tau}^t |A(s)| \, ds + \int_{\tau}^t m(s) \, ds.
\]

Therefore \( \phi \) is a continuous mapping from \( S \) into \( S \) [4]. Let \( (x_n)_{n \in N \cup \{0\}} \) be a sequence such that \( \phi(x_n) = x_{n+1} \) with \( x_0 \) is an arbitrary element in \( S \). Thus \( D \subset S \) and, from (M4), \( \gamma(D) = \gamma(\phi(D)) \). If \( G \) is the set of all limit points of the sequence \( (x_n) \), then \( \phi(G) = G \). Put \( R(X) = \text{conv} \phi(X) \) for \( X \subset S \) and consider the family \( \Omega \) of all subsets \( X \) of \( S \) such that \( G \subset X \) and \( R(X) \subset X \).

Now \( S \in \Omega \) and so \( \Omega \neq \emptyset \). Let \( V \) be the intersection of all sets of the family \( \Omega \). Then \( V \in \Omega \). Moreover the mapping \( t \mapsto \gamma(\phi(V)(t)) \) is absolutely continuous. Assume that \( t \geq 0 \) and \( \varepsilon > 0 \) thus from the assumptions on the function \( m \) we can find \( T_0 \geq t \) such that \( \| m \|_{[T_0, \infty]} < \frac{\varepsilon}{2r} \). If we put \( I_0 := [0, T_0] \), then by the Scorza-Dragoni theorem there exists a closed subset \( I_\varepsilon \) of \( I_0 \) such that \( \lambda(I_0 - I_\varepsilon) < \delta \) and the function \( w \) is uniformly continuous on \( I_\varepsilon \times [0, 2T_0] \). From our last assumption, we can find a closed subset \( J_\varepsilon \) of \( I_0 \) such that \( \lambda(I_0 - J_\varepsilon) < \delta \) and such that for any compact subset \( C \) of \( J_\varepsilon \) and any bounded subset \( Z \) of \( E \),
\[
\gamma(f(C \times Z)) \leq \sup_{s \in C} w(s, \gamma(Z)).
\]

Since \( \phi \) is continuous and \( w \) is Carathéodory we can find a closed subset \( I_\varepsilon \) of \( I, \delta > 0, \eta > 0 \) \( (\eta < \delta) \) such that if \( s_1, s_2 \in I_\varepsilon \) and \( r_1, r_2 \in [0, 2T_0] \)
satisfy $|s_1 - s_2| < \delta$, $|r_1 - r_2| < \delta$, then $|w(s_1, r_1) - w(s_2, r_2)| < \varepsilon$ and if $|s_1 - s_2| < \eta$, then $|\gamma(V(s_1)) - \gamma(V(s_2))| < \delta$. Let us fix $\tau$ such that $0 \leq t \leq \tau \leq T$ and consider the partition, to $[t, \tau]$, $t = t_0 < t_1 < \cdots < t_m = \tau$ such that $t_i - t_{i-1} < \eta$ for $i = 1, \ldots, n$. Let $T_i = J_i \cap [t_{i-1}, t_i] \cap I_i$, $P = \sum_{i=1}^m T_i = [t, \tau] \cap J_i \cap I_i$ and $Q = [t, \tau] - P$. Since $G(t, .)$ is uniformly continuous on $P$, we can find $\eta' > 0$ (\eta' < \delta) such that if $r_1, r_2 \in P$ and $|r_1 - r_2| < \eta'$, then

$$\|G(t, r_1) - G(t, r_2)\| < \varepsilon$$

and we can find $s_i$ in $T_i$ with

$$\sup_{s \in T_i} \|G(t, s)\| = \|G(t, s_i)\|.$$

Let $S_i = \{x(t) : x \in S, t \in T_i\}$. In virtue of Lemma 1.6, Lemma 1.9, the mean value theorem and Lemma 1.8 if $\rho(t) := \gamma(V(t))$ we get

$$\rho(\tau) - \rho(t) \leq \gamma \int_t^\tau G(t, s)f(s, V(s))\, ds \leq 2\gamma(B_1) \int_t^\tau \|G(t, s)\|w(s, \rho(s))\, ds.$$

