

ON AREA STATIONARY SURFACES IN THE SPACE OF ORIENTED GEODESICS OF HYPERBOLIC 3-SPACE

NIKOS GEORGIU*

Abstract

We study area-stationary surfaces in the space $L(\mathbb{H}^3)$ of oriented geodesics of hyperbolic 3-space, endowed with the canonical neutral Kähler structure. We prove that every holomorphic curve in $L(\mathbb{H}^3)$ is an area-stationary surface. We then classify Lagrangian area-stationary surfaces Σ in $L(\mathbb{H}^3)$ and prove that the family of parallel surfaces in \mathbb{H}^3 orthogonal to the geodesics $\gamma \in \Sigma$ form a family of equidistant tubes around a geodesic. Finally we find an example of a two parameter family of rotationally symmetric area-stationary surfaces that are neither Lagrangian nor holomorphic.

The last two decades has seen increasing interest in spaces $L(M)$ of oriented geodesics of a manifold M , with particular attention to their rich geometric structure. In the case of the space $L(\mathbb{E}^3)$ of oriented affine lines in Euclidean 3-space this interest can be traced back over a hundred years to Weierstrass's construction of minimal surfaces [16] and Whittaker's solutions to the wave equation [17].

A natural complex structure on $L(\mathbb{E}^3)$ was considered by Hitchin to construct monopoles in \mathbb{E}^3 [11], and then Guilfoyle and Klingenberg understood that the canonical symplectic structure on $L(\mathbb{E}^3)$ is compatible with this complex structure [7], [8] and that the associated Kähler metric is of neutral signature. Salvai subsequently proved that this neutral Kähler metric is (up to addition of the round metric) the unique metric on $L(\mathbb{E}^3)$ that is invariant under Euclidean motions [12]. This Kähler structure has recently been used by Guilfoyle and Klingenberg to solve an 80 year old conjecture of Carathéodory [10].

More recently, Anciaux, Guilfoyle and Romon [2] have studied Lagrangian area-stationary surfaces in TN , with N being an oriented Riemannian surface and the neutral Kähler structure generalising that of the space of oriented geodesics in Euclidean and Lorentzian 3-space.

In addition, Salvai constructed a neutral Kähler metric on the space $L(\mathbb{H}^3)$ of oriented geodesics in hyperbolic 3-space [13], while the geometry of $L(\mathbb{H}^3)$

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was explored by Georgiou, Guilfoyle and Klingenberg [4], [5], [6].

Recently Alekseevsky, Guilfoyle and Klingenberg have given a description of all metrics defined on the space $L(M)$ of oriented geodesics of space forms or rank 1 Riemannian symmetric spaces, that are invariant under the isometry group of M [1].

The current paper can be considered as a continuation of the work of Georgiou and Guilfoyle (see [6]) on surface theory of $L(H^3)$. Here we study immersed area stationary surfaces, that is, immersed surfaces that area stationary for the area functional.

Since $L(H^3) = S^2 \times S^2 - \{\text{diag}\}$, we can consider surfaces in $L(H^3)$ of rank 0, 1 or 2, depending on the rank of the projection onto the first factor.

In the first two sections we illustrate the geometric background on the construction of $L(H^3)$ and its submanifold theory, including geodesics and surfaces (further details can be found in [4], [5], [6]).

In section 3 we investigate rank 1 surfaces in $L(H^3)$ and prove that there are no holomorphic curves of rank 1. Moreover, in the case of a surface being Lagrangian of rank 1, we prove that it can not be area stationary.

Surfaces of rank 2 are studied in section 4 and, in particular, we prove that every holomorphic curve of rank 2 is area stationary and, since the only holomorphic curve of rank 0 are orthogonal to a horosphere, we conclude that every holomorphic curve is area stationary.

We also classify all Lagrangian area stationary surfaces. We summarize this result as:

MAIN THEOREM. *Let $S \subset H^3$ be a C^3 smooth immersed oriented surface and $\Sigma \subset L(H^3)$ be the Lagrangian surface formed by the oriented geodesics normal to S .*

The surface Σ is area stationary iff S is an equidistant tube around a geodesic in H^3 . In terms of holomorphic coordinates (μ_1, μ_2) on $L(H^3)$, the surface Σ is given by

$$\bar{\mu}_2 = \frac{1 + \lambda_1 \mu_1}{\lambda_2 + \mu_1},$$

where $\lambda_1, \lambda_2 \in \mathbb{C}$ with $\lambda_1 \lambda_2 \neq 1$.

Finally, in section 5 we obtain a two parameter family of area stationary rotationally symmetric surfaces that are neither Lagrangian nor holomorphic.

1. The Neutral Kähler Metric on $L(H^3)$

We now describe the construction of the canonical Kähler metric on the space $L(H^3)$ of oriented geodesics in Hyperbolic 3-space – further details can be found in [4].

Consider the four dimensional manifold $S^2 \times S^2$ endowed with the canonical complex structure $J = j \oplus j$. Here, and throughout, we write \mathbf{P}^1 for S^2 with the standard complex structure. By taking local holomorphic coordinates (μ_1, μ_2) , we set $\bar{\Delta} = \{(\mu_1, \mu_2) \in \mathbf{P}^1 \times \mathbf{P}^1 \mid \mu_1 \bar{\mu}_2 = -1\}$. Then $L(\mathbf{H}^3)$ can be identified with $\mathbf{P}^1 \times \mathbf{P}^1 - \bar{\Delta}$.

The Kähler metric G and the symplectic form Ω on $L(\mathbf{H}^3)$ are expressed locally by:

$$G = \text{Im} \left(\frac{2}{(1 + \mu_1 \bar{\mu}_2)^2} d\mu_1 \otimes d\bar{\mu}_2 \right), \quad \Omega = -\text{Re} \left(\frac{2}{(1 + \mu_1 \bar{\mu}_2)^2} d\mu_1 \wedge d\bar{\mu}_2 \right).$$

It has been proven that the Kähler metric G is scalar flat, conformally flat and is of signature $(++--)$.

Moreover, G is invariant under the action induced on $L(\mathbf{H}^3)$ by the isometry group of \mathbf{H}^3 . Indeed, this has been shown to be the unique Kähler metric on $L(\mathbf{H}^3)$ with this property [13].

In order to transfer geometric data between $L(\mathbf{H}^3)$ and \mathbf{H}^3 we use the correspondence space:

$$\begin{array}{ccc} L(\mathbf{H}^3) \times \mathbf{R} & & \\ \pi_1 \downarrow & \searrow \Phi & \\ L(\mathbf{H}^3) & & \mathbf{H}^3 \end{array}$$

Given $\gamma \in L(\mathbf{H}^3)$, the set $\Phi \circ \pi_1^{-1}(\gamma)$ is the oriented geodesic in \mathbf{H}^3 , while, for a point $p \in \mathbf{H}^3$, $\pi_1 \circ \Phi^{-1}(p)$ is the set of oriented geodesics in $L(\mathbf{H}^3)$ that pass through p .

The map Φ takes an oriented geodesic γ in $L(\mathbf{H}^3)$ and a real number r to the point on γ an affine parameter distance r from some fixed point on the geodesic. This choice of point on each geodesic can be made globally, but we more often just use a local choice, which is sufficient for our purposes.

Let (x_0, x_1, x_2) be the local coordinates of the upper-half space model of \mathbf{H}^3 . Then, in terms of holomorphic coordinates (μ_1, μ_2) on $L(\mathbf{H}^3)$ the map Φ has expression:

$$(1.1) \quad z = \frac{1 - \mu_1 \bar{\mu}_2}{2\bar{\mu}_2} + \left(\frac{1 + \mu_1 \bar{\mu}_2}{2\bar{\mu}_2} \right) \tanh r, \quad t = \frac{|1 + \bar{\mu}_1 \mu_2|}{2|\mu_2| \cosh r},$$

where $z = x_1 + ix_2$ and $t = x_0$.

