

EXISTENCE AND CONCENTRATION OF SOLUTIONS FOR A CLASS OF ELLIPTIC PROBLEMS WITH DISCONTINUOUS NONLINEARITY IN \mathbb{R}^N

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Abstract

Using variational methods we establish existence and concentration of positive solutions for a class of elliptic problems in \mathbb{R}^N , whose nonlinearity is discontinuous.

1. Introduction

Many recent studies have focused on the nonlinear Schrödinger equation

$$(NLS) \quad i\epsilon \frac{\partial \Psi}{\partial t} = -\epsilon^2 \Delta \Psi + (V(x) + E)\Psi - f(\Psi) \quad \text{for all } x \in \mathbb{R}^N,$$

where $\epsilon > 0$. Knowledge of the solutions for the elliptic equation

$$(P_\epsilon) \quad -\epsilon^2 \Delta u + V(x)u = f(u) \quad \text{in } \mathbb{R}^N,$$

which can be rewritten of the form

$$-\Delta u + V(\epsilon x)u = f(u) \quad \text{in } \mathbb{R}^N,$$

has great importance in the study of standing-wave solutions of (NLS). In recent years, the existence and concentration of positive solutions for general semilinear elliptic equations (P_ϵ) for the case $N \geq 3$ have been extensively studied, see for example, Bartsch, Pankov & Wang [6], Floer & Weinstein [17], Oh [21], Rabinowitz [22], Wang [24], Alves & Souto [5], Alves & Figueiredo [3], [4], del Pino & Felmer [15] and their references.

In [22], by a mountain pass argument, Rabinowitz proves the existence of positive solutions of (P_ϵ) , for $\epsilon > 0$ small, whenever

$$(V_0) \quad V_\infty = \liminf_{|x| \rightarrow \infty} V(x) > \inf_{x \in \mathbb{R}^N} V(x) = \gamma > 0.$$

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Later Wang [24] showed that these solutions concentrate at global minimum points of $V(x)$ as ϵ tends to 0.

In [15], del Pino and Felmer find solutions which concentrate around a local minimum of $V(x)$ by introducing a penalization method. More precisely, they assume that there is an open and bounded set Λ compactly contained in Ω such that

$$(V_1) \quad 0 < \gamma \leq V_0 = \inf_{x \in \Lambda} V(x) < \min_{x \in \partial \Lambda} V(x).$$

In [18], Gazzola & Radulescu have considered the existence of solution for the following class of problem

$$\begin{cases} Lu + V(x)u = f(x, u) & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \end{cases}$$

where L is a general elliptic operator of second order and f is a discontinuous function with subcritical growth and $V : \mathbb{R}^N \rightarrow \mathbb{R}$ is a coercive continuous function, that is,

$$V(x) \rightarrow +\infty \quad \text{as} \quad |x| \rightarrow +\infty.$$

In that paper, by using variational methods, the authors show that the coercivity of V implies that the nonlinearity defines a compact operator. Such a property is crucial to conclude that weak limit of (PS) sequence of the energy functional are solutions of the problem. Still related to elliptic problems with discontinuous nonlinearity, we cite the papers of Alves, Bertone & Gonçalves [1], Alves, Santos & Gonçalves [2], Badiale [7], [8], Dinu [16], Radulescu [23] and their references.

Motivated by the papers [18], [22] and [24], we study the existence and concentration of solutions for the following class of problem

$$(P)_{\epsilon, \beta} \quad \begin{cases} -\Delta u + V(\epsilon x)u = H(u - \beta)u^p & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \end{cases}$$

where $\epsilon, \beta > 0$ are positive parameters, H is the Heaviside function given by

$$H(t) = \begin{cases} 1 & \text{if } t > 0, \\ 0 & \text{if } t \leq 0, \end{cases}$$

and $p \in (1, \frac{N+2}{N-2})$ if $N \geq 3$ or $p \in (1, +\infty)$ if $N = 1, 2$. Moreover, $V : \mathbb{R}^N \rightarrow \mathbb{R}$ is a positive continuous function verifying the assumption (V_0) .

By a solution for $(P)_{\epsilon, \beta}$, we understand as a function $u \in W_{\text{loc}}^{2, \frac{p+1}{p}}(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ verifying

$$(1.1) \quad -\Delta u(x) + V(\epsilon x)u(x) \in [g(u(x)), \bar{g}(u(x))] \quad \text{a.e in } \mathbb{R}^N$$

where $g(t) = H(t - \beta)t^p$, $\bar{g}(t) = \lim_{\delta \rightarrow 0^+} g(t + \delta)$ and $\underline{g}(t) = \lim_{\delta \rightarrow 0^+} g(t - \delta)$.

The interest in the study of nonlinear partial differential equations with discontinuous nonlinearities has increased because many free boundary problems arising in mathematical physics may be stated in this form. Among these problems, we have the obstacle problem, the seepage surface problem, and the Elenbaas equation, see for example [10], [11], [12].

