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INVOLUTIONS ON A TROPICAL LINE

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Abstract

Tropical involutive linear maps and pencils of degree 2 on the tropical projective line are introduced and studied. Both concepts are related in a similar way to linear involutions and g^1_2's in the classical (algebraic) projective line, and tropicalization relates the algebraic phenomena to the tropical ones.

1. Introduction

1.1. *Complete linear series*

Complete linear series on tropical curves have been recently object of intense study, and are relatively well understood thanks to Riemann-Roch type theorems and to combinatoric descriptions, see [5], [1], [2]. These linear series turn out to be polyhedral complexes which do not have pure dimension, and this fact has been an obstacle to define and study non-complete linear series. As far as we know, [6] and [3], which study linear series of plane curves (determined by imposing a singular point in the first case, three base points in the second) are the only works dealing with non-complete linear series so far. On projective tropical space, complete linear series are specially simple, as they consist of a single polytope, a simple quotient of a free tropical module. Thus it seems reasonable to consider sublinear series of these (as in [6] and [3]), and the simplest case is that of pencils of degree 2 on the projective line. In classical algebraic geometry, these are called g^1_2's on P^1, and they are very closely related to *linear involutions*, i.e., projectivities of P^1 onto itself whose square is the identity, and to (ramified) double covers of P^1 by itself. In this note we introduce tropical notions of g^1_2 and involutive linear maps, and show that both are closely related as in the algebraic case. Our description suggests that a double cover of the tropical line by itself could have a segment of "ramification points"; in the forthcoming work [7] we study involutions on the Berkovich projective line and show that its (ramified) double covers by itself have a whole (tropical) line of ramification points.

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1.2. *Setting*

Let $\mathsf{R}_{\text{trop}} = \mathsf{R} \cup \{-\infty\}$ be the tropical semifield, with operations $a \oplus b = \max(a, b)$ and $a \odot b = a + b$. Natural number exponents in tropical expressions should be interpreted tropically, i.e., $a^n = a \odot \cdots \odot a$. A *tropical polynomial* in the variables $x_1, \ldots, x_n$ is a formal expression $\bigoplus_\alpha a_\alpha \odot x^\alpha$ where $\alpha \in \mathsf{Z}^n_{\geq 0}$ is a multi-index, $x^\alpha = x_1^{\alpha_1} \odot \cdots \odot x_n^{\alpha_n}$ and $a_\alpha \in \mathsf{R}_{\text{trop}}$ for all α. Such a tropical polynomial defines, by interpreting sums and products tropically, a piecewise linear function with integer slopes

$$(x_1, \ldots, x_n) \mapsto \bigoplus_\alpha a_\alpha \odot x^\alpha = \max_\alpha \left(a_\alpha + \sum_{i=1}^n \alpha_i x_i \right),$$

whose corner locus, denoted $\mathscr{T}\left(\bigoplus_\alpha a_\alpha x^\alpha\right)$, is a *tropical affine hypersurface*.

Tropical projective n-space, TP^n, is the quotient of $\mathsf{R}^{n+1}_{\text{trop}} \setminus \{(-\infty, \ldots, -\infty)\}$ by the equivalence relation $(x_1, \ldots, x_n) \sim (y_1, \ldots, y_n)$ when $\exists t \in \mathsf{R}, x_i = y_i + t$ for all i.

We use the notation $(x_0 : \ldots : x_n)$ for the point of TP^n corresponding to $(x_0, \ldots, x_n) \in \mathsf{R}^n_{\text{trop}}$. A tropical polynomial P does not define a function on TP^n, but if it is homogeneous then its corner locus is well defined, and we also denote it $\mathscr{T}(P)$. A pair of homogeneous tropical polynomials of the same degree, not both equal to $-\infty$, define a rational function $(x_0 : \ldots : x_n) \mapsto (P(x_0, \ldots, x_n) : Q(x_0, \ldots, x_n))$ (a continuous piecewise linear map to the projective line). Since a polynomial different from $-\infty$ can take the value ∞ only at the borders of projective space, i.e., when at least one of the $x_i = -\infty$, rational function can have a nonempty indeterminacy locus (where $(P : Q)$ takes the non-existent value $(-\infty : -\infty)$) but it must be a subset of the border; in the case $n = 1$ such indeterminacies can always be resolved, extending the function by continuity.