2. Surfaces in $L(\mathbf{H}^3)$

Our interest in this paper is focused on the study of two parameter families of oriented geodesics, or surfaces in $L(\mathbf{H}^3)$. Therefore, we recall some basic

results on the surface theory of the space of oriented geodesics in hyperbolic 3-space – further details can be found in [6].

In order to avoid any confusion, whether we use the expression of *geodesic congruence* we mean a surface in $L(\mathbb{H}^3)$.

For computational purposes, we give explicit local parameterizations of the geodesic congruence. The dual picture of a surface in $L(\mathbb{H}^3)$ is to consider the surfaces in \mathbb{H}^3 that are orthogonal to a given set of geodesics. However, not every geodesic congruence has such orthogonal surfaces – indeed, most don't. To explain this further, we consider the first order properties of Σ , which can be described by two complex functions, the optical scalars: $\rho, \sigma : \Sigma \times \mathbb{R} \mapsto \mathbb{C}$. The real part Θ and imaginary part λ of ρ are the *divergence* and *twist* of the geodesic congruence, while σ is the *shear*.

Given a surface $\Sigma \subset L(\mathbb{H}^3)$, a trio $\{e_0, e_+, e_-\}$ of complex vector fields in $\mathbb{C} \otimes T\mathbb{H}^3$ is said to be an *adapted null frame* if for each $\gamma \in \Sigma$, we have $e_0 = \dot{\gamma}$ and e_+ is the complex conjugate of e_- such that the orientation of $\{e_0, \text{Re}(e_+), \text{Im}(e_+)\}$ is the standard orientation on \mathbb{H}^3 , and:

$$\langle e_0, e_0 \rangle = \langle e_+, e_- \rangle = 1, \quad \langle e_0, e_+ \rangle = \langle e_+, e_+ \rangle = 0.$$

The optical scalars are defined by:

$$\rho = \langle \nabla_{e_-} e_+, e_0 \rangle, \quad \sigma = \langle \nabla_{e_+} e_+, e_0 \rangle,$$

where ∇ denotes the Levi-Civita connection of the hyperbolic metric.

In terms of the holomorphic coordinates (μ_1, μ_2) , the optical scalars have the following local expressions:

$$\sigma = \frac{8\mu_2 J_{2\bar{1}}}{\bar{\mu}_2 \Delta |1 + \mu_1 \bar{\mu}_2|^2}$$

$$\rho = -1 - \frac{8e^{-r}}{\Delta} \left[\frac{J_{2\bar{1}}}{(1 + \bar{\mu}_1 \mu_2)^2} e^r - \frac{|\mu_2|^2 J_{1\bar{1}}}{|1 + \bar{\mu}_1 \mu_2|^2} e^{-r} \right],$$

where

$$J_{kl} = \partial \mu_k \bar{\partial} \mu_l - \bar{\partial} \mu_k \partial \mu_l,$$

and

$$\frac{1}{4} \Delta = \frac{J_{2\bar{2}}}{|\mu_2|^2 |1 + \mu_1 \bar{\mu}_2|^2} e^{2r} + \frac{J_{\bar{2}1}}{(1 + \mu_1 \bar{\mu}_2)^2}$$

$$+ \frac{J_{\bar{1}2}}{(1 + \bar{\mu}_1 \mu_2)^2} + \frac{|\mu_2|^2 J_{1\bar{1}}}{|1 + \mu_1 \bar{\mu}_2|^2} e^{-2r}.$$

A surface Σ in $L(\mathbb{H}^3)$, is said to be *Lagrangian* if the symplectic form Ω pulled back to Σ vanishes. We now give the following Proposition:

PROPOSITION 1 ([6]). *The following statements are equivalent:*

- (i) *the geodesic congruence Σ is Lagrangian,*
- (ii) *locally there exists an embedded surface S in \mathbb{H}^3 such that the geodesics of Σ are normal to S ,*
- (iii) *the imaginary part λ of ρ (the twist) is zero.*

Let Σ be a Lagrangian surface in $L(\mathbb{H}^3)$ parameterized by $v \mapsto (\mu_1(v, \bar{v}), \mu_2(v, \bar{v}))$. The surfaces S in \mathbb{H}^3 orthogonal to the geodesics of Σ are given by (1.1), where the functions $r = r(v, \bar{v})$ solve:

$$(2.1) \quad 2\partial r = \frac{\mu_2}{\bar{\mu}_1\mu_2 + 1} \left(\partial\bar{\mu}_1 + \frac{\partial\mu_2}{\mu_2^2} \right) + \frac{\bar{\mu}_2}{\mu_1\bar{\mu}_2 + 1} \left(\partial\mu_1 + \frac{\partial\bar{\mu}_2}{\bar{\mu}_2^2} \right),$$

where ∂ denotes the derivative with respect of v .

The set of Lagrangian geodesic congruences is divided into three categories, depending on the rank of the immersion of the geodesic congruence.

DEFINITION 1. Given an immersion $f : \Sigma \rightarrow L(\mathbb{H}^3)$, consider the map $(\pi \circ f)_* : T\Sigma \rightarrow T\mathbb{P}^1$, where π is projection onto the first factor of $L(\mathbb{H}^3) = \mathbb{P}^1 \times \mathbb{P}^1 - \bar{\Delta}$. The *rank* of the immersion f at a point $\gamma \in \Sigma$ is defined to be the rank of this map at γ , which can be 0, 1 or 2.

Note that by reversing the orientation of the geodesics, the rank can be defined by projection onto the second factor. A rank 0 Lagrangian geodesic congruence correspond to a geodesic congruence orthogonal to a horosphere.

In the Lagrangian case, the functions σ and ρ have the following interpretation in terms of the second fundamental form of the orthogonal surfaces in \mathbb{H}^3 .

PROPOSITION 2 ([6]). *Let $S \subset \mathbb{H}^3$ be a C^2 immersed surface and $\Sigma \subset L(\mathbb{H}^3)$ be the oriented normal geodesic congruence. Then*

$$(2.2) \quad |\sigma| = \frac{1}{2}|\lambda_1 - \lambda_2|, \quad \rho = -\frac{1}{2}(\lambda_1 + \lambda_2),$$

where λ_1 and λ_2 are the principal curvatures of S .

The induced metric G_Σ on a Lagrangian surface Σ in $L(\mathbb{H}^3)$ can be described by the functions σ and λ :

THEOREM 1 ([6]). *Let Σ be a surface in $L(\mathbb{H}^3)$. The induced metric is Lorentz (degenerate, Riemannian) iff $|\sigma|^2 - \lambda^2 > 0$ ($= 0, < 0$), where λ and σ are the twist and the shear of Σ .*

The theorem says that if a surface is Lagrangian then is either Lorentz or degenerate, the latter occurring at umbilic points on the orthogonal surfaces in \mathbb{H}^3 .

The following theorem recovers the Weierstrass representation for the flat surfaces in \mathbb{H}^3 :

THEOREM 2 ([6]). *Let S be an oriented C^2 smooth immersed surface in \mathbb{H}^3 with normal geodesic congruence Σ . Assume that the metric G_Σ induced on Σ by the neutral Kähler metric is non-degenerate.*

Then S is flat iff Σ is of rank two and is parameterized by $\mu_2 = \mu_2(\bar{\mu}_1)$, that is, μ_2 is an anti-holomorphic function of μ_1 .

We use the complex structure J of $L(\mathbb{H}^3)$ in order to describe another important class of surfaces:

DEFINITION 2. A point γ on a surface $\Sigma \subset L(\mathbb{H}^3)$ is said to be a *complex point* if the complex structure J acting on $L(\mathbb{H}^3)$ preserves $T_\gamma \Sigma$. A surface $\Sigma \subset L(\mathbb{H}^3)$ is said to be a *holomorphic curve* if all of the points of Σ are complex points.

In particular:

PROPOSITION 3 ([6]). *A point γ on a surface Σ is complex iff the shear vanishes along γ .*

By Proposition 2, observe that complex points on a Lagrangian surface $\Sigma \subset L(\mathbb{H}^3)$ correspond to umbilic points on the surfaces in \mathbb{H}^3 orthogonal to Σ .