Our main result is the following:

THEOREM 1.1. *Assume that (V_0) holds. Then, there are $\epsilon^*, \beta^* > 0$ such that $(P)_{\epsilon, \beta}$ has a positive solution $u_{\epsilon, \beta}$ for $\epsilon \in (0, \epsilon^*)$ and $\beta \in (0, \beta^*)$. Moreover, if $z_{\epsilon, \beta} \in \mathbb{R}^N$ denotes a maximum point of $u_{\epsilon, \beta}$, we have that*

$$\lim_{(\epsilon, \beta) \rightarrow (0, 0)} V(\epsilon z_{\epsilon, \beta}) = \gamma.$$

Theorem 1.1 completes the study made in [18], [22] and [24], in the sense that, in [18] the authors studied the case $\liminf_{|x| \rightarrow \infty} V(x) = \infty$, because in that paper V is coercive. In the present paper, we do not assume this assumption on V , this way, V can not be coercive. Once that V is not necessarily coercive, we use a new argument to overcome the lack of compactness involving the nonlinearity, see Lemma 3.3 in Section 3. Related to the papers [22] and [24], our contribution is directly related to the fact that we are considering a discontinuous nonlinearity, while that in those papers, the nonlinearity considered was continuous. The authors believe that this is the first work related to the study of concentration of the solutions for a class of elliptic problems with discontinuous nonlinearity. A first difficulty involving this class of problem, is the fact that we cannot use the classical variational methods; here it is necessary to use some results for a locally Lipschitz functional. When the nonlinearity is continuous and satisfies some assumptions, the mountain pass level is equal to the minimum of the energy functional on Nehari manifolds, which is a key point in a lot of papers. However, this property is not true for discontinuous nonlinearity. Hence, the arguments used in the above reference cannot be repeated directly, and a careful analysis is necessary to get similar results to those found in [22] and [24].

2. Basic results from convex analysis

In this section, for the reader's convenience, we recall some definitions and basic results on the critical point theory of locally Lipschitz continuous functionals as developed by Chang [10], Clarke [13], [14] and Grossinho & Tersin [19].

Let X be a real Banach space. A functional $I : X \rightarrow \mathbb{R}$ is locally Lipschitz continuous, $I \in \text{Lip}_{\text{loc}}(X, \mathbb{R})$ for short, if given $u \in X$ there is an open neighborhood $V := V_u \subset X$ and some constant $K = K_V > 0$ such that

$$|I(v_2) - I(v_1)| \leq K \|v_2 - v_1\|, \quad v_i \in V, \quad i = 1, 2.$$

The directional derivative of I at u in the direction of $v \in X$ is defined by

$$I^0(u; v) = \limsup_{h \rightarrow 0, \lambda \downarrow 0} \frac{I(u + h + \lambda v) - I(u + h)}{\lambda}.$$

Hence $I^0(u; \cdot)$ is continuous, convex and its subdifferential at $z \in X$ is given by

$$\partial I^0(u; z) = \{ \mu \in X^*; I^0(u; v) \geq I^0(u; z) + \langle \mu, v - z \rangle, v \in X \},$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between X^* and X . The generalized gradient of I at u is the set

$$\partial I(u) = \{ \mu \in X^*; \langle \mu, v \rangle \leq I^0(u; v), v \in X \}.$$

Since $I^0(u; 0) = 0$, $\partial I(u)$ is the subdifferential of $I^0(u; 0)$.

A few definitions and properties will be recalled below.

$\partial I(u) \subset X^*$ is convex, non-empty and weak*-compact,

$$\lambda(u) = \min \{ \| \mu \|_{X^*}; \mu \in \partial I(u) \},$$

and

$$\partial I(u) = \{ I'(u) \}, \quad \text{if } I \in C^1(X, \mathbb{R}).$$

A critical point of I is an element $u_0 \in X$ such that $0 \in \partial I(u_0)$ and a critical value of I is a real number c such that $I(u_0) = c$ for some critical point $u_0 \in X$.

THEOREM 2.1 ([19], [23]). *Let $I \in \text{Lip}_{\text{loc}}(X, \mathbb{R})$ with $I(0) = 0$ and satisfying:*

- (i) *There are $r > 0$ and $\rho > 0$, such that $I(u) \geq \rho$, for $\|u\| = r$, $u \in X$;*
- (ii) *There is $e \in X \setminus B_r(0)$ with $I(e) < 0$.*

If $c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t))$ and

$$\Gamma = \{ \gamma \in C([0, 1], X), \gamma(0) = 0 \text{ and } \gamma(1) = e \},$$

then $c \geq \rho$ and there is a sequence $\{u_n\} \subset X$ verifying

$$I(u_n) \rightarrow c \quad \text{and} \quad \lambda(u_n) \rightarrow 0.$$

PROPOSITION 2.1 ([10], [13]). *Let $\{u_n\} \subset X$ and $\{\rho_n\} \subset X^*$ with $\rho_n \in \partial I(u_n)$. If $u_n \rightarrow u$ in X and $\rho_n \xrightarrow{*} \rho$ in X^* , then $\rho_0 \in \partial I(u)$.*