As a set, TP^1 can be identified with $\mathsf{R} \cup \{-\infty, +\infty\}$ by mapping $(a : b)$ to $b - a$. We use the notation $p_- = (0 : -\infty)$ and $p_+ = (-\infty : 0)$ for the two ends, or points at infinity, of TP^1. Note that TP^1 is a totally ordered set, the order being $(a : b) \leq (c : d)$ when $b - a \leq d - c$. Given two points $p, q \in \mathsf{TP}^1$ we denote by $[p, q]$ the tropical convex hull [4] of the pair, which is just the segment with ends at p and q.

1.3. *A divisor*

A divisor on TP^1 is an element $\sum_{p \in \mathsf{TP}^1} a_p \cdot p$ of the free abelian group on its points, and it is *effective* when all coefficients a_p are non-negative. The degree of a divisor is the sum of the coefficients. The order of a rational function $f = (P : Q)$ at a point $p \in \mathsf{TP}^1$ is defined, following [5], as the sum of the outgoing slopes of f at p, and

$$(f) = \sum_{p \in \mathsf{TP}^1} \mathrm{ord}_p(f) \cdot p$$

is the principal divisor determined by (f). Two divisors are linearly equivalent if they differ by a principal divisor.

Given a divisor D, one considers the space $R(D)$ of all rational functions f such that $(f) + D$ is effective, and the complete linear series $|D| = \{(f) + D \mid f \in R(D)\}$. Actually, this machinery makes sense mostly for curves of higher genus, because in the case of a line $|D|$ is just the set of all effective divisors of the same degree, but it is worth keeping the $\mathsf{R}_{\mathrm{trop}}$-module and geometric structure of $R(D)$ and $|D|$ also in this case.

Thus, for any divisor D of degree 2, $|D| = |2 \cdot p_+|$, and a rational function such that $(f) + 2 \cdot p_+$ is effective can always be written as $f(x : y) = (x^2 : P(x, y))$, where $P(x, y) = \alpha_{20} \odot x^2 \oplus \alpha_{11} \odot x \odot y \oplus \alpha_{02} \odot y^2$ is a homogeneous tropical polynomial of degree 2. So the complete linear series of degree 2 is

$$|D| = \left\{ \mathscr{T}(\alpha_{20} \odot x^2 \oplus \alpha_{11} \odot x \odot y \oplus \alpha_{02} \odot y^2) \;\middle|\; \begin{array}{l} \alpha_{20}, \alpha_{11}, \alpha_{02} \in \mathsf{R}_{\mathrm{trop}} \\ \alpha_{20} \oplus \alpha_{11} \oplus \alpha_{02} \neq -\infty \end{array} \right\}.$$

Observe that this is not isomorphic to the tropical projective plane but to a quotient of it, because two tropical polynomials can have the same roots even if one is not a scalar multiple of the other; indeed, given α_{20}, α_{02}, for all $\alpha_{11} \le (\alpha_{20} + \alpha_{02})/2$, $\mathscr{T}(\alpha_{20} \odot x^2 \oplus \alpha_{11} \odot x \odot y \oplus \alpha_{02} \odot y^2) = (\alpha_{02}/2 : \alpha_{20}/2)$.

2. Tropical involutions

2.1. *Tropical matrices and projectivities*

In the set TM_2 of all 2×2 matrices with entries in $\mathsf{R}_{\mathrm{trop}}$, not all equal to $-\infty$, consider the equivalence relation

$$A \sim B \text{ when } \exists t \in \mathsf{R},\, A = B + \begin{pmatrix} t & t \\ t & t \end{pmatrix}.$$

Let PTM_2 be the set of matrices in TM_2 with no column equal to $\binom{-\infty}{-\infty}$ modulo $\sim$. Given $A \in \mathsf{TM}_2$ with no column equal to $\binom{-\infty}{-\infty}$, there is an associated map $\varphi_A : \mathsf{TP}^1 \to \mathsf{TP}^1$ given by multiplication on the left, and $\varphi_A = \varphi_B \Leftrightarrow A \sim B$.

