MATH. SCAND. 111 (2012), 135–146

NONTRIVIALLY NOETHERIAN C^*-ALGEBRAS

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Abstract

We say that a C^*-algebra is *Noetherian* if it satisfies the ascending chain condition for two-sided closed ideals. A *nontrivially* Noetherian C^*-algebra is one with infinitely many ideals. Here, we show that nontrivially Noetherian C^*-algebras exist, and that a separable C^*-algebra is Noetherian if and only if it contains countably many ideals and has no infinite strictly ascending chain of primitive ideals. Furthermore, we prove that every Noetherian C^*-algebra has a finite-dimensional center. Where possible, we extend results about the ideal structure of C^*-algebras to Artinian C^*-algebras (those satisfying the descending chain condition for closed ideals).

1. Introduction

A ring is *Noetherian* if every ascending chain of ideals stabilizes, that is, given an increasing chain of two-sided ideals $I_1 \subset I_2 \subset I_3 \subset \cdots$, there is an n such that $I_n = I_{n+1} = \cdots$ (also called the *ascending chain condition* for two-sided ideals). In this paper, we extend the notion of a Noetherian ring to C^*-algebras, and investigate the corresponding ideal structure.

Definition 1.1. We say that a C^*-algebra A is *Noetherian*[1] if every ascending chain of closed two-sided ideals stabilizes, that is, if A satisfies the ascending chain condition for closed ideals (from now on, by 'ideal' we mean closed and two-sided unless otherwise specified or obviously false).

The dual notion to a Noetherian C^*-algebra is an *Artinian* C^*-algebra, which satisfies the descending chain condition for closed ideals.

Example 1.2. Most obviously, every C^*-algebra with finitely many ideals is Noetherian, such as $\mathscr{B}(H)$ for separable Hilbert space H or any finite-dimensional C^*-algebra. We are particularly interested in Noetherian C^*-algebras with infinitely many ideals, which we call *nontrivially Noetherian*. By Corollary 4.2, nontrivially Noetherian C^*-algebras do indeed exist.

* The authors would like to express their gratitude to Professor John Quigg for his constant support and guidance.

Received 11 January 2011.

[1] This definition differs from the traditional definition of a Noetherian Banach algebra, which requires every increasing chain of left-sided ideals to stabilize. In [14], it was shown that every Noetherian Banach algebra is finite dimensional. After relaxing the ascending chain condition to two-sided ideals, this result is still true for commutative C^*-algebras (Theorem 3.1), but not in general. In Example 4.3, we give an example of a infinite-dimensional Noetherian C^*-algebra.

In the following section we review basic facts about Noetherian topological spaces. We primarily focus on the topological restrictions imposed on the space of primitive ideals of Noetherian (and Artinian) C^*-algebras.

2. Noetherian Topological Spaces

A *Noetherian topological space* is a space that satisfies the ascending chain condition for open subsets (equivalently, that every family of open subsets contains a maximal element).

EXAMPLE 2.1. Finite spaces are obviously Noetherian. A more enlightening example is a countable set with a single dense point (that is, a single point whose closure is the entire space) and the cofinite topology on the complement of this point. Although any set with the cofinite topology is Noetherian, we will return to this specific example later.

In particular, we are interested in the primitive ideal space of a C^*-algebra, which is the set of primitive ideals (kernels of irreducible representations) denoted $\mathrm{Prim}(A)$. Endowed with the hull-kernel topology,[2] $\mathrm{Prim}(A)$ is a T_0 space[3] wherein the correspondence $I \mapsto \mathrm{hull}(I)$ is an inclusion-reversing bijection from the set of closed ideals of A onto the closed subsets of $\mathrm{Prim}(A)$ [13, Theorem 5.4.7]. This correspondence immediately gives us the following result, which allows us to investigate the (algebraic) property of Noetherianness via the topological properties of its primitive ideal space.

LEMMA 2.2. *A C^*-algebra A is Noetherian if and only if* $\mathrm{Prim}(A)$ *is.*

The dual idea to a Noetherian space is an *Artinian* space, satisfying the descending chain condition for open sets. In particular, every point in an Artinian space has a minimal neighborhood. It is also easy to see that a C^*-algebra is Artinian if and only if its primitive ideal space is Artinian, using the same reasoning.