Therefore $\dot{\rho}(t) \leq c\dot{w}(t, \rho(t))$ a.e. [12] and since $\rho(0) = 0$, then $\rho \equiv 0$ and so $V^w$ is weakly compact in $C_w(\mathbb{R}^+, E)$. But $V$ is closed, hence it is a convex and compact subset in $C_w(\mathbb{R}^+, E)$. From the Schauder-Tichonov theorem, since $\phi$ is a continuous mapping from $V$ to $V$, there is a fixed point $y$ of $\phi$ such that $y$ is the desired weak solution of (P) and satisfies $\sup_{t \in \mathbb{R}^+} \|y(t)\| \leq r$.

In the following theorem we will deal with the differential equation

$$(P') \quad \dot{x}(t) = L(t)x(t) + f'(t, x(t)), \quad t \in I$$

where $f' : I \times B_r \rightarrow E$ is a Carathéodory function, $L : I \rightarrow \mathcal{L}(E)$ is strongly measurable and Bochner integrable operator on $I$ and $\gamma$ is a measure of strong noncompactness. We get a generalization of Theorem 2 in [26] and Theorem 9 in [13].

**Theorem 2.2.** In the setting of Theorem 2.1 we replace the function $f$ by $f'$ such that for each $x \in B_r$, $f'(I \times \{x\})$ is separable; the function $m$ by $m' \in L^1(I, \mathbb{R}^+)$ and the operator $A$ by $L$. Then problem $(P')$ has a solution.

**Proof.** Let

$$S = \left\{ x \in C(I, E) : \|x(t) - x(\tau)\| \leq r \int_t^\tau |A(s)|\, ds + \int_t^\tau m'(s)\, ds, \ 0 \leq t \leq \tau \right\}.$$
Suppose that the mapping \( \phi : S \to S \) is defined by
\[
\phi(x)(t) = G(t, 0)x_0 + \int_0^t G(t, s)f(s, x(s)) \, ds \quad \text{for } t \in I \text{ and } x \in S.
\]
As in Theorem 2.1 we let \((x_n)_{n \in \mathbb{N} \cup \{0\}}\) be a sequence such that \(\phi(x_n) = x_{n+1}\) where \(x_0\) is an arbitrary element in \(S\), \(V = \{x_n : n = 0, 1, 2, \ldots\}\), \(V \subset S\), \(\gamma(V) = \gamma(\phi(V))\) and \(\rho(t) = \gamma(V(t))\). Then by the same argument we get
\[
\rho(\tau) - \rho(t) \leq \gamma \left( \int_t^\tau G(t, s)f(s, V(s)) \, ds \right)
\leq \gamma(B_1) \int_t^\tau \|G(t, s)\| h(s, \rho(s)) \, ds,
\]
\(\rho\) is differentiable a.e. on \(I\) and \(\rho \equiv 0\). Thus the closure of \(V\) is compact in \(C(I, E)\) and so we can find a subsequence \((x_{n_k})\) of \((x_n)\) which converges to a limit \(x\) in \(C(I, E)\). Since \(\|x_n - \phi(x_n)\| \to 0\) as \(n \to \infty\) and \(\phi\) is continuous, then \(x = \phi(x)\) so as \(x\) is the desired solution of \((P')\) and \(\|x\| \leq r\).