We define the *sign* of a matrix as

$$\sigma \begin{pmatrix} a_{00} & a_{10} \\ a_{01} & a_{11} \end{pmatrix} = \mathrm{sign}(a_{00} + a_{11} - a_{10} - a_{01}) \in \{-1, 0, 1\}.$$

2.2. Main properties of the sign *Let $A, B \in \mathsf{TM}_2$ with no column equal to $\binom{-\infty}{-\infty}$.*

1. *If $A \sim B$ then $\sigma(A) = \sigma(B)$, i.e., the sign of a matrix in PTM_2 is well-defined.*
2. *$\sigma(A \odot B) \in \{0, \sigma(A)\sigma(B)\}$.*
3. *$\sigma(A) = 0 \Leftrightarrow \varphi_A$ is constant.*
4. *$\sigma(A) = 1 \Leftrightarrow \varphi_A$ is order-preserving, i.e., $p \geq q \Rightarrow \varphi_A(p) \geq \varphi_A(q)$.*
5. *$\sigma(A) = -1 \Leftrightarrow \varphi_A$ is order-reversing, i.e., $p \geq q \Rightarrow \varphi_A(p) \leq \varphi_A(q)$.*

The proof of 2.2 is elementary.

2.3. *Involutive matrices*

In PTM_2, consider the subset ITM_2 of those matrices with 2 equal values in the diagonal, i.e., of the form $J_{a,b,c} = \begin{pmatrix} b & a \\ c & b \end{pmatrix}$. These form the tropicalization of the locus of involutive matrices in PGL(2), and we call them *involutive tropical matrices*. Denote $\varphi_{a,b,c} = \varphi_{J_{a,b,c}}$.

2.4. Characterization *Let $A \in \mathsf{PTM}_2$, and let $F = \mathrm{Im}(\varphi_A) \subset \mathsf{TP}^1$. A is involutive if and only if F is not reduced to an end of TP^1 and the restriction of φ_A^2 to F is the identity map.*

Proof. By Develin-Sturmfels [4], the image of φ_A is the tropical convex hull of the two points determined by the columns of A, i.e., a point or a segment of positive length. Thus, $\varphi_{a,b,c}$ is surjective (bijective in fact) if and only if $b = -\infty$ or $a = c = -\infty$. Now for all $(x : y) \in \mathsf{TP}^1$, $\varphi_{-\infty,b,-\infty}(x : y) = (b \odot x : b \odot y) = (x : y)$, i.e., $\varphi_{-\infty,b,-\infty}$ is the identity map. On the other hand, $\varphi^2_{a,-\infty,c}(x : y) = \varphi_{a,-\infty,c}(a \odot y : c \odot x) = (a \odot c \odot x : a \odot c \odot y) = (x : y)$, so $\varphi^2_{a,-\infty,c}$ is the identity map, for all $a, c \in \mathsf{R}$.

It is not hard to check that $\varphi_{a,b,c}$, restricted to its image, is always a bijection, but we distinguish three different cases, depending on the sign of the matrix.

$\sigma(J_{a,b,c}) = 1$: This includes all cases when a or c is $-\infty$, i.e., when the image contains an end of TP^1. In this case, a computation as above shows that $\varphi_{a,b,c}$ restricted to $[(b : c), (a : b)]$ is the identity, whereas $[(0 : -\infty), (b : c)]$ maps identically to $(b : c)$ and $[(a : b), (-\infty : 0)]$ maps to $(a : b)$. Moreover $\varphi_{a,b,c}$ is idempotent, i.e., $\varphi^2_{a,b,c} = \varphi_{a,b,c}$. Topologically speaking $\varphi_{a,b,c}$ is the retraction of TP^1 to F.