3. Non-existence of rank one area stationary surfaces

Consider a surface $\Sigma \subset L(\mathbb{H}^3)$ of rank one. Then Σ can be locally parameterized by $\mu_1 = \mu_1(s)$ and $\mu_2 = \mu_2(s, t)$ where $(s, t) \in D$ with D being an open subset of \mathbb{R}^2 .

We have the following:

PROPOSITION 4. *An immersed holomorphic curve in $L(\mathbb{H}^3)$ cannot be of rank 1.*

PROOF. Assume the existence of an immersed rank one surface on $L(\mathbb{H}^3)$ such that in an open neighborhood $U \subset \Sigma$ is holomorphic, which, by Proposition 3, is equivalent to the vanishing of the shear σ in this open set. Considering now the local parametrisation $\Sigma \rightarrow L(\mathbb{H}^3) : (s, t) \mapsto (\mu_1(s), \mu_2(s, t))$, the vanishing of the shear implies the vanishing of $J_{\bar{1}} \bar{1}$, which means $\partial_s \bar{\mu}_1 \partial_t \bar{\mu}_2 = 0$ on U .

Then, in an open subset V of U , either $\partial_s \bar{\mu}_1 = 0$ or $\partial_t \bar{\mu}_2 = 0$. In any case, we have a contradiction since, if $\partial_s \bar{\mu}_1 = 0$ then μ_1 is constant and therefore Σ is of rank 0 on V , and if $\partial_t \bar{\mu}_2 = 0$ then μ_2 only depends on s and Σ would not be immersed.

We now assume that the rank 1 surface Σ is Lagrangian. In this case the induced metric $g = f^*G$ has components in local coordinates (s, t) :

$$g_{ss} = 2 \operatorname{Im} \left[\frac{\partial_s \mu_1 \partial_s \bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^2} \right], \quad g_{st} = \operatorname{Im} \left[\frac{\partial_s \mu_1 \partial_t \bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^2} \right], \quad g_{tt} = 0,$$

and the nonzero Christoffel symbols are given by:

$$\begin{aligned} \Gamma_{ss}^s &= \operatorname{Re} \left(\frac{\partial_s^2 \mu_1}{\partial_s \mu_1} - \frac{2\bar{\mu}_2 \partial_s \mu_1}{1 + \mu_1 \bar{\mu}_2} \right), \\ \Gamma_{st}^t &= \operatorname{Re} \left(\frac{\partial_{st}^2 \mu_2}{\partial_t \mu_2} - \frac{2\bar{\mu}_1 \partial_s \mu_2}{1 + \bar{\mu}_1 \mu_2} \right), \\ \Gamma_{tt}^t &= \operatorname{Re} \left(\frac{\partial_t^2 \mu_2}{\partial_t \mu_2} - \frac{2\bar{\mu}_1 \partial_t \mu_2}{1 + \bar{\mu}_1 \mu_2} \right). \end{aligned}$$

It is already known that the induced metric g of a rank one Lagrangian surface Σ is scalar flat [6].

The second fundamental form $h = h_{ij}^{\mu_k}$ has non-vanishing components:

$$\begin{aligned} h_{ss}^{\mu_1} &= \partial_s^2 \mu_1 - \frac{2\bar{\mu}_2 (\partial_s \mu_1)^2}{1 + \mu_1 \bar{\mu}_2} - \partial_s \mu_1 \Gamma_{ss}^s, \\ h_{ss}^{\mu_2} &= \partial_s^2 \mu_2 - \frac{2\bar{\mu}_1 (\partial_s \mu_2)^2}{1 + \bar{\mu}_1 \mu_2} - \partial_s \mu_2 \Gamma_{ss}^s - \partial_t \mu_2 \Gamma_{ss}^t, \\ h_{st}^{\mu_2} &= \partial_{st}^2 \mu_2 - \frac{2\bar{\mu}_1 \partial_s \mu_2 \partial_t \mu_2}{1 + \bar{\mu}_1 \mu_2} - \partial_t \mu_2 \Gamma_{st}^t, \\ h_{tt}^{\mu_2} &= \partial_t^2 \mu_2 - \frac{2\bar{\mu}_1 (\partial_t \mu_2)^2}{1 + \bar{\mu}_1 \mu_2} - \partial_t \mu_2 \Gamma_{tt}^t, \end{aligned}$$

with $h_{ij}^{\bar{\mu}_k} = \overline{h_{ij}^{\mu_k}}$.

In contrast to what occurs in the space $L(E^3)$ of oriented lines in Euclidean 3-space [9], we have the following:

PROPOSITION 5. *There are no area stationary Lagrangian surfaces in $L(H^3)$ of rank one.*

PROOF. Let Σ be a Lagrangian surface in $L(\mathbb{H}^3)$ of rank one, locally parameterized by $\mu_1 = \mu_1(s)$ and $\mu_2 = \mu_2(s, t)$.

The mean curvature vector $H = 2 \operatorname{Re}(H^{\mu_1} \partial / \partial \mu_1 + H^{\mu_2} \partial / \partial \mu_2)$ in local coordinates (s, t) is given by:

$$H^{\mu_i} = g^{ss} h_{ss}^{\mu_i} + 2g^{st} h_{st}^{\mu_i} + g^{tt} h_{tt}^{\mu_i}$$

Then $H^{\mu_1} = 0$ and it remains to find H^{μ_2} . By using the expressions of $h_{ij}^{\mu_i}$ and by considering the Lagrangian condition:

$$(3.1) \quad \frac{\partial_s \mu_1 \partial_t \bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^2} = - \frac{\partial_s \bar{\mu}_1 \partial_t \mu_2}{(1 + \bar{\mu}_1 \mu_2)^2},$$

we find H^{μ_2} , and finally the mean curvature vector H of Σ is:

$$H = 4 \operatorname{Re} \left[g^{st} \frac{(1 + \bar{\mu}_1 \mu_2)^2}{\partial_s \bar{\mu}_1} \partial_t \left(\frac{\partial_s \mu_1 \partial_s \bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^2} + \frac{\partial_s \bar{\mu}_1 \partial_s \mu_2}{(1 + \bar{\mu}_1 \mu_2)^2} \right) \frac{\partial}{\partial \mu_2} \right],$$

which means that the surface Σ is area stationary iff

$$\partial_t \left[\operatorname{Re} \left(\frac{\partial_s \mu_1 \partial_s \bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^2} \right) \right] = 0.$$

The above condition and the Lagrangian condition give:

$$(3.2) \quad A \partial_s \bar{\mu}_2 + \bar{A} \partial_s \mu_2 = f(s),$$

$$(3.3) \quad A \partial_t \bar{\mu}_2 + \bar{A} \partial_t \mu_2 = 0,$$

where

$$A = \frac{\partial_s \mu_1}{(1 + \mu_1 \bar{\mu}_2)^2}.$$

Differentiate equations (3.2) and (3.6) with respect to t and s , respectively, and then subtract:

$$(3.4) \quad \operatorname{Re}(\partial_t A \partial_s \bar{\mu}_2 - \partial_s \bar{A} \partial_t \bar{\mu}_2) = 0.$$

After a brief computation we get:

$$\begin{aligned} \partial_s A \partial_t \bar{\mu}_2 &= \frac{\partial_s^2 \mu_1 \partial_t \bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^2} - \frac{2\mu_1 \partial_s \mu_1 \partial_s \bar{\mu}_2 \partial_t \bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^3} - \frac{2\bar{\mu}_2 (\partial_s \mu_1)^2 \partial_t \bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^3}, \\ \partial_t A \partial_s \bar{\mu}_2 &= - \frac{2\mu_1 \partial_s \mu_1 \partial_s \bar{\mu}_2 \partial_t \bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^3}, \end{aligned}$$

and then condition (3.4) becomes

$$(3.5) \quad \operatorname{Re} \left(\frac{2\bar{\mu}_2(\partial_s \mu_1)^2 \partial_t \bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^3} - \frac{\partial_s^2 \mu_1 \partial_t \bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^2} \right) = 0.$$

