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A TAYLOR-LIKE EXPANSION OF A COMMUTATOR WITH A FUNCTION OF SELF-ADJOINT, PAIRWISE COMMUTING OPERATORS

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Abstract

Let A be a ν-vector of self-adjoint, pairwise commuting operators and B a bounded operator of class $C^{n_0}(A)$. We prove a Taylor-like expansion of the commutator $[B, f(A)]$ for a large class of functions $f\colon \mathsf{R}^\nu \to \mathsf{R}$, generalising the one-dimensional result where A is just a self-adjoint operator. This is done using almost analytic extensions and the higher-dimensional Helffer-Sjöstrand formula.

1. Introduction

It is well-known that if A is a self-adjoint operator, B is a bounded operator of class $C^{n_0}(A)$ in the sense of [1] and f satisfies $|f^{(n)}(x)| \leq C_n \langle x\rangle^{s-n}$ for all n, then for $0 \leq t_1 \leq n_0$, $0 \leq t_2 \leq 1$ with $s + t_1 + t_2 < n_0$,

$$[B, f(A)] = \sum_{k=1}^{n_0-1} \frac{1}{k!} f^{(k)}(A) \operatorname{ad}_A^k(B) + R_{n_0}(A, B)$$

where $\operatorname{ad}_A^k(B)$ is the k'th iterated commutator, $R_{n_0}(A, B) \in \mathscr{B}(\mathscr{H}_A^{-t_2}; \mathscr{H}_A^{t_1})$ and $\mathscr{H}_A^t$ is defined as $\mathscr{D}(\langle A\rangle^t)$ equipped with the graph-norm $\|v\|_t = \|\langle A\rangle^t v\|$ for $t \geq 0$ and $\mathscr{H}_A^{-t}$ is the dual space of $\mathscr{H}_A^t$. This follows relatively easily from using the (one-dimensional) Helffer-Sjöstrand formula

$$f(A) = \frac{1}{\pi} \int_{\mathsf{C}} \bar\partial \tilde f(z)(A - z)^{-1} dz, \tag{1}$$

where $\bar\partial = \frac{1}{2}(\partial_x + i\partial_y)$ and $\tilde f$ is an almost analytic extension of f, and the identity

$$\begin{aligned}[B, f(A)] = \sum_{k=1}^{n_0-1} \frac{1}{k!} \frac{k!}{\pi} \int_{\mathsf{C}} \bar\partial \tilde f(z)(-1)^k (A - z)^{-k-1} dz \operatorname{ad}_A^k(B) \\ + \frac{(-1)^{n_0}}{\pi} \int_{\mathsf{C}} \bar\partial \tilde f(z)(A - z)^{-n_0} \operatorname{ad}_A^{n_0}(B)(A - z)^{-1} dz\end{aligned}$$

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when $\frac{k!}{\pi}\int_{\mathsf{C}}\bar\partial\tilde f(z)(-1)^k(A-z)^{-k-1}dz$ is recognised as $f^{(k)}(A)$ using (1). Such commutator expansions where first proved in [7]. See, e.g., [4] for details. Due to the higher complexity of the general Helffer-Sjöstrand formula, these calculations do not lead directly to the generalised result where A is a vector of self-adjoint, pairwise commuting operators. However, we will follow the same idea.

The theorem may be viewed as an abstract analogue of pseudo-differential calculus. The one-dimensional version is an often used result, see, e.g., [2] and [4]. Apart from the obvious interest in generalising the result to higher dimensions, our improvement has proven useful in the treatment of models in quantum field theory, see [6]. In particular, a lemma in [6] whose proof depends on our result, extends the results of [5] to a larger class of models.

2. The setting and result

In the following, $A = (A_1, \dots, A_\nu)$ is a vector of self-adjoint, pairwise commuting operators acting on a Hilbert space $\mathscr{H}$, and $B \in \mathscr{B}(\mathscr{H})$ is a bounded operator on $\mathscr{H}$. We shall use the notion of B being of class $C^{n_0}(A)$ introduced in [1]. For notational convenience, we adopt the following convention: If $0 \le j \le \nu$, then δ_j denotes the multi-index $(0, \dots, 0, 1, 0, \dots, 0)$, where the 1 is in the j'th entry.

Definition 1. Let $n_0 \in \mathsf{N} \cup \{\infty\}$. Assume that the multi-commutator form defined iteratively by $\mathrm{ad}_A^0(B) = B$ and $\mathrm{ad}_A^\alpha(B) = [\mathrm{ad}_A^{\alpha-\delta_j}(B), A_j]$ as a form on $\mathscr{D}(A_j)$, where $\alpha \ge \delta_j$ is a multi-index and $1 \le j \le \nu$, can be represented by a bounded operator also denoted by $\mathrm{ad}_A^\alpha(B)$, for all multi-indices α, $|\alpha| < n_0+1$. Then B is said to be of class $C^{n_0}(A)$ and we write $B \in C^{n_0}(A)$.