The following two results are elementary facts concerning Noetherian spaces that are well-known[4], and we include the proof of these facts in order to make this paper self-contained. As these facts demonstrate, being Noetherian is an extremely strong condition to impose on general topological spaces.

LEMMA 2.3. *A topological space is Noetherian if and only if it is hereditarily compact* (*that is, every subspace is compact*).[5]

[2] Recall that $\ker(E) = \bigcap_{I\in E} I$ for nonempty $E \subset \mathrm{Prim}(A)$, and for $F \subset A$, $\mathrm{hull}(F) = \{I \mid I \text{ is primitive and } F \subset I\}$. The hull-kernel topology is the unique topology on $\mathrm{Prim}(A)$ (see [13, §5.4], [3, §2.9]) such that $\overline{E} = \mathrm{hull}(\ker(E))$.

[3] Distinct points have distinct closures.

[4] See, e.g., [9, §1.1], [15] for a discussion of Noetherian spaces.

[5] See [15, Theorem 1] for a more extensive list of equivalent conditions.

PROOF. By contrapositive, let X be a space with noncompact subspace Y, and let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of distinct open sets in Y with no finite subcover. Then we can construct a strictly increasing sequence of open sets by taking unions of elements in this cover, proving that Y is not Noetherian. Because every subspace of a Noetherian space is Noetherian, X is also not Noetherian.

Conversely, say that X is not Noetherian and let $Y_1 \supset Y_2 \supset \cdots$ be a strictly descending chain of closed sets. Define $E = Y_1 \setminus \bigcap_n Y_n$. It is clear that $\{E\cap Y_n\}$ is a family of closed sets in E with the finite intersection property and empty intersection. So E is a noncompact subspace of X.

In the next lemma, recall that an *irreducible set* is a set that is not the union of two proper closed subsets, and an *irreducible component* is a maximal irreducible subspace. Note that the closure of an irreducible subspace is also irreducible, so every irreducible component is closed. Any space can be written as a union of irreducible components, since every singleton is irreducible, and hence is contained in a maximal closed irreducible set (by Zorn's lemma). A *sober* space has the property that every closed irreducible set is the closure of a unique point. Notice that every sober space is T_0, since otherwise there would exist an irreducible set that is the closure of two distinct points. An important result, which we use frequently in Section 4, is that $\mathrm{Prim}(A)$ is sober when A is separable [3, 3.9.1(b)], a property that is equivalent to every prime ideal being primitive.[6]

Note that $\mathrm{Prim}(A)$ is not sober in general, as shown in both [17] and [12, Proposition 31], as each author exhibits a (nonseparable) prime C^*-algebra that is not primitive.[7]

LEMMA 2.4 ([9, Proposition 1.5]). *Every closed subspace of a Noetherian topological space is uniquely expressible as the finite union of its irreducible components.*

PROOF. Let X be a Noetherian space. To show that every closed subspace of X is expressible as the finite union of irreducible closed sets, suppose that some closed set cannot be. Consider the family of all nonempty closed sets that cannot be written as the union of finitely many irreducible closed sets. By assumption, this family is nonempty, and since X is Noetherian it has a minimal element, call it Y. So Y is not irreducible, and hence can be written as

[6] We say that A is *primitive* if the zero ideal is so. See, e.g., [8, Corollary 1.5 & Appx. A] for a complete discussion, as well as a more complete characterization of the primitive ideal spaces of separable nuclear C^*-algebras. In essence, the sobriety of $\mathrm{Prim}(A)$ follows because $\mathrm{Prim}(A)$ is the continuous open image of the space of pure states of A.

[7] The first reference relies on the continuum hypothesis, using a transfinite recursion of length $2^{\aleph_0}$. The second reference uses a simpler construction without CH.

$Y_1 \cup Y_2$ where Y_1 and Y_2 are distinct proper closed subsets. As Y is minimal, Y_1 and Y_2 can be expressed as the union of finitely many irreducible closed sets, so Y can also be, which is a contradiction.

Let Y be a closed subset of X, with $Y = E_1 \cup \cdots \cup E_m$ where each E_i is closed and irreducible. Notice that every irreducible closed set is contained in an irreducible component (a straightforward application of Zorn's lemma) so we may assume without loss of generality (after possibly throwing out several of the E_i's) that each E_i is a component of Y. Now suppose that $Y = F_1 \cup \cdots \cup F_n$ is another decomposition of Y into components. In this case, $E_1 = \bigcup_1^n (E_1 \cap F_j)$, and hence $E_1 \subset F_j$ for some $1 \le j \le n$ (since E_1 is irreducible) and thus $E_1 = F_j$ since E_1 is a component. Proceeding by induction, it is easy to see that the E_i's are unique up to reordering.