PROPOSITION 2.2 ([10], [13]). *Let $R > 0$, $\Psi(u) = \int_{\mathbb{R}^N} G(u) dx$ and $\Psi_R(v) = \int_{B_R(0)} G(v) dx$, where $G(t) = \int_0^t g(s) ds$. Then, $\Psi \in \text{Lip}_{\text{loc}}(L^{p+1}(\mathbb{R}^N), \mathbb{R})$, $\Psi_R \in \text{Lip}_{\text{loc}}(L^{p+1}(B_R(0)), \mathbb{R})$, $\partial\Psi(u) \in L^{\frac{p+1}{p}}(\mathbb{R}^N)$ and $\partial\Psi_R(v) \in L^{\frac{p+1}{p}}(B_R(0))$. Moreover, if $\rho \in \partial\Psi(u)$ and $\zeta \in \partial\Psi_R(v)$, then*

$$\rho(x) \in [\underline{g}(u(x)), \bar{g}(u(x))] \quad \text{a.e in } \mathbb{R}^N$$

and

$$\zeta(x) \in [\underline{g}(v(x)), \bar{g}(v(x))] \quad \text{a.e in } B_R(0).$$

3. Existence of solution for $(P)_{1,\beta}$

In this section, we will suppose that $V_\infty < +\infty$, because the case $V_\infty = +\infty$ has been considered in [18]. Here, we consider the space

$$(3.2) \quad E = \left\{ u \in H^1(\mathbb{R}^N); \int_{\mathbb{R}^N} V(x)u^2 < \infty \right\}$$

endowed with the norm

$$\|u\|^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2),$$

which is associated with inner product

$$\langle u, v \rangle = \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv).$$

The energy functional associated with $(P)_{1,\beta}$ is given by

$$I_\beta(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) - \int_{\mathbb{R}^N} G(u),$$

where $G(t) = \int_0^t g(s) ds$.

LEMMA 3.1. *The functional I_β satisfies the mountain pass geometry.*

PROOF. We begin observing that $I_\beta(0) = 0$ and

$$I_\beta(u) \geq \frac{1}{2}\|u\|^2 - \frac{1}{p+1} \int_\Omega |u|^{p+1}.$$

Hence, by Sobolev embedding

$$I_\beta(u) \geq \frac{1}{2}\|u\|^2 - C\|u\|^{p+1}.$$

Thereby, there exist $r, \rho > 0$ verifying

$$I_\beta(u) \geq \rho > 0 \quad \text{for } \|u\| = r, u \in E.$$

Now, let $\psi \in C_0^\infty(\mathbb{R}^N)$ with $\psi > 0$ and $K = \text{supt } \psi \subset \mathbb{R}^N$. Then, for $t > 0$

$$I_\beta(t\psi) \leq \frac{t^2}{2}\|\psi\|^2 - \frac{t^{p+1}}{p+1} \int_{K \cap \{t\psi > \beta\}} \psi^{p+1} + \int_{K \cap \{t\psi > \beta\}} \frac{\beta^{p+1}}{p+1} + C \text{med}(K).$$

Once that $p > 1$,

$$I_\beta(t\psi) \rightarrow -\infty \quad \text{as } t \rightarrow +\infty.$$

Consequently, for $t_0 > 0$ large enough, we derive that $e = t_0\psi$ satisfies

$$I_\beta(e) < 0 \quad \text{with } e \in E \setminus B_r(0).$$

Combining Lemma 3.1 with Theorem 2.1, there is a sequence $\{u_n\} \subset E$ satisfying

$$(3.3) \quad I_\beta(u_n) \rightarrow c_\beta \quad \text{and} \quad \lambda_\beta(u_n) \rightarrow 0,$$

where c_β is the mountain pass level of the functional I_β .

In what follows, we will show that $\{u_n\}$ given in (3.3) is bounded in E . To this end, we will use the following notation for the functional I_β :

$$I_\beta(u) = Q(u) - \Psi(u)$$

where $Q(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)$ and $\Psi(u) = \int_{\mathbb{R}^N} G(u)$.

LEMMA 3.2. *The sequence $\{u_n\}$ is bounded in E .*

PROOF. From (3.3),

$$I_\beta(u_n) \rightarrow c_\beta \quad \text{and} \quad \lambda_\beta(u_n) \rightarrow 0.$$

From now on, we consider $\{\omega_n\} \subset (H^1(\mathbb{R}^N))^*$ such that $\lambda_\beta(u_n) = \|\omega_n\|_*$ and

$$\omega_n = Q'(u_n) - \rho_n$$

where $\{\rho_n\} \subset \partial\Psi(u_n)$. Then,

$$(3.4) \quad \langle \omega_n + \rho_n, u_n \rangle = \langle Q'(u_n), u_n \rangle = \|u_n\|^2.$$

Once that $0 \leq (p+1)G(t) \leq t\underline{g}(t) \forall t \in \mathbb{R}$, it follows that

$$\Psi(u_n) = \int_{\mathbb{R}^N} G(u_n) \leq \frac{1}{p+1} \int_{\mathbb{R}^N} u_n \underline{g}(u_n).$$

From Proposition 2.2,

$$\underline{g}(u_n(x)) \leq \rho_n(x) \leq \bar{g}(u_n(x)) \quad \text{a.e in } \mathbb{R}^N$$

leading to

$$u_n(x) \underline{g}(u_n(x)) \leq \rho_n(x) u_n(x) \quad \text{a.e in } \mathbb{R}^N,$$

which implies that

$$\int_{\mathbb{R}^N} u_n \underline{g}(u_n) \leq \int_{\mathbb{R}^N} \rho_n u_n = \langle \rho_n, u_n \rangle.$$