$\sigma(J_{a,b,c}) = 0$: In this case $\varphi_{a,b,c}$ is constant and the image point is not an end of TP^1. Therefore, trivially, it is idempotent and restricted to its image it is the identity map.

$\sigma(J_{a,b,c}) = -1$: This includes all cases with $b = -\infty$; these are involutions and have been considered before, so assume $b \in \mathsf{R}$. In this case $a, c \in \mathsf{R}$, and one checks that $\varphi_{a,b,c}$ restricted to $[(a : b), (b : c)]$ is the unique orientation-reversing isometry, whereas $[(0 : -\infty), (a : b)]$ maps identically to $(b : c)$ and $[(b : c), (-\infty : 0)]$ maps to $(a : b)$. Moreover $\varphi_{a,b,c}^2 = \varphi_{-a,-b,-c}$ is the retraction of TP^1 to F.

Conversely, let $A = \begin{pmatrix} a_{00} & a_{10} \\ a_{01} & a_{11} \end{pmatrix}$ be a matrix such that the restriction of φ_A^2 to $F = \mathrm{Im}(\varphi_A)$ is the identity map. Write $A^2 = \begin{pmatrix} b_{00} & b_{10} \\ b_{01} & b_{11} \end{pmatrix}$. If

$$A^2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_{00} \odot x \oplus b_{10} \odot y \\ b_{01} \odot x \oplus b_{11} \odot y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

in a segment of positive length, we must have $b_{00} \odot x \oplus b_{10} \odot y = b_{00} \odot x$ (i.e., $y - x < b_{00} - b_{10}$) and $b_{01} \odot x \oplus b_{11} \odot y = b_{11} \odot y$ (i.e., $y - x > b_{01} - b_{11}$) in that segment, and $b_{00} = b_{11}$. So we can assume without loss of generality that $b_{00} = b_{11} = 0$ and $b_{10} + b_{01} < 0$, i.e., $A^2 = J_{b_{10},0,b_{01}}$. The equation

$$A^2 = \begin{pmatrix} a_{00}^2 \oplus a_{10} \odot a_{01} & a_{10} \odot (a_{00} \oplus a_{11}) \\ a_{01} \odot (a_{00} \oplus a_{11}) & a_{10} \odot a_{01} \oplus a_{11}^2 \end{pmatrix} = \begin{pmatrix} 0 & b_{10} \\ b_{01} & 0 \end{pmatrix}$$

has two types of solutions, depending on whether $0 = 2a_{00} > a_{10} + a_{01}$ or $2a_{00} \le a_{10} + a_{01} = 0$. If $0 = 2a_{00} > a_{10} + a_{01}$ then one is forced to have $2a_{11} = 0$ and so $A = A^2 = J_{a_{10},0,a_{01}}$ is one of the retractions above, $\sigma(A) = 1$.

All other solutions have $a_{10} + a_{01} = 0$ and $a_{00}, a_{11} \le 0$. The condition that φ_A^2 is the identity on F implies that each column of A determines a point in the segment $[(0 : b_{01}), (b_{10} : 0)]$ where φ_A^2 is the identity, therefore

$$\begin{aligned} a_{01} + \max(a_{00}, a_{11}) &= a_{01} \odot (a_{00} \oplus a_{11}) = b_{01} - 0 \le a_{11} - a_{10}, \\ -a_{10} - \max(a_{00}, a_{11}) &= -(a_{10} \odot (a_{00} \oplus a_{11})) = 0 - b_{10} \le a_{01} - a_{00}, \end{aligned}$$

so $\max(a_{00}, a_{11}) \le a_{11} - a_{01} - a_{10} = a_{11}$ and $\max(a_{00}, a_{11}) \le a_{00} - a_{01} - a_{10} = a_{00}$. Thus $a_{00} = a_{11} \le a_{10} + a_{01}$, which shows A is involutive, and either constant or of negative sign.