It follows that the only Hausdorff Noetherian spaces are finite, because the only irreducible sets in a Hausdorff space are the singletons. This also shows that a Noetherian space cannot contain an infinite Hausdorff subspace, as every subspace of a Noetherian space is Noetherian.

Now that the basic topological properties of Noetherian spaces (and hence the primitive ideals space of Noetherian C^*-algebras) have been given, we investigate their implications. Although our primary interest, as stated in Example 1.2, are the Noetherian C^*-algebras with infinitely many ideals, in the following section we first give some preliminary results for C^*-algebras that trivially satisfy the condition of being Noetherian.

3. Trivially Noetherian C^*-algebras

In this section, we investigate the algebras that have only finitely many ideals, and are thus clearly Noetherian. As will be shown here, trivially Noetherian C^*-algebras can vary wildly in structure (e.g., $\mathbb{C}$ vs. $\mathscr{B}(H)$), and the nontriviality condition imposes considerably more control.

THEOREM 3.1. *For a commutative C^*-algebra A, the following are equivalent:*

(i) *A is finite-dimensional,*

(ii) *A has finitely many ideals,*

(iii) *A is Noetherian.*

PROOF. Since (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is obvious, it remains to show (iii) $\Rightarrow$ (i).

Since A is commutative, $\mathrm{Prim}(A)$ is Hausdorff. Furthermore, since A is Noetherian, $\mathrm{Prim}(A)$ must be finite (say $\#\,\mathrm{Prim}(A) = n$) by the discussion following Lemma 2.4. Therefore, thinking of A as $C(\mathrm{Prim}\, A)$, we have $A \cong \mathbb{C}^n$.

In order to extend results of this type to noncommutative algebras, first notice by definition (see [11, Definition 1.5]) that every unital C^*-algebra A with center Z is a $C(\mathrm{Prim}\, Z)$-algebra, and hence A is the C_0-section algebra of an upper semi-continuous C^*-bundle over $\mathrm{Prim}(Z)$ (see, e.g., [18, Theorem C.26]). Therefore all ideals in Z (corresponding to the functions supported in an open set $U \subset \mathrm{Prim}(Z)$) can be uniquely extended to ideals in A. This fact is summarized in the following lemma.

LEMMA 3.2. *Let A be a unital C^*-algebra with center Z. Then $I \mapsto IA$ is an injective map from the ideals of Z to the ideals of A.*

As stated above, this fact can be used to extend results about commutative algebras to the noncommutative case. Although it relies on facts that have not yet been shown, we include the following result here to illustrate this point.

COROLLARY 3.3. *A C^*-algebra A with countably many ideals* (*in particular, any separable Noetherian C^*-algebra, by Theorem 4.10*) *has a finite dimensional center.*[8]

PROOF. Denote by Z the center of the (minimal) unitization of A (denoted $\tilde{A}$) and assume that Z is infinite dimensional. Then $\mathrm{Prim}(Z)$ is an infinite Hausdorff space, and thus (by Lemma 4.11) contains uncountably many open sets. So Z contains uncountably many ideals, and hence so does $\tilde{A}$, by Lemma 3.2. But unitization adds only a single point to $\mathrm{Prim}(A)$, and so A has uncountably many ideals if and only if the same is true of $\tilde{A}$.

4. Separable Noetherian C^*-algebras

In this section, we first review a theorem of Bratteli and Elliott which gives sufficient conditions for a topological space to be the primitive ideal space of a C^*-algebra. By exhibiting a few examples of infinite Noetherian (and Artinian) spaces satisfying the prescribed conditions, this theorem implies the existence of nontrivially Noetherian (and Artinian) C^*-algebras. In fact, we show that this result is exhaustive in the case of separable Noetherian C^*-algebras, in the sense that these algebras must have a countable sober Noetherian primitive ideal space, and conversely any countable sober Noetherian space is in fact the primitive ideal space of such an algebra.