Using the Lagrangian condition (3.1) in (3.5), we have

$$(3.6) \quad \frac{\bar{\mu}_2 \partial_s \mu_1}{1 + \mu_1 \bar{\mu}_2} - \frac{\mu_2 \partial_s \bar{\mu}_1}{1 + \bar{\mu}_1 \mu_2} = \frac{1}{2} \left(\frac{\partial_s^2 \mu_1}{\partial_s \mu_1} - \frac{\partial_s^2 \bar{\mu}_1}{\partial_s \bar{\mu}_1} \right) = h(s).$$

Integration of (3.1) with respect of t gives

$$(3.7) \quad \frac{\bar{\mu}_2 \partial_s \mu_1}{1 + \mu_1 \bar{\mu}_2} + \frac{\mu_2 \partial_s \bar{\mu}_1}{1 + \bar{\mu}_1 \mu_2} = g(s),$$

and then the sum (3.6) + (3.7), is

$$\frac{\bar{\mu}_2 \partial_s \mu_1}{1 + \mu_1 \bar{\mu}_2} = h(s) + g(s) = m(s) \partial_s \mu_1.$$

Hence

$$\bar{\mu}_2 = \frac{m}{1 - m \mu_1} = \bar{\mu}_2(s),$$

which is a contradiction, since Σ is of rank one.

4. Rank two area stationary surfaces

Consider a rank 2 surface Σ in $L(\mathbb{H}^3)$. That is, a surface Σ given locally by $\mu_1 \rightarrow (\mu_1, \mu_2(\mu_1, \bar{\mu}_1))$ for some smooth function $\mu_2 : \mathbb{C} \rightarrow \mathbb{C}$. We are interested in area stationary surfaces in $L(\mathbb{H}^3)$ of rank 2 and therefore we consider variations of the area integral

$$\mathcal{A}(\Sigma) = \int_{\Sigma} |\mathbf{G}|^{\frac{1}{2}} d\mu_1 d\bar{\mu}_1.$$

For an arbitrary parameterization $\mu_1 \rightarrow (\mu_1, \mu_2(\mu_1, \bar{\mu}_1))$ the area integral is

$$|\mathbf{G}| = \frac{\Delta^2}{64} (\lambda^2 - |\sigma|^2),$$

where

$$(4.1) \quad \lambda = \frac{4i}{\Delta} \left[\frac{\partial \mu_2}{(1 + \bar{\mu}_1 \mu_2)^2} - \frac{\bar{\partial} \bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^2} \right],$$

$$(4.2) \quad \sigma = \frac{8\mu_2 \partial \bar{\mu}_2}{\bar{\mu}_2 \Delta |1 + \mu_1 \bar{\mu}_2|^2},$$

and

$$(4.3) \quad \frac{1}{4}\Delta = \frac{\partial\mu_2\bar{\partial}\bar{\mu}_2 - \bar{\partial}\bar{\mu}_2\partial\bar{\mu}_2}{|\mu_2|^2|1 + \mu_1\bar{\mu}_2|^2}e^{2r} \\ - 2\operatorname{Re}\left(\frac{\partial\mu_2}{(1 + \bar{\mu}_1\mu_2)^2}\right) + \frac{|\mu_2|^2}{|1 + \mu_1\bar{\mu}_2|^2}e^{-2r},$$

with ∂ denotes the differentiation with respect to μ_1 .

A surface is area stationary if $\delta\mathcal{A}(\Sigma) = 0$. In order to compute this quantity note that

$$\frac{\Delta^2\lambda^2}{64} = -\frac{1}{4}\left[\frac{\partial\mu_2}{(1 + \bar{\mu}_1\mu_2)^2} - \frac{\bar{\partial}\bar{\mu}_2}{(1 + \mu_1\bar{\mu}_2)^2}\right]^2,$$

and so

$$\delta\left(\frac{\Delta^2\lambda^2}{64}\right) \\ = -\operatorname{Re}\left[\frac{\partial\mu_2}{(1 + \bar{\mu}_1\mu_2)^2} - \frac{\bar{\partial}\bar{\mu}_2}{(1 + \mu_1\bar{\mu}_2)^2}\right]\left[\frac{\partial\delta\mu_2}{(1 + \bar{\mu}_1\mu_2)^2} - \frac{2\bar{\mu}_1\partial\mu_2\delta\mu_2}{(1 + \bar{\mu}_1\mu_2)^3}\right],$$

while, since

$$\frac{\Delta^2|\sigma|^2}{64} = \frac{\partial\bar{\mu}_2\bar{\partial}\mu_2}{|1 + \bar{\mu}_1\mu_2|^4},$$

we have

$$\delta\left(\frac{\Delta^2|\sigma|^2}{64}\right) = 2\operatorname{Re}\left(\frac{\partial\bar{\mu}_2\bar{\partial}\delta\mu_2}{|1 + \bar{\mu}_1\mu_2|^4} - \frac{2\bar{\mu}_1}{1 + \bar{\mu}_1\mu_2} \frac{\partial\bar{\mu}_2\bar{\partial}\mu_2\delta\mu_2}{|1 + \bar{\mu}_1\mu_2|^4}\right).$$

Combining these we find that

$$\delta|G|^{\frac{1}{2}} = \frac{16}{\Delta\sqrt{\lambda^2 - |\sigma|^2}} \operatorname{Re}\left\{-\frac{1}{2}\left[\frac{\partial\mu_2}{(1 + \bar{\mu}_1\mu_2)^2} - \frac{\bar{\partial}\bar{\mu}_2}{(1 + \mu_1\bar{\mu}_2)^2}\right] \cdot \partial\left(\frac{\delta\mu_2}{(1 + \bar{\mu}_1\mu_2)^2}\right) - \frac{\partial\bar{\mu}_2\bar{\partial}\delta\mu_2}{|1 + \bar{\mu}_1\mu_2|^4} + \frac{2\bar{\mu}_1}{1 + \bar{\mu}_1\mu_2} \frac{\partial\bar{\mu}_2\bar{\partial}\mu_2\delta\mu_2}{|1 + \bar{\mu}_1\mu_2|^4}\right\} \\ = 2\operatorname{Re}\left[\frac{\lambda i}{\sqrt{\lambda^2 - |\sigma|^2}} \partial\left(\frac{\delta\mu_2}{(1 + \bar{\mu}_1\mu_2)^2}\right) - \frac{\bar{\mu}_2\sigma\bar{\partial}\delta\mu_2}{\mu_2|1 + \bar{\mu}_1\mu_2|^2\sqrt{\lambda^2 - |\sigma|^2}} \right. \\ \left. + \frac{16\bar{\mu}_1}{1 + \bar{\mu}_1\mu_2} \frac{\partial\bar{\mu}_2\bar{\partial}\mu_2\delta\mu_2}{\Delta|1 + \bar{\mu}_1\mu_2|^4\sqrt{\lambda^2 - |\sigma|^2}}\right].$$

Integrating by parts we have established the following:

PROPOSITION 6. *A rank two surface is area stationary if and only if*

$$(4.4) \quad \frac{-i}{(1 + \bar{\mu}_1\mu_2)^2} \partial \left(\frac{\lambda}{\sqrt{\lambda^2 - |\sigma|^2}} \right) + \bar{\partial} \left(\frac{\bar{\mu}_2\sigma}{\mu_2|1 + \bar{\mu}_1\mu_2|^2\sqrt{\lambda^2 - |\sigma|^2}} \right) + \frac{\bar{\mu}_1\Delta|\sigma|^2}{4(1 + \bar{\mu}_1\mu_2)\sqrt{\lambda^2 - |\sigma|^2}} = 0.$$

The following proposition shows that all holomorphic curves on $L(\mathbb{H}^3)$ are area stationary:

PROPOSITION 7. *Every holomorphic curve Σ in $L(\mathbb{H}^3)$ such that the metric G_Σ induced on Σ by the neutral Kähler metric is non-degenerate, is area stationary.*

PROOF. Consider a holomorphic curve Σ in $L(\mathbb{H}^3)$. Then by Proposition 3 the shear σ vanishes throughout the surface Σ .