In the following theorem we let \(h : I \times \mathbb{R}^+ \to \mathbb{R}^+\) be a Carathéodory function, such that for each bounded subset \(Z\) of \(I \times \mathbb{R}^+\) there exists a function \(\varphi : I \to \mathbb{R}^+\) such that \(h(t, s) \leq \varphi(t), (t, s) \in Z\) and \(\varphi\) is integrable function on \([c, T]\) for each \(c, 0 < c \leq T\). Moreover we assume that the identically zero function is the only absolutely continuous function on \([0, c]\) which satisfies \(\dot{u}(t) = h(t, u(t))\) a.e. on \([0, c]\) such that the right derivative \(D_+u(0)\) of \(u(t)\) at \(t = 0\) exists and \(D_+u(0) = u(0) = 0\).

We note that the assumptions on \(h\) are weaker than that on a Kamke function \(w\).

**Theorem 2.3.** If we replace in the setting of Theorem 2.2 a Kamke function \(w\) by a function \(h\) and we suppose that \(f'\) is bounded and continuous, then problem \((P')\) has a solution.

**Proof.** By the same argument as in Theorem 2.2 we get
\[
\rho(\tau) - \rho(t) \leq \gamma \left( \int_t^\tau G(t, s)f(s, V(s)) \, ds \right)
\leq \gamma(B_1) \int_t^\tau \|G(t, s)\| h(s, \rho(s)) \, ds,
\]
where \(\rho(t) = \gamma(V(t))\). Since \(f'\) is a bounded function, we can find a constant \(M > 0\) such that \(\|f'(t, x)\| \leq M\) for each \((t, x) \in I \times B_r\). Let \(N' : I \to \mathbb{R}\) be defined by \(N'(t) = \sup_{\|x\|, \|y\| \leq M \|t\|} \|f'(t, x) - f'(t, y)\|\). We see that \(N'\) is
lower semicontinuous on $]0, T]$ and continuous at $0$ [22]. Let $\epsilon > 0$ and $t_0$ be fixed in $I$. Then, there exist $x_1, y_1 \in B_r; \|x_1\|, \|y_1\| \leq Mt$ such that

$$\mathcal{N}(t_0) - \frac{\epsilon}{2} \leq \|f'(t_0, x_1) - f'(t_0, y_1)\|.$$  

Moreover, $f'$ is continuous. Thus $\exists \delta > 0$ such that if $|t - t_0| < \delta$, $\|x_1 - x\| < \delta$, $\|y_1 - y\| < \delta$, we have

$$\|f'(t_0, x_1) - f'(t, x)\| < \frac{\epsilon}{4} \quad \text{and} \quad \|f'(t_0, y_1) - f'(t, y)\| < \frac{\epsilon}{4}.$$  

From relations (4) and (5), we get

$$\mathcal{N}(t_0) - \frac{\epsilon}{2} \leq \|f'(t_0, x_1) - f'(t_0, y_1)\|$$

$$\leq \|f'(t_0, x_1) - f'(t, x)\|$$

$$+ \|f'(t, x) - f'(t, y)\| + \|f'(t, y) - f'(t_0, y_1)\|$$

$$\leq \|f'(t, x) - f'(t, y)\| + \frac{\epsilon}{2},$$

and so,

$$\mathcal{N}(t_0) - \epsilon \leq \|f'(t, x) - f'(t, y)\|.$$  

Thus, for each $t$ with $|t - t_0| < \delta$, there exist $x_1, y_1$ with $\|x_1\|, \|y_1\| \leq Mt$ such that $\mathcal{N}(t_0) - \epsilon \leq \|f'(t, x_1) - f'(t, y_1)\| \leq \mathcal{N}(t)$. We conclude that $\mathcal{N}$ is lower semicontinuous. Moreover from the continuity of $f'$, $\mathcal{N}$ is continuous at $0$. Consequently we can say that $\|\int_t^\tau f'(s, x(s)) - \int_t^\tau f'(s, y(s)) ds\| \leq \int_t^\tau \mathcal{N}(s) ds$ for each $x, y \in V$. Then from relation (3) we have

$$\rho(\tau) - \rho(t)$$

$$\leq \min \left( \int_t^\tau \|G(t, s)\| \mathcal{N}(s) ds, \gamma(B_1) \int_t^\tau \|G(t, s)\| h(s, \rho(s)) ds \right),$$

where $0 < t \leq \tau \leq T$. Therefore $\rho$ is an absolutely continuous function on $I$ and so

$$\dot{\rho}(t) \leq \min \left( \|G(t, s)\| \mathcal{N}(t), \|G(t, s)\| h(t, \rho(t)) \right), \quad \text{a.e. on } I.$$  