Therefore,

$$(3.5) \quad \Psi(u_n) \leq \frac{1}{p+1} \int_{\mathbb{R}^N} u_n \underline{g}(u_n) \leq \frac{1}{p+1} \langle \rho_n, u_n \rangle.$$

Combining (3.4) and (3.5),

$$I_\beta(u_n) - \frac{1}{p+1} \langle \omega_n + \rho_n, u_n \rangle \geq \left(\frac{1}{2} - \frac{1}{p+1} \right) \|u_n\|^2 - \frac{1}{p+1} \langle \rho_n, u_n \rangle,$$

or equivalently,

$$I_\beta(u_n) - \frac{1}{p+1} \langle \omega_n, u_n \rangle \geq \left(\frac{1}{2} - \frac{1}{p+1} \right) \|u_n\|^2.$$

Using the fact that $\{I_\beta(u_n)\}$ is bounded, there is $M > 0$ such that

$$M - \frac{1}{p+1} \langle \omega_n, u_n \rangle \geq \left(\frac{1}{2} - \frac{1}{p+1} \right) \|u_n\|^2.$$

Recalling that $\langle \omega_n, u_n \rangle = o_n(1)$, we have

$$M + o_n(1)\|u_n\| \geq \left(\frac{1}{2} - \frac{1}{p+1}\right)\|u_n\|^2,$$

showing that $\{u_n\}$ is bounded in E .

The next lemma is a key point in our argument, because the functional Ψ is not compact. In what follows, for each $R > 0$, we set $\Psi_R : L^{p+1}(B_R(0)) \rightarrow \mathbb{R}$ the following functional

$$\Psi_R(u) = \int_{B_R(0)} G(u).$$

Moreover, for each $\psi \in L^{p+1}(B_R(0))$, we set the function $\tilde{\psi} \in L^{p+1}(\mathbb{R}^N)$ given by

$$\tilde{\psi}(x) = \begin{cases} \psi(x), & x \in B_R(0) \\ 0, & x \in B_R^c(0). \end{cases}$$

LEMMA 3.3. *Let $\{u_n\} \subset E$ with $u_n \rightharpoonup u$ in E and $\{\rho_n\} \subset \partial\Psi(u_n)$ with $\rho_n \rightharpoonup \rho_0$ in $L^{\frac{p+1}{p}}(\mathbb{R}^N)$. Then,*

$$\rho_0(x) \in [\underline{g}(u(x)), \overline{g}(u(x))] \quad a.e \text{ in } \mathbb{R}^N.$$

PROOF. Hereafter, for each $R > 0$, we denote by $u_{n,R}$, $\rho_{n,R}$, u_R and $\rho_{0,R}$ the restriction of the functions u_n , ρ_n , u and ρ_0 to $B_R(0)$. For each $\psi \in L^{p+1}(B_R(0))$, a simple computation implies that

$$\int_{B_R(0)} \rho_{n,R} \psi = \int_{\mathbb{R}^N} \rho_n \tilde{\psi}$$

and

$$\Psi_R^0(u_{n,R}, \psi) = \Psi^0(u_n, \tilde{\psi}).$$

Once that

$$\int_{\mathbb{R}^N} \rho_n \tilde{\psi} \leq \Psi^0(u_n, \tilde{\psi}),$$

we derive

$$\int_{B_R(0)} \rho_{n,R} \psi \leq \Psi_R^0(u_{n,R}, \psi), \quad \forall \psi \in L^{p+1}(B_R(0)),$$

which yields

$$\rho_{n,R} \in \partial\Psi_R(u_{n,R}).$$

Since, $u_{n,R} \rightarrow u_R$ in $L^{p+1}(B_R(0))$ and $\rho_{n,R} \rightarrow \rho_{0,R}$ in $L^{\frac{p+1}{p}}(B_R(0))$, from Proposition 2.1

$$\rho_{0,R} \in \partial\Psi_R(u_R)$$

and so, by Proposition 2.2

$$\rho_{0,R}(x) \in [\underline{g}(u_R(x)), \bar{g}(u_R(x))] \quad \text{a.e in } B_R(0),$$

or equivalently,

$$\rho_0(x) \in [\underline{g}(u(x)), \bar{g}(u(x))] \quad \text{a.e in } B_R(0).$$

Now, using the fact that $R > 0$ is arbitrary, we get

$$\rho_0(x) \in [\underline{g}(u(x)), \bar{g}(u(x))] \quad \text{a.e in } \mathbb{R}^N.$$

THEOREM 3.1. *Assume that $c_\beta < c_\infty$, where c_∞ is the mountain pass level associated with the functional*

$$I_\infty(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V_\infty u^2) - \frac{1}{p+1} \int_{\mathbb{R}^N} |u|^{p+1}, \quad \forall u \in H^1(\mathbb{R}^N).$$

Then, $(P)_{1,\beta}$ has a nontrivial solution.