Not only do Noetherian C^*-algebras have countably many ideals, but we also believe that C^*-algebras with only countably many ideals are not too far

[8] We hope to push this result further. Assuming that A is a unital Noetherian algebra, we can use the Dauns-Hofmann Theorem (specifically, the version given in [4, Theorem, p. 272]) to write $A = \bigoplus_{I \in \mathrm{Prim}(Z)} A/IA$, where Z is the center of A, since $\mathrm{Prim}(Z)$ is finite (and discrete). Of particular interest is the case where Z is one-dimensional.

removed from being Noetherian (especially if they are approximately finite-dimensional[9]) and we give a weakened converse to this fact. Of course, there are plenty of nontrivially Noetherian C^*-algebras that are not AF, since the (minimal) tensor product of any separable C^*-algebra with a simple, separable, exact C^*-algebra does not change the primitive ideal space (see [16, Theorem 5]).[10]

THEOREM 4.1 ([2, §5]). *If X is a sober space wherein the compact open sets form a countable base, then X is homeomorphic to the primitive ideal space of an AF algebra.*

It will be useful to note that the proof of Theorem 4.1 is constructive, in that it describes an algorithm for creating a Bratteli diagram that gives an AF algebra with the desired primitive ideal space. More specifically, if X is a space satisfying the conditions of Theorem 4.1, then [2, §4] gives a way of constructing a sequence $\mathscr{F}_0, \mathscr{F}_1, \ldots$ satisfying the following properties:

(i) Each $\mathscr{F}_n$ is a finite family of compact open subsets of X.
(ii) If an element $U \in \mathscr{F}_{n+1}$ is contained in the union $\bigcup_\alpha V_\alpha$ with each $V_\alpha \in \mathscr{F}_n$, then $U \subset V_{\alpha_0}$ for some α_0,
(iii) Each element of $\mathscr{F}_n$ is a union of elements of $\mathscr{F}_{n+1}$,
(iv) The elements $\bigcup_n \mathscr{F}_n$ form a basis for X.

If we define a relation between the elements of $\mathscr{F}_n$ and $\mathscr{F}_{n+1}$ (denoted $\mathscr{F}_n \searrow \mathscr{F}_{n+1}$) by saying that $V \in \mathscr{F}_n$ is related to $U \in \mathscr{F}_{n+1}$ if $U \subset V$, then the lattice $\mathscr{F}_0 \searrow \mathscr{F}_1 \searrow \cdots$ corresponds to a Bratteli diagram for an AF algebra with primitive ideal space X.

COROLLARY 4.2. *Nontrivially Noetherian C^*-algebras exist.*

EXAMPLE 4.3. Let k be a countable field (for example, the algebraic closure of a finite field). Our space is k^n. A basis for the family of closed sets is $\{(x_1, \ldots, x_n) \in k^n \mid p(x_1, \ldots, x_n) = 0\}$ where p ranges through the polynomials in $k[x_1, \ldots, x_n]$. It follows from general theorems in algebraic geometry (see, e.g., [9, §1.1]) that this space is Noetherian and T_1. However it is not sober in general, since non-singleton irreducible closed sets exist (e.g. solutions to $X_1 = 0$) but all points are closed. So we can adjoin dense points to all the irreducible closed sets (called *generic points* in [9]) to get a Noetherian, T_1

[9] Recall that an approximately finite-dimensional (AF) algebra A is a separable C^*-algebra that is the direct limit of finite-dimensional algebras.

[10] In [16], the result is shown when $A \otimes B$ satisfies the so-called 'Property (F)', which is satisfied if either A or B is exact. A recent proof using more modern language can be found by combining [1, Proposition 2.17 (2) and Proposition 2.16 (iii)].

and sober space with countably many points. As explained in Corollary 4.2, this space satisfies all the necessary properties of Theorem 4.1, and hence generates a Noetherian C^*-algebra. Similar constructions give many other spaces with the same properties, giving us a large supply of separable nontrivially Noetherian C^*-algebras.[11]

EXAMPLE 4.4. For a simpler example, it is straightforward to check that the space described in Example 2.1 (a countable space with one dense point and the cofinite topology on the complement of that point) is sober, Noetherian, and has a countable base of compact open sets. Notice, in particular, that (the complement of) each closed point corresponds to a maximal ideal. Arbitrary ideals are finite intersections of these maximal ideals, and any infinite intersection of maximal ideals is zero. In fact, since the proof of [2, §5] is constructive, we can give a more detailed description of AF algebras with this primitive ideal space in the following few paragraphs.[12]