Thus $\rho \equiv 0$ on $I$, see Lemma 1 in [22]. We can complete the proof as in the proof of Theorem 2.2.
3. Existence results for problem (Q)

We consider the problem

\( Q \quad \dot{x}(t) = L(t)x(t) + f^d(t, \theta_1 x), \quad t \in I. \)

Let \( B_r = \{ x \in E : \|x\| \leq r \}, \) \( L(t) \in \mathcal{L}(E) \) and for \( t \in I \) we define \( \theta_1 x(s) = x(t + s) \) for all \( s \in [-d, 0] \). We assume that \( C([-d, 0], B_r) \) is the Banach space of continuous functions from \([-d, 0]\) into \( B_r \) and \( f^d : I \times C([-d, 0], B_r) \rightarrow E \).

In the following theorem we deal with problem (Q) and we have a generalization of Theorem 2.1.

**Theorem 3.1.** If we replace in the setting of Theorem 2.1 the function \( f \) by \( f^d \); the function \( m \) by \( m' \in L^1(I, \mathbb{R}^+) \) and the operator \( A \) by \( L \), then problem (Q) has a weak solution.

**Proof.** We apply some methods for functional equations similar to those of [10]. For any arbitrary \( n \in \mathbb{N} \), we define \( \gamma_1 : [-d, \frac{L}{n}] \times E \rightarrow E \) by

\[
\gamma_1(t, x) = \begin{cases} 
\xi(t) & \text{if } t \in [-d, 0] \\
\xi(0) + nt(x - \xi(0)) & \text{if } t \in [0, \frac{L}{n}]
\end{cases}
\]

and also we define \( f_1 : [0, \frac{L}{n}] \times E \rightarrow E \) by \( f_1(t, x) = f^d(t, \theta_1 \gamma_1(., x)) \).

Arguing as in the proof of Theorem 2.1, there is a continuous function \( y_1 = \xi \) on \([-d, 0] \) and for each \( t \in [0, \frac{L}{n}] \)

\[
y_1(t) = G(t, 0)\xi(0) + \int_0^t G(t, s) f_1(s, y_1(s)) \, ds.
\]

Moreover \( \sup_{t \in [0, \frac{L}{n}]} \|y_1(t)\| \leq r \). Set \( k' = k - 1 \). By induction, for each \( k \in \{2, 3, \ldots, n\} \), there exists a bounded function \( y_{k'} \) such that \( y_{k'} = \xi \) on \([-d, 0] \) and for each \( t \in [0, \frac{k'T}{n}] \)

\[
y_{k'}(t) = G(t, 0)\xi(0) + \int_0^t G(t, s) f_{k'}(s, y_{k'}(s)) \, ds,
\]

where \( f_{k'}(t, x) = f^d(t, \theta_1 \gamma_k(., x)) \). Assume that \( \gamma_k : [-d, \frac{k'T}{n}] \times E \rightarrow E \) is such that

\[
\gamma_k(t, x) = \begin{cases} 
\gamma_{k'}(t) & \text{if } t \in [-d, \frac{k'T}{n}] \\
\gamma_{k'}\left(\frac{k'T}{n}\right) + n(t - \frac{k'T}{n})(x - \gamma_{k'}\left(\frac{k'T}{n}\right)) & \text{if } t \in [\frac{k'T}{n}, \frac{k'T}{n}].
\end{cases}
\]
Thus if \( f_k : \left[ \frac{kT}{n}, \frac{kT}{n} \right] \times E \to E \) is defined by \( f_k(t, x) = f^d \left( t, \theta_{\frac{k}{n}} (y_k(., x)) \right) \), then we have a continuous function \( y_k \) defined on \( \left[ \frac{kT}{n}, \frac{kT}{n} \right] \) by

\[
y_k(t) = G(t, \frac{kT}{n}) y_k'(\frac{kT}{n}) + \int_{\frac{kT}{n}}^{t} G(t, s) f_k(s, y_k(s)) \, ds.
\]