PROOF. From Lemma 3.1 and Theorem 2.1, there is a sequence $\{u_n\} \subset E$ satisfying

$$(3.6) \quad I_\beta(u_n) \rightarrow c_\beta \quad \text{and} \quad \lambda_\beta(u_n) \rightarrow 0.$$

By using standard arguments, we can assume, without loss of generality, that $\{u_n\}$ is bounded in E and $u_n(x) \geq 0$ for all $x \in \mathbb{R}^N$. This way, there is $u \in H^1(\mathbb{R}^N)$ such that, for some subsequence,

$$(3.7) \quad u_n \rightharpoonup u \quad \text{in } E$$

and

$$(3.8) \quad u_n \rightarrow u \quad \text{in } L^p_{\text{loc}}(\mathbb{R}^N).$$

CLAIM 3.1. *The weak limit u is nontrivial, that is, $u \neq 0$.*

In fact, if $u \equiv 0$, the limit $u_n \rightarrow 0$ in E does not hold, because $c_\beta > 0$. From Lions' lemma [20], there are $\{y_n\} \subset \mathbb{R}^N$ and $r, \alpha > 0$ verifying

$$\liminf_{n \rightarrow \infty} \int_{B_r(y_n)} |u_n|^2 \geq \alpha > 0.$$

Since we are supposing $u = 0$, the Sobolev embedding combined with the above inequality yields $\{y_n\}$ is unbounded. Now, we set

$$(3.9) \quad v_n(x) = u_n(x + y_n).$$

Using the boundedness of $\{u_n\}$ in E , it follows that $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Hence, there is $v \in H^1(\mathbb{R}^N) \setminus \{0\}$ and a subsequence of $\{v_n\}$, still denote by itself, such that

$$v_n \rightharpoonup v \quad \text{in } E$$

and

$$v_n \rightarrow v \quad \text{in } L^s_{\text{loc}}(\mathbb{R}^N),$$

$1 \leq s < 2^*$ if $N \geq 3$ and $1 \leq s$ if $N = 1, 2$.

Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ verifying $\varphi(x) = 1$ for $x \in B_1(0)$, $\varphi(x) = 0$ for $x \in B_2^c(0)$, $0 \leq \varphi(x) \leq 1$ and $\varphi_R(x) = \varphi(\frac{x}{R})$ for $R > 0$. Then,

$$\langle \omega_n, (\varphi_R v_n)(\cdot - y_n) \rangle = o_n(1),$$

because the sequence $\{(\varphi_R v_n)(\cdot - y_n)\}$ is bounded in E . Thereby,

$$\begin{aligned} \int_{\mathbb{R}^N} \nabla u_n \nabla ((\varphi_R u_n)(x - y_n)) + V u_n (\varphi_R v_n)(x - y_n) \\ = \int_{\mathbb{R}^N} \rho_n ((\varphi_R v_n)(x - y_n)) + o_n(1), \end{aligned}$$

and so,

$$\int_{B_{2R}} |\nabla v_n|^2 \varphi_R(x) + \int_{B_{2R}} V(x + y_n) v_n^2 \varphi_R + \int_{B_{2R}} v_n \nabla v_n \nabla \varphi_R \leq \int_{B_{2R}} v_n^{p+1} \varphi_R.$$

Combining Fatou's lemma with the Sobolev embedding, we get

$$\int_{B_{2R}} |\nabla v|^2 \varphi_R + \int_{B_{2R}} V_\infty v^2 \varphi_R + \int_{B_{2R}} v \nabla v \nabla \varphi_R \leq \int_{B_{2R}} v^{p+1} \varphi_R.$$

Now, taking the limit of $R \rightarrow +\infty$,

$$(3.10) \quad \int_{\mathbb{R}^N} |\nabla v|^2 + V_\infty v^2 \leq \int_{\mathbb{R}^N} v^{p+1}.$$

Once that $v \neq 0$, there is $t > 0$, such that $tv \in \mathcal{N}_\infty$, where \mathcal{N}_∞ is the Nehari manifold associated with I_∞ given by

$$\mathcal{N}_\infty = \{u \in H^1(\mathbb{R}^N) \setminus \{0\}; I'_\infty(u)u = 0\}.$$

Then,

$$(3.11) \quad \int_{\mathbb{R}^N} |\nabla v|^2 + V_\infty v^2 = t^{p-1} \int_{\mathbb{R}^N} v^{p+1}.$$

From (3.10) and (3.11), we conclude that $t \leq 1$.

By a result found in Willem [25],

$$c_\infty \leq \inf_{u \in \mathcal{N}_\infty} I_\infty(u),$$

from where it follows that $c_\infty \leq I_\infty(tv)$. Consequently,

$$\begin{aligned} c_\infty \leq I_\infty(tv) &= I_\infty(tv) - \frac{1}{p+1} I'_\infty(tv)tv \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right) t^2 \int_{\mathbb{R}^N} |\nabla v|^2 + V_\infty v^2, \end{aligned}$$

and since $t \leq 1$,

$$c_\infty \leq \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\mathbb{R}^N} |\nabla v|^2 + V_\infty v^2.$$

Combining Fatou's lemma with the inequality $\underline{g}(s)s \geq (p+1)G(s)$ for all $s \geq 0$, we obtain

$$\begin{aligned} c_\infty \leq \liminf \left[\left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\mathbb{R}^N} |\nabla u_n|^2 + V(x)u_n^2 \right. \\ \left. + \int_{\mathbb{R}^N} \left(\frac{\underline{g}(u_n)u_n}{p+1} - G(u_n)\right) \right]. \end{aligned}$$