First, label X by $X = \mathsf{N} \sqcup \{x_0\}$ (as a set), where N has the cofinite topology and x_0 is dense in X. We can construct one possible lattice satisfying conditions described after Theorem 4.1 by letting each $\mathscr{F}_n = \{U_E\}$ be the family of 2^n compact open subsets indexed by the subsets of $\{1, \dots, n\}$ (using the convention that $\mathscr{F}_0 = \{\emptyset\}$) defined by setting $U_E = X \setminus E$. Then a set $U_E \in \mathscr{F}_n$ is related to a set $U_F \in \mathscr{F}_{n+1}$ if $U_F \subset U_E$ (or equivalently if $E \subset F$). To simplify matters, we just set $\mathscr{F}_n = \mathscr{P}(\{1, \dots, n\})$ and say $E \searrow F$ if $E \subset F$. The corresponding Bratteli diagram (giving the AF algebra in Example 4.4) is shown in Figure 1. We label each node with the corresponding set, and the superscript at each node is the size of the matrix algebra (the smallest size possible suffices).

Note that many of the choices made in the construction of this particular Bratteli diagram are completely arbitrary. In general, there are many ways to construct a lattice of compact opens subsets of X in order to define an AF algebra with X as its primitive ideal space.

Using Figure 1, we can see that the corresponding dimension group is given by $\varinjlim G_n$, where $G_n \cong \mathsf{Z}^{2^n}$ (with basis $\{[S] \mid S \subset \{1, \dots, n\}\}$) and connecting homomorphisms $\varphi_n : G_n \to G_{n+1}$ given on the basis elements by $\varphi_n([S]) = \sum_{S \subset T} [T]$.

[11] More generally (as explained in e.g [6, p. 187]), any T_0 space X has a unique sobrification X^s, the space of irreducible closed subsets of X. The open subsets of X^s are of the form $\{E \in X^s \mid E \cap U \neq \emptyset\}$, where U ranges through the open subsets of X. X can naturally be embedded into X^s via the map $x \mapsto \overline{\{x\}}$. Since sobriety is an important condition in many of the results in Section 4, one interesting question is to investigate the sobrification of the primitive ideal space of a non-separable C^*-algebra.

[12] Many thanks to the reviewer for pointing out these facts, and the facts that follow.

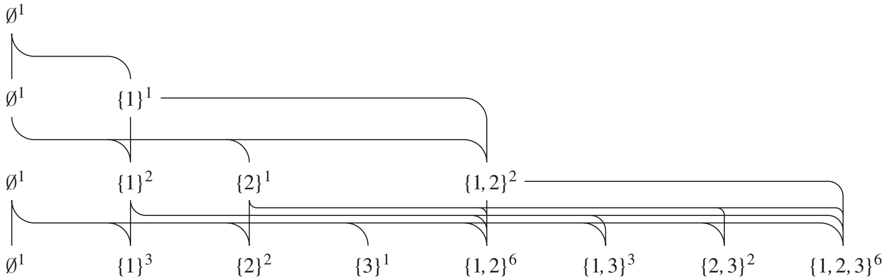

FIGURE 1. The Bratteli diagram of an AF algebra with primitive ideal space homeomorphic to the space described in Example 4.4.

We have so far not been able to come up with a more explicit description of the dimension group, and different lattice constructions have not made this computation much easier. However, if it can be shown in the case of Noetherian spaces that the dimension group corresponding to any lattice constructed using the method described in the paragraph following Example 4.4 depends only on the underlying space, then perhaps it can be shown that the primitive ideal space is a good (or even complete) invariant for Noetherian AF algebras.

COROLLARY 4.5. *Nontrivially Artinian C^*-algebras exist.*

EXAMPLE 4.6. Consider the lower topology (see, e.g., [7, Definition O-5.4]) on the natural numbers, where the basic open sets are the subsets of the form $U_n = \{k \mid k \leq n\}$ for $n \in \mathsf{N}$ (i.e. the closed sets are the rays, and the open sets have the form $\{0, \dots, n\}$.) Since any closed set has only finitely many closed supersets, this space is Artinian. Sobriety is clear since $\overline{\{n\}} = \{k \mid n \leq k\}$. A similar example was considered in the context of C^*-algebras in [10, Example 1.1].