By Proposition 4 we know that a holomorphic surface Σ can be either rank 0 or 2. In the case of rank 0, the surface Σ is totally null and, in particular, it is orthogonal to a horosphere, which is not our case. Then Σ must be of rank 2 and therefore Proposition 6 shows that is area stationary.

Consider now a Lagrangian surface Σ in $L(\mathbb{H}^3)$. We are interested in area stationary Lagrangian surfaces of rank 2. In this case, the twist λ vanishes on Σ and then Proposition 6 implies that a Lagrangian surface Σ of rank 2 will be area stationary iff

$$\partial \ln \left(\frac{\bar{\sigma}_0}{\sigma_0} \right) - \frac{4\bar{\mu}_2}{1 + \mu_1\bar{\mu}_2} = 0,$$

where

$$(4.5) \quad \sigma_0 = \frac{\partial \bar{\mu}_2}{(1 + \mu_1\bar{\mu}_2)^2}.$$

DEFINITION 3. The *Lagrangian angle* ϕ of the surface $\Sigma \subset L(\mathbb{H}^3)$ is defined by

$$\sigma_0 = |\sigma_0|e^{2i\phi},$$

where σ_0 is given by (4.5).

An equivalent condition that characterizes Lagrangian area stationary surfaces in $L(\mathbb{H}^3)$ is given by the following proposition:

PROPOSITION 8. *Let $\Sigma \subset \mathbb{L}(\mathbb{H}^3)$ be a Lagrangian surface of rank two. Then Σ is a area stationary surface iff Σ is locally orthogonal to a flat surface in \mathbb{H}^3 and the Lagrangian angle ϕ , satisfies the following PDE:*

$$(4.6) \quad e^{-i\phi} \partial^2 e^{-i\phi} = e^{i\phi} \bar{\partial}^2 e^{i\phi} = |\sigma_0|.$$

PROOF. Assume that Σ is a Lagrangian area stationary surface of rank two. Then $H^{\mu_1} = 0$ which means that

$$\partial \ln \left(\frac{\bar{\sigma}_0}{\sigma_0} \right) - \frac{4\bar{\mu}_2}{1 + \mu_1 \bar{\mu}_2} = 0$$

and by introducing the Lagrangian angle ϕ , the above gives

$$(4.7) \quad \mu_2 = \frac{i \bar{\partial} \phi}{1 - i \bar{\mu}_1 \bar{\partial} \phi}.$$

By derivation of the above with respect of μ_1 we obtain

$$i \partial \bar{\partial} \phi = \rho_0, \quad -i \partial \bar{\partial} \phi = \bar{\rho}_0.$$

The Lagrangian condition $\rho_0 = \bar{\rho}_0$ implies that $\rho_0 = 0$ and therefore μ_2 is anti-holomorphic, which means that Σ is locally orthogonal to a flat surface in \mathbb{H}^3 .

Because of $\rho_0 = 0$ we obtain

$$(4.8) \quad \partial \bar{\partial} \phi = 0.$$

The fact that μ_2 is an anti-holomorphic function of μ_1 implies that $\ln \bar{\sigma}_0$ is anti-holomorphic too, which means that $\partial \ln \bar{\sigma}_0 = 0$ and then

$$(4.9) \quad \partial \ln |\sigma_0| = 2i \partial \phi.$$

The expression of σ_0 in terms of ϕ is

$$(4.10) \quad \sigma_0 = \frac{\partial \bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^2} = -(\partial \phi)^2 - i \partial^2 \phi.$$

Then we have

$$|\sigma_0| e^{2i\phi} = -(\partial \phi)^2 - i \partial^2 \phi$$

which gives

$$|\sigma_0| e^{i\phi} = [-(\partial \phi)^2 - i \partial^2 \phi] e^{-i\phi}$$

and therefore $|\sigma_0| = e^{-i\phi} \partial^2 e^{-i\phi}$, which implies equation (4.6).

We now prove the converse. If ϕ is a real solution of (4.6), it satisfies

$$(4.11) \quad -i\partial^2\phi - (\partial\phi)^2 = |\sigma_0|e^{2i\phi} = \sigma_0.$$

By the assumption that Σ is locally orthogonal to a flat in \mathbb{H}^3 , μ_2 is anti-holomorphic and therefore σ_0 is a holomorphic function. Then

$$\bar{\partial}[-i\partial^2\phi - (\partial\phi)^2] = 0,$$

which implies

$$\partial[(\bar{\partial}\partial\phi)e^{-2i\phi}] = 0,$$

and hence there is a holomorphic function β such that

$$(\bar{\partial}\partial\phi)e^{-2i\phi} = \beta, \quad (\bar{\partial}\partial\phi)e^{2i\phi} = \bar{\beta}.$$

Therefore, the function ϕ can be written as

$$(4.12) \quad \phi = a + \bar{a},$$

where a is a holomorphic function. In other words, we have proved that $\bar{\partial}\partial\phi = 0$.

On the other hand, using equations (4.11) and (4.12), μ_2 must satisfies the following equation:

$$(4.13) \quad -i\partial^2a - (\partial a)^2 = \frac{\partial\bar{\mu}_2}{(1 + \mu_1\bar{\mu}_2)^2},$$

but because of the fact that $\bar{\mu}_2$ and a are holomorphic, the above equation is equivalent to an ordinary differential equation of first order. In addition, we observe that

$$\bar{\mu}_2 = -\frac{i\partial a}{1 + i\mu_1\partial a}$$

is a solution of (4.13) and because this equation is equivalent to an ODE of first order, it is unique.

Then it is easy to see that $H^{\mu_1} = 0$ and therefore Σ is a Lagrangian area stationary surface.

In the following proposition we give an explicit local expression of all Lagrangian area stationary surfaces in $\mathbb{L}(\mathbb{H}^3)$ in terms of the holomorphic coordinates (μ_1, μ_2) on $\mathbb{P}^1 \times \mathbb{P}^1 - \bar{\Delta}$:

PROPOSITION 9. *Every Lagrangian area stationary surface Σ in $\mathbb{L}(\mathbb{H}^3)$ of rank two can be locally parameterized by*

$$(4.14) \quad \Sigma \rightarrow \mathbb{L}(\mathbb{H}^3) : (\mu_1, \bar{\mu}_1) \mapsto \left(\mu_1, \mu_2 = \frac{\bar{\lambda}_1\bar{\mu}_1 + 1}{\bar{\mu}_1 + \bar{\lambda}_2} \right),$$

where $\lambda_1, \lambda_2 \in \mathbb{C}$ with $\lambda_1\lambda_2 \neq 1$.

PROOF. Let Σ be a Lagrangian area stationary surface of rank two in $L(\mathbb{H}^3)$. By Proposition 8 the surface Σ is locally orthogonal to a flat surface in \mathbb{H}^3 , which allows us to obtain the holomorphic parameterization $(\mu_1, \bar{\mu}_1) \mapsto (\mu_1, \mu_2(\bar{\mu}_1))$. In addition, the Lagrangian angle ϕ must satisfy equation (4.6).

There is a holomorphic function a such that $\phi = a + \bar{a}$, and applying this to equation (4.6), we get:

$$e^{3i\bar{a}}\bar{\partial}^2 e^{i\bar{a}} = e^{-3ia}\partial^2 e^{-ia} = c_0,$$

where $c_0 \in \mathbb{R}$ is a real constant.

Then the holomorphic function a satisfies

$$(4.15) \quad \partial^2 e^{-ia} = c_0 e^{3ia},$$

which is equivalent to the following ordinary differential equation of second order:

$$\ddot{x} = c_0 x^{-3}.$$

The unique solution of (4.15) is

$$(4.16) \quad a = \frac{i}{2} \log[(\alpha_0 \mu_1 + \beta_0)^2 - c_0] - \frac{i}{2} \log \alpha_0,$$

and the Lagrangian angle is $\phi = a + \bar{a}$.