3. Tropical pencils

3.1. *A* g_2^1 *on* TP^1

A g_2^1 on TP^1 must be a linear subseries of the complete linear series of degree 2 which, as explained in 1.3, is

$$|2 \cdot p_+| = \{\mathscr{T}(\alpha_{20} \odot x^2 \oplus \alpha_{11} \odot x \odot y \oplus \alpha_{02} \odot y^2) \mid \alpha_{20}, \alpha_{11}, \alpha_{02} \in \mathsf{R}_{\mathrm{trop}}\},$$

Since a tropical line in a plane is given by a tropical linear equation, we define g^1_2's as determined by a tropical linear equation in the coefficients $\alpha_{20}, \alpha_{11}, \alpha_{02} \in \mathsf{R}_{\text{trop}}$. Thus, if $L_{a,b,c}$ denotes the tropical line defined as the corner locus of the tropical polynomial function $a \odot \alpha_{20} \oplus b \odot \alpha_{11} \oplus c \odot \alpha_{02}$, the corresponding g^1_2 is

$$g_{a,b,c} = \{\mathscr{T}(\alpha_{20} \odot x^2 \oplus \alpha_{11} \odot x \odot y \oplus \alpha_{02} \odot y^2) \mid (\alpha_{20} : \alpha_{11} : \alpha_{02}) \in L_{a,b,c}\} \subset |2 \cdot p_+|.$$

3.2. Correspondence pencils $\leftrightarrow$ involutions *Let $a, b, c \in \mathsf{R}_{\text{trop}}$, with $a \oplus b \neq -\infty \neq b \oplus c$. Then the set $\{p + \varphi_{a,b,c}(p)\}_{p \in \mathsf{TP}^1} \subset |2p_+|$ is equal to $g_{a,b,c}$.*

Proof. The proof is slightly different depending on the sign of the involutive matrix $J_{a,b,c}$. Let us prove the statement for $\sigma(J_{a,b,c}) \geq 0$, and leave the negative case to the reader. Let F be the image of $\varphi_{a,b,c}$; its end points are $(b : c)$ and $(a : b)$, and its mid point is $(a/2 : c/2)$.

We first prove that $\{p + \varphi_{a,b,c}(p)\}_{p \in \mathsf{TP}^1} \subset g_{a,b,c}$. Distinguish four cases, depending on the position of the point $p \in \mathsf{TP}^1$ relative to these three points.

$p \leq (b : c)$: Put $p = (b : t)$ with $t \leq c$, then $p + \varphi_{a,b,c}(p) = (b : t) + (b : c) = \mathscr{T}(c \odot t \odot x^2 \oplus b \odot c \odot x \odot y \oplus b^2 \odot y^2)$, and $(c \odot t : b \odot c : b^2) \in L_{a,b,c}$ because $a \odot c \odot t \leq b \odot b \odot c = c \odot b^2$.

$(b : c) \leq p \leq (a/2 : c/2)$: Put $p = (b : t)$ with $c \leq t \leq b + (c - a)/2$, then $p + \varphi_{a,b,c}(p) = 2(b : t) = \mathscr{T}(t^2 \odot x^2 \oplus b \odot c \odot x \odot y \oplus b^2 \odot y^2)$, and $(t^2 : b \odot c : b^2) \in L_{a,b,c}$ because $a \odot t^2 \leq b \odot b \odot c = c \odot b^2$.

$(a/2 : c/2) \leq p \leq (a : b)$: Put $p = (t : b)$ with $a \leq t \leq b + (a - c)/2$, then $p + \varphi_{a,b,c}(p) = 2(t : b) = \mathscr{T}(b^2 \odot x^2 \oplus a \odot b \odot x \odot y \oplus t^2 \odot y^2)$, and $(b^2 : a \odot b : t^2) \in L_{a,b,c}$ because $a \odot b^2 = b \odot a \odot b \geq c \odot t^2$.