THEOREM 4.7. *Every countable sober Noetherian (Artinian) space is the primitive ideal space of a Noetherian (Artinian) C^*-algebra.*

We prove the Noetherian case first, using the following topological lemma.

LEMMA 4.8. *Every infinite sober Noetherian space has the same number of points as open sets.*

PROOF. If X is a Noetherian space, then any closed subset is the union of its irreducible components, as shown in Lemma 2.4. Since X is sober, each of the terms in this decomposition is a point closure. So we see that there is a bijective correspondence between closed sets and finite sets of points. If X is infinite, then the family of finite sets of points has the same cardinality as X.

PROOF OF THEOREM 4.7 IN THE NOETHERIAN CASE. By Theorem 4.1, it suffices to show that the compact open subsets form a countable base. But any countable Noetherian space has at most countably many open sets by Lemma 4.8, and each of these is compact by Lemma 2.3.

For the Artinian case, we use a similar technical lemma.

LEMMA 4.9. *A sober Artinian space with countably many points has only countably many compact open subsets.*

PROOF. We show that each compact open set can be decomposed into a finite union in a 'canonical' way. Let V be compact and open, and for each $x \in X$ denote by U_x the minimal neighborhood of x. Then $\{U_x\}_{x \in V}$ is an open cover of V. This has a finite subcover, and so V can be written as finite union of minimal neighborhoods. Since the collection of all finite unions of the U_x's is countable (since X is), X has only countably many compact open subsets.

PROOF OF THEOREM 4.7 IN THE ARTINIAN CASE. By Theorem 4.1, it suffices to show that the compact open subsets form a countable base. But this follows immediately by Lemma 4.9, after noting that the minimal neighborhood around each point is compact.

Now that we know nontrivially Noetherian (and Artinian) separable C^*-algebras exist, we begin an investigation into their properties. As is the case with purely algebraic rings, the condition of being Noetherian puts a control on their ideal structure. However, due to the stringent topological conditions that are put on the primitive ideal space of Noetherian C^*-algebras, being Noetherian is a much stronger condition in this context. A result that demonstrates this fact is given below.

THEOREM 4.10. *Every separable Noetherian C^*-algebra has only countably many ideals.*[13]

PROOF. Let A be separable Noetherian C^*-algebra containing uncountably many ideals. It follows from the Lemma 4.8 that $\mathrm{Prim}(A)$ contains uncountably many points. Let Π be the collection of uncountable irreducible subsets of $\mathrm{Prim}(A)$. By Lemma 2.4, $\mathrm{Prim}(A)$ can be decomposed into finitely many irreducible components. At least one of these must be uncountable, so Π is nonempty. Let E be a minimal element of Π. So E is uncountable and every proper closed subset of E is countable, and hence every closed subset of E is either countable or cocountable.

[13] We would like to extend this result to Artinian C^*-algebras as well. However, a 'finite decomposition' result analogous to Lemma 2.4 is missing.

Now, by a result due primarily to Effros [5, Theorem 2.4],[14] the Borel structure on the primitive ideal space (with hull-kernel topology) of a separable C^*-algebra with uncountably many ideals is isomorphic to the unit interval with the standard Borel structure. Such an isomorphism takes E to an uncountable Borel subset of the unit interval. But the countable-cocountable σ-algebra on E does not agree with the standard Borel σ-algebra on any uncountable subset of the unit interval, so we have a contradiction.

Notice that Theorem 4.10 does not have a direct converse, since there exist non-Noetherian C^*-algebras with only countably many ideals. One counter-example is in the proof of Corollary 4.5, where we define a non-Noetherian (although Artinian) topological space which is the primitive ideal space of an AF algebra with only countably many ideals.

However, it seems that having only countably many ideals is fairly close to being Noetherian. We can show a partial equivalence, which is that a separable C^*-algebra is Noetherian if and only if it has only countably many ideals and it has no infinite strictly ascending sequence of primitive ideals. The latter condition can be viewed as a weak version of being Noetherian. We will first show that if A is a separable C^*-algebra with only countably many ideals, than any ideal in A is a finite intersection of primitive ideals, which is to say that any closed set in $\mathrm{Prim}(A)$ can be decomposed into finitely many irreducible components. We use the following lemma.