Further, for \( 0 \leq s \leq r \leq t \), \( G(t, s)G(s, r) = G(t, r) \) and for each \( t \in \left[ \frac{kT}{n}, \frac{kT}{n} \right] \) we have

\[
y_k'(\frac{kT}{n}) = G\left( \frac{kT}{n}, 0 \right) \xi(0) + \int_{\frac{kT}{n}}^{\frac{kT}{n}} G\left( \frac{kT}{n}, s \right) f_k'(s, y_k(s)) \, ds.
\]

Hence

\[
y_k(t) = G(t, \frac{kT}{n}) G\left( \frac{kT}{n}, 0 \right) \xi(0) + \int_{0}^{\frac{kT}{n}} G(t, s) G\left( \frac{kT}{n}, s \right) f_k'(s, y_k(s)) \, ds
\]

\[
+ \int_{\frac{kT}{n}}^{t} G(t, s) f_k(s, x(s)) \, ds
\]

\[
= G(t, 0) \xi(0) + \int_{0}^{\frac{kT}{n}} G(t, s) f_k'(s, y_k(s)) \, ds
\]

\[
+ \int_{\frac{kT}{n}}^{t} G(t, s) f_k(s, y_k(s)) \, ds
\]

\[
= G(t, 0) \xi(0) + \int_{0}^{t} G(t, s) g_k(s, y_k(s)) \, ds,
\]

where

\[
g_k(t, y_k(t)) = \begin{cases} f_k'(t, y_k(t)) & \text{if } t \in \left[ 0, \frac{kT}{n} \right] \\ f_k(t, y_k(t)) & \text{if } t \in \left[ \frac{kT}{n}, \frac{kT}{n} \right]. \end{cases}
\]

Consequently, for all \( n \in \mathbb{N} \), we have a continuous bounded function \( \nu_n \) such that \( \nu_n = \xi \) on \([d, 0] \) and for each \( t \in I, \frac{kT}{n} \leq t \leq \frac{kT}{n} \) for some \( k \in \{1, 2, 3, \ldots, n\} \), we have

\[
\nu_n(t) = G(t, 0) \xi(0) + \int_{0}^{t} G(t, s) h_n(s) \, ds
\]
where \( h_n(t) = f^d(t, \theta_{\frac{kT}{n}} y_k(., v_n(t))) \). Let \( t_1, t_2 \in I \) and \( t_1 < t_2 \). Then

\[
\|v_n(t_1) - v_n(t_2)\| \\
\leq \|G(t_1, 0) - G(t_2, 0)\| \|\xi(0)\| + \int_{t_1}^{t_2} \|G(t_1, s) - G(t_2, s)\| \|h_n(s, v_n(s))\| ds \\
+ \int_{t_1}^{t_2} \|G(t_2, s)\| \|h_n(s, v_n(s))\| ds \\
\leq \|G(t_1, 0) - G(t_2, 0)\| \|\xi(0)\| + \int_{t_1}^{t_2} \|G(t_1, s) - G(t_2, s)\| \|m'(s)\| ds \\
+ c \int_{t_1}^{t_2} \|m'(s)\| ds,
\]