By Proposition 2.2,

$$\begin{aligned} c_\infty \leq \liminf_{n \rightarrow +\infty} \left[\left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\mathbb{R}^N} |\nabla u_n|^2 + V(x)u_n^2 \right. \\ \left. + \frac{1}{p+1} \int_{\mathbb{R}^N} \rho_n u_n - \int_{\mathbb{R}^N} G(u_n) \right] \end{aligned}$$

that is,

$$c_\infty \leq \liminf_{n \rightarrow +\infty} \left[I_\beta(u_n) - \frac{1}{p+1} \langle \omega_n, u_n \rangle \right] = \liminf_{n \rightarrow +\infty} [I_\beta(u_n) + o_n(1)] = c_\beta,$$

obtaining a contradiction. Therefore, $u \geq 0$ and $u \neq 0$. Next, we will show that u is a solution for $(P)_{1,\beta}$. To this end, we need to prove that $u \in W_{\text{loc}}^{2, \frac{p+1}{p}}(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ and

$$-\Delta u(x) + V(x)u(x) \in [\underline{g}(u(x)), \bar{g}(u(x))] \quad \text{a.e in } \mathbb{R}^N.$$

Since $\{u_n\} \subset E$ is a $(PS)_{c_\beta}$ sequence, there are $\{\omega_n\} \subset \partial I_\beta(u_n)$ and $\{\rho_n\} \subset \partial \Psi(u_n)$ verifying

$$(3.12) \quad \|\omega_n\|_* \rightarrow 0$$

and

$$(3.13) \quad \langle \omega_n, \varphi \rangle = \int_{\mathbb{R}^N} \nabla u_n \nabla \varphi + V(x)u_n \varphi - \int_{\mathbb{R}^N} \rho_n \varphi, \quad \varphi \in E,$$

with

$$(3.14) \quad \rho_n(x) \in [\underline{g}(u_n(x)), \bar{g}(u_n(x))] \quad \text{a.e in } \mathbb{R}^N.$$

The boundedness of $\{u_n\}$ combined with (3.14) implies that $\{\rho_n\}$ is bounded in $L^{\frac{p+1}{p}}(\mathbb{R}^N)$. Thus, there is $\rho_0 \in L^{\frac{p+1}{p}}(\mathbb{R}^N)$ and a subsequence of $\{\rho_n\}$, such that

$$(3.15) \quad \rho_n \rightharpoonup \rho_0 \quad \text{in } L^{\frac{p+1}{p}}(\mathbb{R}^N).$$

From (3.7) and (3.15)

$$\int_{\mathbb{R}^N} \nabla u \nabla \varphi + V(x)u \varphi = \int_{\mathbb{R}^N} \rho_0 \varphi, \quad \forall \varphi \in E.$$

Moreover, by Lemma 3.3,

$$(3.16) \quad \rho_0 \in [\underline{g}(u(x)), \bar{g}(u(x))] \quad \text{a.e in } \mathbb{R}^N,$$

showing that u is a nonnegative weak solution of the problem

$$-\Delta u + V(x)u = \rho_0 \quad \text{in } \mathbb{R}^N.$$

Using regularity theory, it follows that $u \in W_{\text{loc}}^{2, \frac{p+1}{p}}(\mathbb{R}^N)$, and so

$$(3.17) \quad -\Delta u(x) + V(x)u(x) = \rho_0(x) \quad \text{a.e in } \mathbb{R}^N.$$

Therefore, from (3.16) and (3.17) it follows that u is a solution for $(P)_{1,\beta}$.

REMARK 3.1. Using the fact that $V(x) \geq \gamma$ for all $x \in \mathbb{R}^N$, it is immediate to check, by using the Stampachia theorem, that $\{x \in \mathbb{R}^N; u(x) = \beta\}$ has null measure for β small enough, this way, the solution u verifies

$$-\Delta u + V(x)u = H(u - \beta)u^p \quad \text{a.e in } \mathbb{R}^N.$$

This information is very important in many applications.

4. Existence and concentration of solutions for $(P)_{\epsilon,\beta}$

In this section, we consider the space

$$(4.18) \quad E_\epsilon = \left\{ u \in H^1(\mathbb{R}^N); \int_{\mathbb{R}^N} V(\epsilon x)u^2 < \infty \right\}$$

endowed with the norm

$$\|u\|^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + V(\epsilon x)u^2),$$

which is associated with inner product

$$\langle u, v \rangle = \int_{\mathbb{R}^N} (\nabla u \nabla v + V(\epsilon x)uv).$$

The energy functional associated with $(P)_{\epsilon,\beta}$ is given by

$$I_{\epsilon,\beta}(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(\epsilon x)u^2) - \int_{\mathbb{R}^N} G(u),$$

and its mountain pass level is denoted by $c_{\epsilon,\beta}$.

PROOF OF THEOREM 1.1. We will divide the proof in two steps.