The immersion of Σ is obtained by substituting (4.16) into (4.7) and then

$$\mu_2 = \frac{i\bar{\partial}\phi}{1 - i\bar{\mu}_1\bar{\partial}\phi} = \frac{i\bar{\partial}\bar{a}}{1 - i\bar{\mu}_1\bar{\partial}\bar{a}} = \frac{\alpha_0^2\bar{\mu}_1 + \alpha_0\beta_0}{\alpha_0\beta_0\bar{\mu}_1 + \beta_0^2 - c_0}.$$

If we set $\bar{\lambda}_1 = \alpha_0\beta_0^{-1}$ and $\bar{\lambda}_2 = (\beta_0^2 - c_0)(\alpha_0\beta_0)^{-1}$ then the area stationary surface Σ is given by the immersion (4.14).

If $\lambda_1\lambda_2 = 1$ we find that Σ is a totally null surface given by the immersion $\mu_2 = \lambda_1$, and so it is not of rank two.

For a given Lagrangian area stationary surface Σ in $L(\mathbb{H}^3)$, there is locally a family of parallel flat surfaces in \mathbb{H}^3 such that their oriented normals are contained in Σ . We recall the classification of complete flat surfaces in hyperbolic 3-space:

PROPOSITION 10 ([14], [15]). *Let S be a complete flat surface in hyperbolic 3-space \mathbb{H}^3 . Then S is either a horosphere or an equidistant tube of a geodesic in \mathbb{H}^3 .*

To prove the main theorem we need to introduce a particular class of surface in hyperbolic 3-space \mathbb{H}^3 :

DEFINITION 4. A surface S in hyperbolic 3-space \mathbb{H}^3 is called *isoparametric* if the principal curvatures of S are constant.

Note that all parallel surfaces $\{S_t\}_{t \in I}$ to the isoparametric surface S are also isoparametric.

The following proposition gives a classification of the isoparametric surfaces in hyperbolic 3-space:

PROPOSITION 11 ([3]). *Let S be an isoparametric surface in \mathbb{H}^3 . Then S is either a totally geodesic hyperbolic 2-space, or a totally umbilical surface or an equidistant tube around a geodesic.*

We now prove our main result:

THEOREM 3. *Let $S \subset \mathbb{H}^3$ be a C^3 smooth immersed oriented surface and $\Sigma \subset L(\mathbb{H}^3)$ be the Lagrangian surface formed by the oriented geodesics normal to S .*

The surface Σ is area stationary iff S is an equidistant tube around a geodesic.

PROOF. Let Σ be a Lagrangian geodesic congruence formed by the oriented geodesics normal to S .

First assume that Σ is area stationary. Since it cannot be of rank 0, as that would mean that it is totally null, and by Proposition 5 it cannot be of rank 1, we conclude that Σ is of rank two. Thus it is given locally by the graph:

$$\bar{\mu}_2 = \frac{1 + \lambda_1 \mu_1}{\mu_1 + \lambda_2},$$

where $\lambda_1, \lambda_2 \in \mathbb{C}$.

The non-degeneracy condition of the induced metric G_Σ implies that $\lambda_1 \lambda_2 \neq 1$.

In this case, an orthogonal surface $S \subset \mathbb{H}^3$ can be obtained by solving the following differential equation

$$2\partial r = \frac{\partial \mu_2}{\mu_2(1 + \bar{\mu}_1 \mu_2)} + \frac{\partial \bar{\mu}_2}{\bar{\mu}_2(1 + \mu_1 \bar{\mu}_2)} + \frac{\bar{\mu}_2}{1 + \mu_1 \bar{\mu}_2},$$

and, by using the fact that μ_2 is holomorphic, we obtain, after a brief computation, that

$$2\partial r = \frac{\lambda_1}{1 + \lambda_1 \mu_1},$$

which implies

$$(4.17) \quad r = \frac{1}{2} \log |1 + \lambda_1 \mu_1|^2 + r_0.$$

The function Δ , given by (4.3), is:

$$\Delta = \frac{4|\lambda_1|^2[e^{-2r_0} - |\lambda_1\lambda_2 - 1|^2e^{2r_0}]}{|\lambda_1\mu_1 + 1|^2 + \lambda_1\lambda_2 - 1|^2}.$$

The optical scalars ρ and σ of the Lagrangian area stationary surface Σ given by (4.1) and (4.2) are:

$$\sigma = \frac{2(\lambda_1\lambda_2 - 1)}{e^{-2r_0} - |\lambda_1\lambda_2 - 1|^2e^{2r_0}} \cdot \frac{1 + \bar{\lambda}_1\bar{\mu}_1}{1 + \lambda_1\mu_1}$$

and

$$\rho = -1 + \frac{2}{1 - e^{4r_0}|\lambda_1\lambda_2 - 1|^2}.$$

If we denote by h the mean curvature of the surface $S \subset \mathbb{H}^3$, Proposition 2 gives:

$$(4.18) \quad h = 1 + \frac{2}{e^{4r_0}|\lambda_1\lambda_2 - 1|^2 - 1}.$$

Consider now the principal curvatures m_1 and m_2 of the surface S . The fact that S is flat means that $m_1m_2 = 1$. Then the mean curvature of the surface S is

$$h = \frac{m_1 + m_2}{2} = \frac{m_1 + m_1^{-1}}{2},$$

and by using the relation (4.18), we observe that m_1 must satisfy the following quadratic equation

$$(4.19) \quad m_1^2 - 2\left(1 + \frac{2}{e^{4r_0}|\lambda_1\lambda_2 - 1|^2 - 1}\right)m_1 + 1 = 0.$$

Therefore the principal curvatures of the surface S are constant and in particular are given by:

$$m_1 = \frac{e^{2r_0}|\lambda_1\lambda_2 - 1| + 1}{e^{2r_0}|\lambda_1\lambda_2 - 1| - 1}, \quad m_2 = \frac{e^{2r_0}|\lambda_1\lambda_2 - 1| - 1}{e^{2r_0}|\lambda_1\lambda_2 - 1| + 1},$$

and hence the surface S is isoparametric. Propositions 10 and 11 tell us that the surface S can be either a horosphere or an equidistant tube around a geodesic. By previous work (see the papers [5] and [6]) we have seen that geodesic congruences orthogonal to horospheres are totally null (the induced metric is degenerate). Therefore the surface S must be an equidistant tube around a geodesic γ .

In fact, every area stationary surface Σ is orthogonal to the set $\{S_{r_0}\}_{r_0 \in \mathbb{R}}$ of all parallel equidistant tubes around a geodesic γ and each such a surface S_{r_0} is of hyperbolic distance r_0 from the surface S_0 .

Conversely, assume that the surface $S \subset \mathbb{H}^3$ is an equidistant tube around a geodesic γ' with holomorphic coordinates $(\mu_1 = \mu'_1, \mu_2 = \mu'_2)$. Then S belongs to the set of all parallel equidistant tubes $\mathcal{U}_{\gamma'} = \{S_{r_0}\}_{r_0 \in \mathbb{R}}$ around the geodesic γ' . We first find an explicit expression of the orthogonal geodesic congruence $\Sigma \subset \mathbb{L}(\mathbb{H}^3)$ to all surfaces in $\mathcal{U}_{\gamma'}$.

Consider the hyperbolic 3-space \mathbb{H}^3 in the half space model with local coordinates (x_0, x_1, x_2) .