$p \geq (a : b)$: Put $p = (t : b)$ with $t \leq a$, then $p + \varphi_{a,b,c}(p) = \{(a : b), (t : b)\} = \mathscr{T}(b^2 \odot x^2 \oplus a \odot b \odot x \odot y \oplus a \odot t \odot y^2)$, and $(b^2 : a \odot b : a \odot t) \in L_{a,b,c}$ because $a \odot b^2 = b \odot a \odot b \geq c \odot a \odot t$.

Along the way we have shown that equations parameterized by two of the three rays in $L_{a,b,c}$ (namely, $a + \alpha_{20} \leq b + \alpha_{11} = c + \alpha_{02}$ and $a + \alpha_{20} = b + \alpha_{11} \geq c + \alpha_{02}$) do give pairs of points $p + \varphi_{a,b,c}(p)$. The third ray, with equations $a + \alpha_{20} = c + \alpha_{02} \geq b + \alpha_{11}$ corresponds to the polynomials $c \odot x^2 \oplus t \odot x \odot y \oplus a \odot y^2$ with $t \leq (a + c)/2$, whose unique double root is the mid point $(a/2 : c/2) \in F$, fixed by $\varphi_{a,b,c}$, so we are done.

4. Tropicalization

4.1. *Tropicalization of involutions*

Next we check that all tropical involutive maps (and g_2^1's) as defined above arise as tropicalizations of algebraic involutions. Let K be a valued field, with value group $G \subset \mathsf{R}$, and denote the valuation by v. Assume furthermore that $v(2) = 0$. Given $a, b, c \in G \cup \{-\infty\}$, with $a \oplus b \neq -\infty \neq b \oplus c$, choose $\alpha, \beta, \gamma \in K$ such that $v(\alpha) = -a$, $v(\beta) = -b$, $v(\gamma) = -c$. Since there are fields with value grup equal to R, the restriction to G is not actually relevant; rather, it is considered here only for completeness.

Consider the matrix $A = \begin{pmatrix} \beta & \alpha \\ \gamma & -\beta \end{pmatrix}$; if it turns out to be singular (which can indeed happen if $a + c = 2b$) replace α by some $\alpha + x$ with $x \neq 0$, $v(x) > a$, which certainly exist. Since $A^2 = \begin{pmatrix} \beta^2+\alpha\gamma & 0 \\ 0 & \beta^2+\alpha\gamma \end{pmatrix}$, A is the matrix of an involution in P^2_K, and its tropicalization is the involutive tropical matrix $\begin{pmatrix} b & a \\ c & b \end{pmatrix}$.

It is clear that the g_2^1 on P^1_K corresponding to the involution with matrix A tropicalizes to the tropical g_2^1 corresponding to the tropicalized matrix. It is more illustrative to pay some attention to the fixed points of the involution.

4.2. Tropicalization of fixed points *Let $\bar{K}$ be an algebraic closure of K, and let $p, q \in \mathsf{P}^1_{\bar{K}}$ be the fixed points of the involution $\varphi_A : \mathsf{P}^1_{\bar{K}}$ determined by $A = \begin{pmatrix} \beta & \alpha \\ \gamma & -\beta \end{pmatrix}$. Let $A^t = \begin{pmatrix} b & a \\ c & b \end{pmatrix} = -v(A)$ be the tropicalization of A. Then $\sigma(A^t) = 1$ if and only if $v(p) \neq v(q)$, and in this case the ends of the segment fixed by the tropical involutive map are $-v(p)$ and $-v(q)$.*

Proof. $p = (x : y)$ and $q = (x' : y')$ are the zeros of the homogeneous polynomial $\gamma x^2 - 2\beta xy - \alpha y^2$, so their tropicalizations $-v(p)$, $-v(q)$ are the tropical roots of the tropicalization $F = c \odot x^2 \oplus b \odot x \odot y \oplus a \odot y^2$. These are distinct if and only if $2b \geq c + a$, i.e., if and only if $\sigma(A^t) = 1$. In such a case, the columns of A^t are exactly the tropical roots of F, and they are also the ends of the segment fixed by the tropical involutive map.

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