Step 1: Existence. In what follows, $\omega \in H^1(\mathbb{R}^N)$ denotes a positive ground state solution of the problem

$$(4.19) \quad \begin{cases} -\Delta \omega + \gamma \omega = \omega^p & \text{in } \mathbb{R}^N, \\ \omega > 0 & \text{in } \mathbb{R}^N, \end{cases}$$

that is, if $I_0 : H^1(\mathbb{R}^N) \rightarrow \mathbb{R}$ is the functional energy associated with (4.19) given by

$$I_0(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \gamma |u|^2 - \frac{1}{p+1} \int_{\mathbb{R}^N} |u|^{p+1},$$

we have that $I_0(\omega) = c_0$ and $I'_0(\omega) = 0$, where c_0 is the mountain pass level.

Hereafter, we set $\varphi \in C_0^\infty(\mathbb{R}^N)$ verifying

$$0 \leq \varphi(x) \leq 1, \quad \varphi(x) = 1 \quad \forall x \in B_1(0) \quad \text{and} \quad \varphi(x) = 0 \quad \forall x \in B_2^c(0).$$

Furthermore, for each $R > 1$, we denote by φ_R and ω_R the functions

$$\varphi_R(x) = \varphi\left(\frac{x}{R}\right) \quad \text{and} \quad \omega_R(x) = \varphi_R(x)\omega(x).$$

A direct computation shows that

$$(4.20) \quad \omega_R \rightarrow \omega \quad \text{in} \quad H^1(\mathbb{R}^N) \quad \text{as} \quad R \rightarrow +\infty.$$

Hence, $\omega_R \neq 0$ for R large enough. From this, there is $t_R > 0$, such that

$$I_0(t_R \omega_R) = \max_{t \geq 0} I_0(t \omega_R)$$

and so,

$$(4.21) \quad \int_{B_{2R}} |\nabla \omega_R|^2 + \gamma |\omega_R|^2 = t_R^{p-1} \int_{B_{2R}} \omega_R^{p+1}$$

and

$$\lim_{R \rightarrow \infty} t_R = 1.$$

These facts imply that

$$\hat{\omega}_R = \omega_R t_R \rightarrow \omega \quad \text{in} \quad H^1(\mathbb{R}^N) \quad \text{as} \quad R \rightarrow \infty.$$

Once that $c_0 < c_\infty$, see [22], we can choose $\delta, R > 0$ such that

$$c_0 + \delta < c_\infty \quad \text{and} \quad I_0(\hat{\omega}_R) < c_0 + \frac{\delta}{2},$$

and $t_* > 0$ verifying $I_{\epsilon, \beta}(t_* \hat{\omega}_R) < 0$ uniformly in $\epsilon, \beta > 0$ small enough.

Now, we consider $\hat{\gamma}(t) = t(t_* \hat{\omega}_R)$ for $t \in [0, 1]$, which belongs to $\hat{\gamma} \in \Gamma$. From the definition of $c_{\epsilon, \beta}$

$$c_{\epsilon, \beta} \leq \max_{t \in [0, 1]} I_{\epsilon, \beta}(\hat{\gamma}(t)) = \max_{t \geq 0} I_{\epsilon, \beta}(t \hat{\omega}_R) = I_{\epsilon, \beta}(\hat{t} \hat{\omega}_R)$$

for some $\hat{t} = \hat{t}(\epsilon, \beta, R) > 0$.

A straightforward computation implies that for each $R > 0$ given, there are positive constants C, K such that $C \leq \hat{t} \leq K$, for $\epsilon, \beta > 0$ small enough.

Now, since $\gamma \leq V(x)$ for all $x \in \mathbb{R}^N$, it is easy to check that

$$(4.22) \quad c_0 \leq c_{\epsilon, \beta} \leq \max_{t \geq 0} I_{\epsilon, \beta}(t \hat{\omega}_R).$$

Without loss of generality, we assume that $V(0) = \gamma$. Thus, for each $\eta > 0$, there is $\epsilon_0 > 0$ such that,

$$0 < V(\epsilon x) - \gamma < \eta \quad \text{for } \epsilon \in (0, \epsilon_0) \text{ and } x \in \text{supt } \omega_R = \hat{B}_{2R}(0),$$

from where it follows that

$$\int_{\mathbb{R}^N} V(\epsilon x) \hat{\omega}_R^2 < \int_{\mathbb{R}^N} (\gamma + \eta) \hat{\omega}_R^2.$$

Using these informations,

$$c_{\epsilon, \beta} \leq I_0(\hat{t} \hat{\omega}_R) + \frac{\hat{t}^2}{2} \eta \int_{B_{2R}} \hat{\omega}_R^2 + \frac{\hat{t}^{p+1}}{p+1} \int_{B_{2R} \cap [\hat{t} \hat{\omega}_R \leq \beta]} \hat{\omega}_R^{p+1} + \int_{B_{2R} \cap [\hat{t} \hat{\omega}_R > \beta]} \frac{\beta^{p+1}}{p+1},$$

that is,

$$c_{\epsilon, \beta} \leq c_0 + \frac{\delta}{2} + \eta C_1 + C_2 \beta^{p+1}$$

where C_1, C_2 do not depend of $\epsilon, \beta > 0$. Hence, for $\eta, \beta > 0$ small enough

$$(4.23) \quad c_{\epsilon, \beta} \leq c_0 + \frac{\delta}{2} + \frac{\delta}{4} + \frac{\delta}{4} \leq c_0 + \delta < c_\infty.$$

From Theorem 3.1, the problem $(P)_{\epsilon, \beta}$ has a nontrivial solution for $\epsilon, \beta > 0$ small enough.