By direct computation we get that for a given point $p = (p_0, p_1, p_2)$ in \mathbb{H}^3 and a given vector $e_0 = a_0 \frac{\partial}{\partial x_0} + a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} \in T_p \mathbb{H}^3$ the unique geodesic $\gamma : I \subset \mathbb{R} \rightarrow \mathbb{H}^3 : r \mapsto \gamma(r) \in \mathbb{H}^3$ such that

$$\gamma(0) = p, \quad \dot{\gamma}(0) = e_0,$$

where I is an open interval containing 0 and the dot denotes the differentiation with respect of r , is defined by:

$$\begin{aligned} x_0 &= p_0 \sqrt{\frac{a_0^2 + a_1^2 + a_2^2}{a_1^2 + a_2^2}} \operatorname{sech} \left[\frac{\sqrt{a_0^2 + a_1^2 + a_2^2}}{p_0} (r + r_0) \right], \\ x_1 &= \frac{a_1 p_0 \sqrt{a_0^2 + a_1^2 + a_2^2}}{a_1^2 + a_2^2} \tanh \left[\frac{\sqrt{a_0^2 + a_1^2 + a_2^2}}{p_0} (r + r_0) \right] + c_3, \\ x_2 &= \frac{a_2 p_0 \sqrt{a_0^2 + a_1^2 + a_2^2}}{a_1^2 + a_2^2} \tanh \left[\frac{\sqrt{a_0^2 + a_1^2 + a_2^2}}{p_0} (r + r_0) \right] + c_4. \end{aligned}$$

Introduce complex coordinate $z = x_1 + ix_2$ and set $t = x_0$. We then obtain

$$\xi = c_1 + ic_2 = \frac{\beta}{t_0^2}, \quad \eta = c_3 + ic_4 = z_0 + t_0 \frac{a}{\beta},$$

where $t_0 = t(0)$, $z_0 = z(0)$, $\beta = a_1 + ia_2$ and $a = a_0$.

Therefore for a given point $p = (z_0, t_0)$ and a given vector $e_0 = a \frac{\partial}{\partial t} + \beta \frac{\partial}{\partial z} + \bar{\beta} \frac{\partial}{\partial \bar{z}}$ the unique oriented geodesic $\gamma = (\xi, \eta)$ with the initial conditions $\gamma(0) = p$ and $\dot{\gamma}(0) = e_0$ is given by

$$(4.20) \quad \xi = \frac{\beta}{t_0^2}, \quad \eta = z_0 + t_0 \frac{a}{\beta}$$

Fix the point p on the given oriented geodesic $\gamma' = (\xi', \eta')$. Let $\gamma = (\xi, \eta)$ be an oriented geodesic that intersects γ' orthogonally at p . Denote the unit tangent vectors of γ, γ' at p by e_0, e'_0 respectively. The orthogonality condition gives the following relation:

$$e_0 = \frac{1}{\sqrt{2}}(e^{-i\theta} e'_+ + e^{i\theta} e'_-),$$

for some $\theta \in [0, 2\pi)$ where

$$e'_+ = \frac{1}{\sqrt{2}|\xi'| \cosh^2 r_0} \frac{\partial}{\partial t} + \frac{1}{\sqrt{2} \cosh^2 r_0} \left(-\frac{e^{-r_0}}{\bar{\xi}'} \frac{\partial}{\partial z} + \frac{e^{r_0}}{\xi'} \frac{\partial}{\partial \bar{z}} \right), \quad e'_- = \bar{e}'_+.$$

Thus the unit tangent vector of γ is

$$e_0 = \frac{\cos \theta}{|\xi'| \cosh^2 r_0} \frac{\partial}{\partial t} + \frac{\sinh(r_0 + i\theta)}{\bar{\xi}' \cosh^2 r_0} \frac{\partial}{\partial z} + \frac{\sinh(r_0 - i\theta)}{\xi' \cosh^2 r_0} \frac{\partial}{\partial \bar{z}}.$$

Applying (4.20), the oriented geodesic $\gamma = (\xi, \eta)$ is

$$\xi = \xi' \sinh(r_0 + i\theta), \quad \eta = \eta' + \frac{1}{\bar{\xi}' \tanh(r_0 - i\theta)}$$

Moving the point p along the geodesic (ξ', η') , it is equivalent to an affine shift of r_0 .

Therefore we obtain the surface Σ given by the immersion $f : \mathbb{C} \rightarrow \mathbb{L}(\mathbb{H}^3)$: $(v, \bar{v}) \mapsto (\xi(v, \bar{v}), \eta(v, \bar{v}))$ where

$$\xi = \xi' \sinh v, \quad \eta = \eta' + \frac{1}{\bar{\xi}' \tanh \bar{v}},$$

with $v = r + i\theta$.

If we change the coordinates from (ξ, η) to holomorphic coordinates (μ_1, μ_2) on $\mathbb{L}(\mathbb{H}^3)$, the surface Σ is given by the following immersion

$$\mu_1(v, \bar{v}) = \frac{1 - \cosh \bar{v} - \eta' \bar{\xi}' \sinh \bar{v}}{\bar{\xi}' \sinh \bar{v}},$$

$$\mu_2(v, \bar{v}) = \frac{\xi' \sinh v}{1 + \cosh v + \bar{\eta}' \xi' \sinh v}.$$

We can easily see that

$$\sinh v = \frac{2(\xi')^{-1} \mu_2}{1 + \bar{\mu}_1 \mu_2} \quad \text{and} \quad \cosh v = \frac{1 - \bar{\mu}_1 \mu_2 - 2\bar{\eta}' \mu_2}{1 + \bar{\mu}_1 \mu_2},$$

and from the identity $\cosh^2 v - \sinh^2 v = 1$ we find that the Lagrangian surface Σ is an area stationary surface since it can be written

$$\bar{\mu}_2 = \frac{\lambda_1 \mu_1 + 1}{\mu_1 + \lambda_2}, \quad \text{with } \lambda_1 = \frac{1}{\eta'}, \quad \lambda_2 = \frac{1}{\eta'} \left[(\eta')^2 - \frac{1}{(\bar{\xi}')^2} \right],$$

which completes the proof.

NOTE: We have proved on the main theorem that every area stationary Lagrangian surface is given by the graph (4.14) and is orthogonal to a family of parallel equidistant tubes $\{S_t\}_{t \in I}$ around to the following oriented geodesics $\gamma' = (\mu'_1, \mu'_2)$ and $\tilde{\gamma}' = (\tilde{\mu}'_1, \tilde{\mu}'_2)$, given by

$$\mu'_1 = \frac{-1 + \sqrt{1 - \lambda_1 \lambda_2}}{\lambda_1}, \quad \mu'_2 = \frac{\bar{\lambda}_1}{1 + \sqrt{1 - \bar{\lambda}_1 \bar{\lambda}_2}},$$

and

$$\tilde{\mu}'_1 = -\frac{1 + \sqrt{1 - \lambda_1 \lambda_2}}{\lambda_1}, \quad \tilde{\mu}'_2 = \frac{\bar{\lambda}_1}{1 - \sqrt{1 - \bar{\lambda}_1 \bar{\lambda}_2}}.$$

Consider now the antipodal map $\tau : \mathbf{P}^1 \rightarrow \mathbf{P}^1 : x \mapsto -\bar{x}^{-1}$ and observe that $\tilde{\mu}'_1 = \tau(\mu'_2)$ and $\tilde{\mu}'_2 = \tau(\mu'_1)$ which means that the geodesic $\tilde{\gamma}$ is obtained by reversing the orientation of the geodesic γ . In other words $\tilde{\gamma}$ and γ describe the same geodesic, up to orientation.

5. Rotationally symmetric maximal graphs

We now investigate area stationary surfaces of rank two that are not Lagrangian or holomorphic. The simplest case is the rotationally symmetric ones, see also [9].

DEFINITION 5. A surface in $L(\mathbf{H}^3)$ is *rotationally symmetric* if it is invariant under the induced isometry of G .

LEMMA 1. A graph $\Sigma \rightarrow L(\mathbf{H}^3) : \mu_1 \mapsto (\mu_1, \mu_2 = F(\mu_1, \bar{\mu}_1))$ is *rotationally symmetric if and only if* $F(\mu_1, \bar{\mu}_1) = G(R)e^{i\theta}$ *for some complex-valued function* G , *where* $\mu_1 = Re^{i\theta}$.