LEMMA 4.11. *Any infinite Hausdorff space contains uncountably many open sets.*

PROOF. It suffices to show that an infinite Hausdorff space has infinitely many disjoint open sets, since if $\{U_\alpha\}_{\alpha\in A}$ were such a collection, then $\left\{\bigcup_{\alpha\in E} U_\alpha\right\}_{E\in 2^A}$ would be an uncountable collection of open sets.

Let X be an infinite Hausdorff space. If X contains infinitely many isolated points then these points form an infinite family of disjoint open sets. So we may suppose that X contains only finitely many isolated points. So X contains a limit point, call it z. Let x_1 be a point other than z. Let U_1 and V_1 be open sets separating x_1 and z respectively. As z is a limit point, V_1 contains a point other than z, call it x_2. Again, let U_2 and V_2 be open subsets of V_1 separating x_2 and z. We continue in this manner and produce a sequence of open sets $\{U_n\}_{n\in\mathbb{N}}$. It is easy to see that these open sets are disjoint.

Next, recall that a point x is *separated* if and only if it can be separated from each point outside its closure by disjoint open sets.

A theorem of Dixmier [3, 3.9.4] states that in any separable C^*-algebra, the separated points form a dense set in $\mathrm{Prim}(A)$. Before we use this result to

[14] A very nice proof of this result is due to Dixmier, and is given in [18, Theorem H.39].

prove Lemma 4.12, first note that a separated point in a T_0 space cannot lie in the closure of another point. If a point x is separated and lies in the closure of a point y, then y cannot be in the closure of x, so x and y can be separated with open sets, which is impossible.

LEMMA 4.12. *If a separable C^*-algebra A has countably many ideals, then every closed subset of* $\mathrm{Prim}(A)$ *is the union of finitely many irreducible subsets.*

PROOF. Let X be the set of separated points in $\mathrm{Prim}(A)$. Since no point in X can lie in the closure of any other point in X, any two points in X can be separated with open sets. Thus X is Hausdorff, and hence finite by Lemma 4.11. Every space is the union of closed irreducible components[15] (in particular, $\mathrm{Prim}(A)$ is) and since A is separable, $\mathrm{Prim}(A)$ is sober, and hence $\mathrm{Prim}(A)$ can be written as a union of point-closures. Since X is finite, it can be decomposed into finitely many irreducible subsets, the closures of the separated points. Therefore, since $\mathrm{Prim}(A) = \overline{X} = \bigcup_{x \in X} \overline{\{x\}}$ by [3, 3.9.4], the same is true for $\mathrm{Prim}(A)$. Now if A contains only countably many ideals then any quotient of A contains only countably many ideals, so we can apply the same argument to any closed set in $\mathrm{Prim}(A)$ and deduce the general statement.

THEOREM 4.13. *A separable C^*-algebra A is Noetherian if and only if it contains countably many ideals and has no infinite strictly ascending chain of primitive ideals.*[16]

PROOF. One direction is obvious by Theorem 4.10. We show the converse using Lemma 4.12.

Let A be a separable C^*-algebra with countably many ideals and no infinite strictly ascending chain of primitive ideals. By contradiction, say A is not Noetherian, with $E_1 \supset E_2 \supset \cdots$ an infinite strictly descending sequence of closed sets in $\mathrm{Prim}(A)$. By Lemma 4.12, we can express each E_n as the union of a finite family $\mathscr{F}_n$ of irreducibles. Let $\mathscr{T}$ be the family of all the sets in all of the $\mathscr{F}_n$'s. We construct a tree whose nodes are the sets in $\mathscr{T}$. Two elements of $\mathscr{T}$ are connected when one is contained in the other and there are no sets in $\mathscr{T}$ that lie between them. It is easy to see that this is in fact a tree and that every node has finite degree. So by Kőnig's Lemma this tree contains an infinite branch, which is nothing other than an infinite descending chain of irreducible closed subsets.

[15] This can be seen since, in particular, every singleton is irreducible, and hence is contained in a maximal closed irreducible set (by Zorn's lemma). As shown in Lemma 2.4, this union is unique (and finite) in the case of Noetherian spaces, although not in general.

[16] The latter notion can be viewed as a weak version of being Noetherian. It is equivalent to saying that $\mathrm{Prim}(A)$ contains no infinite strictly descending sequence of irreducible closed subsets.

Since each irreducible closed subset is the closure of a unique point (because A was assumed to be separable, and hence $\mathrm{Prim}(A)$ sober), we have an infinite ascending chain of primitive ideals in A, and thus a contradiction.

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