Step 2: Concentration of the solutions. Hereafter, we denote by $u_{\epsilon, \beta}$ the solution given by Step 1. Thereby, there exists $\rho_{\epsilon, \beta} \in L^{\frac{p+1}{p}}(\mathbb{R}^N)$ such that

$$(4.24) \quad -\Delta u_{\epsilon, \beta}(x) + V(x)u_{\epsilon, \beta}(x) = \rho_{\epsilon, \beta}(x) \quad \text{a.e in } \mathbb{R}^N$$

with $\rho_{\epsilon, \beta} \in [g(u_{\epsilon, \beta}(x)), \bar{g}(u_{\epsilon, \beta}(x))]$ a.e in \mathbb{R}^N .

Next, we fix $\epsilon_n \rightarrow 0, \beta_n \rightarrow 0, u_n = u_{\epsilon_n, \beta_n}$ and $\rho_n = \rho_{\epsilon_n, \beta_n}$. Our goal is to study the behavior of the maximum points related to $\{u_n\}$, more precisely, if $z_n \in \mathbb{R}^N$ denotes a maximum point of u_n , we will prove that

$$\lim_{n \rightarrow \infty} V(\epsilon_n z_n) = \gamma.$$

The arguments used in the proof of Step 1, more precisely in (4.22) and (4.23), lead to

$$(4.25) \quad \lim_{n \rightarrow \infty} c_{\epsilon_n, \beta_n} = c_0 > 0.$$

CLAIM 4.1. *There are $\{y_n\} \subset \mathbb{R}^N$ and $r, \eta > 0$ such that*

$$\liminf_{n \rightarrow \infty} \int_{B_r(y_n)} |u_n|^2 \geq \eta > 0.$$

In fact, if the claim does not hold, then by a result due to Lions

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n|^{p+1} = 0.$$

This limit combined with the fact that u_n is a solution of $(P)_{\epsilon_n, \beta_n}$ implies that

$$\lim_{n \rightarrow \infty} c_{\epsilon_n, \beta_n} = \lim_{n \rightarrow \infty} I_{\epsilon_n, \beta_n}(u_n) = 0,$$

which is a contradiction with (4.25).

CLAIM 4.2. *The sequence $v_n = u_n(\cdot - y_n)$ is strongly convergent in $H^1(\mathbb{R}^N)$. Moreover,*

$$\lim_{|x| \rightarrow \infty} v_n(x) = 0$$

uniformly in $n \in \mathbb{N}$, that is, for each $\eta > 0$, there is $R > 0$ such that

$$|v_n(x)| < \eta \quad \forall x \in \mathbb{R}^N \setminus B_R(0) \quad \text{and} \quad n \in \mathbb{N}.$$

In fact, repeating the same arguments used in the proof of Theorem 3.1, more precisely Claim 3.1, we can assume without loss of generality that $\{\epsilon_n y_n\}$ is a convergent sequence in \mathbb{R}^N with $\epsilon_n y_n \rightarrow y^* \in V^{-1}(\gamma)$. Furthermore, we also derive that if v is the weak limit of $\{v_n\}$, then

$$v_n \rightarrow v \quad \text{in} \quad H^1(\mathbb{R}^N).$$

Now, following the arguments found in Alves & Figueiredo [3], the Moser interaction implies that

$$(4.26) \quad \lim_{|x| \rightarrow \infty} v_n(x) = 0$$

uniformly in $n \in \mathbb{N}$. The basic idea to prove the last limit is to consider the following functions

$$u_{L,n}(x) = \begin{cases} u_n(x), & u_n(x) \leq L \\ L, & u_n(x) \geq L, \end{cases}$$

$$z_{L,n} = \eta_n^p u_{L,n}^{p(\xi-1)} u_n$$

with $\xi > 1$ to be determined later. The limit in (4.26) follows taking $z_{L,n}(\cdot - y_n)$ as a test function in (4.24) combined with the inequality $\rho_n(x) \leq u_n^p(x)$ a.e in \mathbb{R}^N .

On the other hand, it follows from (4.25) that $\liminf_{n \rightarrow \infty} \|v_n\|_{\infty, \mathbb{R}^N} > 0$, then there are $\delta^* > 0$ and $n_0 \in \mathbb{N}$ such that

$$\|v_n\|_{\infty, \mathbb{R}^N} \geq \delta^*, \quad \forall n \geq n_0.$$

Choosing $\eta = \frac{\delta^*}{2}$, there is $R > 0$ such that

$$v_n(x) < \frac{\delta^*}{2} \quad \forall x \in \mathbb{R}^N \setminus B_R(0) \quad \text{and} \quad n \in \mathbb{N}$$

and so, if x_n denotes a maximum point of v_n , we get

$$v_n(x_n) \geq \delta^* \quad \text{and} \quad x_n \in B_R(0).$$

Now, if z_n denote the maximum point of u_n , we have that $z_n = x_n + y_n$, this way, $\epsilon_n z_n = \epsilon_n x_n + \epsilon_n y_n \rightarrow y^*$. Since V is a continuous function,

$$\lim_{n \rightarrow \infty} V(\epsilon_n z_n) = V(y^*) = \gamma$$

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