To find a complete characterisation of all area stationary rotationally symmetric surfaces in $L(\mathbf{H}^3)$ of rank two, seems to be a very difficult problem because the correspondent ordinary differential equations are too complicated. We then investigate the case of $\text{Re } G = 0$:

PROPOSITION 12. Consider a rank two surface Σ immersed by $\mu_1 = Re^{i\theta}$ and $F(\mu_1, \bar{\mu}_1) = iAe^{i\theta}$, where $A = A(R)$ is real-valued function. Then

the surface Σ is a holomorphic curve iff $A = c_0 R$, while is Lagrangian iff $A = c_0 R^{-1}$.

The following proposition gives a two parameter family of rotationally symmetric surfaces of rank two in $L(\mathbb{H}^3)$ that are area stationary:

PROPOSITION 13. *Consider the rotationally symmetric $\Sigma \subset L(\mathbb{H}^3)$ of rank two, given locally immersed by $\mu_1 = Re^{i\theta}$ and $\mu_2 = F(\mu_1, \bar{\mu}_1) = iZ^{1/2}e^{i\theta}$, where $Z = Z(R)$ is a real-valued function.*

Then Σ is area stationary if and only if the function Z is of the form:

$$(5.1) \quad Z(R) = c + \frac{bR^2}{cR^2 - 1},$$

where b and c are real constants that aren't both 0.

PROOF. We first compute the functions Δ , λ and σ of the rotationally symmetric surface Σ . Since the surface Σ is of rank two, then by Proposition 6, it is area stationary iff it satisfies the equation (4.4) which implies that the function Z is satisfying the following ordinary differential equation:

$$(5.2) \quad R(R^2 Z - 1)\ddot{Z} - 2R^3 \dot{Z}^2 + (3R^2 Z + 1)\dot{Z} = 0,$$

where the dot denotes differentiation with respect of the real variable R .

We first observe that the expression of the real-valued function $Z = Z(R)$ given in (5.1) satisfies the ordinary differential equation (5.2). Furthermore, we have obtained a two parameter family of such solutions depended of the constants b and c , which implies that the functions $Z = Z(R)$ given in (5.1) form the complete solutions of (5.2).

Note that for every choice of $b, c \in \mathbb{R}$, the area stationary rotationally symmetric surface Σ_{bc} , immersed by $\mu_1 = Re^{i\theta}$ and $\mu_2 = F(\mu_1, \bar{\mu}_1) = iZ^{1/2}e^{i\theta}$, where $Z = Z(R)$ is given by (5.1), is neither Lagrangian nor holomorphic curve, since for

$$Z = A^2 = c_0^2 R^2 \quad \text{or} \quad Z = A^2 = c_0^2 R^{-2},$$

are not solutions of the ordinary differential equation (5.2).

We now investigate the induced metric of area stationary rotationally symmetric surfaces. Such a surface Σ can be given locally by the immersion $\mu_1 = Re^{i\theta}$, $\mu_2 = iz^{1/2}e^{i\theta}$, where z is given by (5.1).

PROPOSITION 14. *Let Σ be an area stationary rotationally symmetric surface in $L(\mathbb{H}^3)$. Such a surface can be determined by the constants $b, c \in \mathbb{R}$ ($b \neq 0$), given by (5.1).*

If $b > 0$ the induced metric G_Σ is Lorentz, while for $b < 0$ the surface Σ has at least two degenerate curves.

PROOF. From Theorem 1 we need to find the expression $|\sigma|^2 - \lambda^2$. After a brief computation we find that

$$(5.3) \quad |\sigma|^2 - \lambda^2 = A[bR^4 + (cR^2 - 1)^2][((c^2 + b)R^2 - 1)^2 + b],$$

where $A = 64\Delta^{-2}|1 + \mu_1\bar{\mu}_2|^{-2}(1 + R^2z)^{-2}(cR^2 - 1)^{-4} > 0$.

Observe that $|\sigma|^2 - \lambda^2 > 0$ for $b > 0$ and therefore the rotationally symmetric area stationary surface Σ is Lorentzian.

We consider the case of $b = -k^2 < 0$ and $c^2 + b > 0$. Then the induced metric G_Σ of the surface Σ is degenerate at the following least four curves,

$$R = \pm(c - k)^{-1/2}, \quad R = \pm\sqrt{\frac{k + 1}{c^2 - k^2}}, \quad \text{for } k > -1,$$

$$R = \pm(c - k)^{-1/2}, \quad R = \pm\sqrt{\frac{1 - k}{c^2 - k^2}}, \quad \text{for } k < 1,$$

$$R = \pm(c - k)^{-1/2}, \quad R = \pm\sqrt{\frac{k + 1}{c^2 - k^2}}, \quad R = \pm\sqrt{\frac{1 - k}{c^2 - k^2}}, \quad \text{for } |k| < 1.$$

In case of $b = -k^2 < 0$ and $c^2 + b < 0$ the induced metric G_Σ is degenerate at the following least two curves,

$$R = \pm(c + k)^{-1/2}, \quad R = \pm\sqrt{\frac{k + 1}{c^2 - k^2}}, \quad \text{for } k < -1,$$

$$R = \pm(c + k)^{-1/2}, \quad R = \pm\sqrt{\frac{1 - k}{c^2 - k^2}}, \quad \text{for } k > 1,$$

$$R = \pm(c + k)^{-1/2}, \quad R = \pm\sqrt{\frac{k + 1}{c^2 - k^2}}, \quad R = \pm\sqrt{\frac{1 - k}{c^2 - k^2}}, \quad \text{for } |k| > 1.$$

$$R = \pm(c + k)^{-1/2}, \quad \text{for } |k| < 1,$$

and the proposition follows.

We know that $L(H^3)$ can be identified with $\mathbb{P}^1 \times \mathbb{P}^1 - \bar{D}$, where $\bar{D} = \{(x, y) \in \mathbb{P}^1 \times \mathbb{P}^1 : y = -\bar{x}^{-1}\}$ is the reflected diagonal of \mathbb{P}^1 . We would like to see whether an area stationary rotationally symmetric surface Σ intersects \bar{D} .

To do this we consider the local immersion $\mu_1 = Re^{i\theta}$ and $\mu_2 = iZ^{1/2}e^{i\theta}$, of Σ , where $Z = Z(R)$ is given by (5.1). Assume that a point $(R, \theta) \in \Sigma$ is very near to \bar{D} . Then the equation $\mu_1 = -\bar{\mu}_2^{-1}$ gives the following quadratic equation

$$(c^2 + b)R^4 - 1 = 0.$$

Then, we conclude that for b positive the Lorentzian area stationary rotationally symmetric surface Σ must be an open subset of $L(\mathbb{H}^3)$.

Consider the surface $T \subset L(\mathbb{H}^3)$ locally parameterised by

$$\mu_1(R, \theta) = Re^{i\theta}, \quad \mu_2(R, \theta) = \pm i \left(c + \frac{bR^2}{cR^2 - 1} \right)^{1/2} e^{i\theta}.$$

In order T to be an immersed complete surface we require $c < 0$. If in addition we assume that T must be a closed surface, we require $c^2 + b < 0$. We can see now that for $R = 0$ and $R \rightarrow \infty$, we obtain the following two curves on T :

$$\mu_2 = i\sqrt{c}e^{i\theta} \quad \text{and} \quad \mu_2 = i\sqrt{\frac{c^2 + b}{c}}e^{i\theta}.$$

Under the i -th projection $L(\mathbb{H}^3) = \mathbb{S}^2 \times \mathbb{S}^2 - D \rightarrow \mathbb{S}^2$ for $i = 1, 2$ we observe that these surfaces T double cover the sphere, except the north and south pole, where the inverse image of each of these points, is a circle. Therefore, the surface T must be an area stationary tori in $L(\mathbb{H}^3)$.

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UNIVERSIDADE DE SÃO PAULO
IME, BLOCO A
1010 RUA DO MATÃO
CIDADE UNIVERSITÁRIA
05508-090 SÃO PAULO
BRAZIL
E-mail: nikos@ime.usp.br