since \( v_n = \xi \) on \([-d, 0]\) and for all \( s \in I \) \( G(., s) \) is uniformly continuous, then \( A \) is equicontinuous in \( C([-d, T], E) \). \( \gamma(A(t)) = \gamma(\{v_n(t) : n \in \mathbb{N}\}) \) is such that \( \gamma(A(0)) = 0 \) and, as in the proof of Theorem 2.1, \( \gamma(A(t)) = 0 \) for all \( t \in I \). Thus by Ascoli’s theorem, the sequence \( \{v_n : n \in \mathbb{N}\} \) converges uniformly to a function \( v \) which belongs to \( C([-d, T], E) \) such that \( y = \xi \) on \([-d, 0]\). But \( \gamma(\{h_n(t) : n \in \mathbb{N}\}) = 0 \) and so \( \{h_n(t) : n \in \mathbb{N}\} \) is relatively compact. Let \( \mathcal{F}(t) = \text{conv}\{h_n(t) : n \in \mathbb{N}\} \). Thus \( \mathcal{F}(t) \) is nonempty convex and compact. Moreover \( \delta^1_{\mathcal{F}} = \{l \in L^1(I, E) : l(t) \in \mathcal{F}(t)\} \) is nonempty convex and weakly compact. Therefore, there exists a subsequence \( (h_{n_k}) \) of \( (h_n) \) such that \( h_{n_k} \rightarrow l \) weakly, \( l \in \delta^1_{\mathcal{F}} \). Thus \( \{v_n : n \in \mathbb{N}\} \) tends weakly to \( v(t) := G(t, 0)\xi(0) + \int_0^t G(t, s)l(s) ds \). Now \( v \) is uniformly continuous on \([-d, 0]\) and for each \( t \in I \), there exists \( n > \frac{T}{d} \) with \( t \in [\frac{kT}{n}, \frac{(k+1)T}{n}] \) for \( k \in \{1, 2, \ldots, n-1\} \). Hence

\[
\|\theta_{\frac{kT}{n}} y_k(., v_n(t)) - \theta_{\frac{kT}{n}} v\| \\
\leq \sup_{s \in [-d, -\frac{T}{n}]} \left[ \|y_k(\frac{kT}{n} + s, v_n(t)) - v(\frac{kT}{n} + s)\| + \|v(\frac{kT}{n} + s) - v(t + s)\| \right] \\
+ \sup_{s \in [-\frac{T}{n}, 0]} \left[ \|v_n(\frac{kT}{n}) + n(\frac{kT}{n} + s - \frac{kT}{n})v_n(t) - v_n(\frac{kT}{n}) - v(\frac{kT}{n} + s)\| \\
+ \|v(\frac{kT}{n} + s) - v(t + s)\| \right] \\
\leq \sup_{s \in [-d, -\frac{T}{n}]} \left[ \|v_n(\frac{kT}{n} + s) - v(\frac{kT}{n} + s)\| + \|v(\frac{kT}{n} + s) - v(t + s)\| \right] \\
+ \sup_{s \in [-\frac{T}{n}, 0]} \left[ T \|v_n(t) - v_n(\frac{kT}{n})\| + \|v_n(\frac{kT}{n}) - v(\frac{kT}{n} + s)\| \\
+ \|v(\frac{kT}{n} + s) - v(t + s)\| \right]
\]
as \( n \to \infty \). So from Lemma 1.7, problem (Q) has a weak solution \( v \).

In the following theorem we use a measure of strong noncompactness \( \gamma \) so we have a generalization of Theorem 3.1 and an improvement to Theorem 2 in [26] and Theorem 9 in [13].

**Theorem 3.2.** In the setting of Theorem 2.2 if we replace the function \( f' \) by \( f^d \) such that for all \( \varphi \in C([-d, 0], B_r) f^d(I \times \{ \varphi \}) \) is separable, then problem (Q) has a solution.

**Proof.** For \( n \in \mathbb{N} \) we define \( \gamma_1 : [-d, \frac{T}{n}] \times E \to E \), as in the proof of Theorem 3.1, by

\[
\gamma_1(t, x) = \begin{cases} 
\xi(t) & \text{if } t \in [-d, 0] \\
\xi(0) + nt(x - \xi(0)) & \text{if } t \in \left[0, \frac{T}{n}\right]
\end{cases}
\]

and \( f_1 : \left[0, \frac{T}{n}\right] \times E \to E \) by \( f_1(t, x) = f^d(t, \theta_{\frac{T}{n}}(\gamma_1(., x))) \). By Theorem 2.2 there exists a continuous function \( y_1 \) such that \( y_1 = \xi \) on \([-d, 0]\) and for each \( t \in \left[0, \frac{T}{n}\right] \)

\[
y_1(t) = G(t, 0)\xi(0) + \int_0^t G(t, s) f_1(s, y_1(s)) \, ds.
\]

Then we can construct, for each \( n \in \mathbb{N} \), a continuous bounded function \( v_n \) such that \( v_n = \xi \) on \([-d, 0]\) and for each \( t \in I \) \( v_n \) is defined by

\[
v_n(t) = G(t, 0)\xi(0) + \int_0^t G(t, s) h_n(s) \, ds,
\]

where \( h_n(t) = f^d(t, \theta_{\frac{T}{n}}(\gamma_1(., v_n(t)))) \) with \( k \in \{1, 2, 3, \ldots, n\} \) and \( \frac{(k-1)T}{n} \leq t \leq \frac{kt}{n} \). We can complete the proof as in the proof of Theorem 3.1.

In the next theorem we let \( h : I \times \mathbb{R}^+ \to \mathbb{R}^+ \) be a Carathéodory function. Also for each bounded subset \( Z \) of \( I \times \mathbb{R}^+ \) we suppose that there exists a function \( m : I \to \mathbb{R}^+ \) such that \( h(t, s) \leq m(t), (t, s) \in Z \) and \( m \) is integrable on \([c, T]\) for each \( c, 0 < c \leq T \). Moreover, assume that the identically zero function is the only absolutely continuous function on \([0, c]\) which satisfies \( \dot{u}(t) = h(t, u(t)) \) a.e. on \([0, c]\) and for which the right derivative \( D_+ u(t) \) of \( u(t) \) at \( t = 0 \) exists and is 0.

**Theorem 3.3.** If we replace in the setting of Theorem 3.2 a Kamke function \( w \) by a function \( h \) and we suppose that \( f^d \) is bounded and continuous, then problem (Q) has a solution.
We omit the proof since it runs as in the proof of Theorem 3.2 except that we replace the use of Theorem 2.2 by that of Theorem 2.3 to find a continuous function $y_1$ such that $y_1 = \xi$ on $[-d, 0]$ and for each $t \in [0, \frac{T}{n}]$

\[ y_1(t) = G(t, 0)\xi(0) + \int_0^t G(t, s)f_1(s, y_1(s))\,ds. \]

In fact, if $L(t) \neq 0$ our results generalize that of Gomaa [10] and Cichon [4], since we have a generalization of the compactness assumptions and in [4] the results are stated without delay. For the important case $L(t) = 0$ we have, as a special case, a generalization of the existence theorems of Gomaa [13], Ibrahim-Gomaa [15], Papageorgiou [23], Cramer-Lakshmikantham-Mitchell [7], Szep [25] and Boudourides [2] in all of which the results are stated without delay. Szep in [25] studied the special case of problem (P) in a reflexive Banach space, Boudourides [2] and Cramer-Lakshmikantham-Mitchell [7] studied the special case of problem (P) in a nonreflexive Banach space, Papageorgiou [23] found weak solutions for the special case of problem (P) on a finite interval $I$ with $0 < T < \infty$, Ibrahim-Gomaa [15] found weak solutions for the special case of problem (P) on a finite interval $I$ and in [13] we give a generalization to recent results on the Cauchy problem by using weak and strong measures of noncompactness. Moreover in [11], [12] we study the nonlinear differential equations with and without delay while in [9] we study the differential inclusions with moving constraints.

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REFERENCES


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