Remark 2. The definition of $\mathrm{ad}_A^\alpha(B)$ does not depend on the order of the iteration since the A_j are pairwise commuting. We call $|\alpha|$ the *degree* of $\mathrm{ad}_A^\alpha(B)$.

In the following, $\mathscr{H}_A^s := D(\langle A\rangle^s)$ for $s \ge 0$ will be used to denote the scale of spaces associated to A. For negative s, we define $\mathscr{H}_A^s := (\mathscr{H}_A^{-s})^*$.

Theorem 3. *Assume that* $B \in C^{n_0}(A)$ *for some* $n_0 \ge n+1 \ge 1$, $0 \le t_1 \le n+1$, $0 \le t_2 \le 1$ *and that* $\{f_\lambda\}_{\lambda\in I}$ *satisfies*

$$\forall\alpha\ \exists C_\alpha\colon |\partial^\alpha f_\lambda(x)| \le C_\alpha \langle x\rangle^{s-|\alpha|}$$

uniformly in λ *for some* $s \in \mathsf{R}$ *such that* $t_1 + t_2 + s < n+1$. *Then*

$$[B, f_\lambda(A)] = \sum_{|\alpha|=1}^{n} \frac{1}{\alpha!}\partial^\alpha f_\lambda(A)\ \mathrm{ad}_A^\alpha(B) + R_{\lambda,n}(A,B)$$

as an identity on $\mathscr{D}(\langle A\rangle^s)$, where $R_{\lambda,n}(A,B) \in \mathscr{B}(\mathscr{H}_A^{-t_2}, \mathscr{H}_A^{t_1})$ and there exist a constant C independent of A, B and λ such that

$$\|R_{\lambda,n}(A,B)\|_{\mathscr{B}(\mathscr{H}_A^{-t_2},\mathscr{H}_A^{t_1})} \le C \sum_{|\alpha|=n+1} \|\mathrm{ad}_A^\alpha(B)\|.$$

REMARK 4. A similar statement holds with the $\mathrm{ad}_A^\alpha(B)$ and $\partial^\alpha f_\lambda(A)$ interchanged at the cost of a sign correction given by $(-1)^{|\alpha|-1}$, and the corresponding remainder term $R'_{\lambda,n}(A,B) \in \mathscr{B}(\mathscr{H}_A^{-t_1}, \mathscr{H}_A^{t_2})$. This can be seen either by proving it analogously or by taking the adjoint equation and replacing B by $-B$.

REMARK 5. If $k \le t_1$ and $n_0 \ge n+1+k$, then $R_{\lambda,n}(A,B)$ can be replaced by $R^k_{\lambda,n}(A,B) \in \mathscr{B}(\mathscr{H}_A^{-t_2+k}, \mathscr{H}_A^{t_1-k})$. This can be seen by commuting $|A-z|^{-2}$ and $\mathrm{ad}_A^\alpha(B)$ in the terms of the remainder, see page 114.

3. The Proof

Let $z \in \mathsf{C}^\nu$, $\operatorname{Im} z \ne 0$, $1 \le \ell \le \nu$ and $g, g_\ell \colon \mathsf{R}^\nu \to \mathsf{C}$ be given as $g(t) = |t-z|^{-2}$ and $g_\ell(t) = t_\ell - \bar{z}_\ell$. Write for $2\beta \le \alpha$

$$T_\alpha^\beta(t,z) := \tfrac{(-2)^{|\alpha-\beta|}|\alpha-\beta|!}{2^{|\beta|}\beta!(\alpha-2\beta)!}(t-\operatorname{Re} z)^{\alpha-2\beta}|t-z|^{-2|\alpha-\beta|}.$$

LEMMA 6. *Let g be as above and α be any multi-index. Then*

$$\partial^\alpha g(t) = \sum_{2\beta\le\alpha} \alpha! T_\alpha^\beta(t,z)|t-z|^{-2}.$$

PROOF. For brevity, we will write α^i or β^i for $\alpha+\delta_i$ or $\beta+\delta_i$, respectively. The formula is obviously true for $\alpha = 0$. Now assume that we have proven the formula for $|\alpha| \le k$. Let $|\alpha| = k$ and $0 \le i \le \nu$ be arbitrary. It suffices to prove the formula for α^i. One easily verifies using the chain rule that

$$(\partial^{\delta_i} g^n)(t) = -2n(t_i - \operatorname{Re} z_i)|t-z|^{-2n-2}. \tag{2}$$

Now by the induction hypothesis, we see that

$$\partial^{\alpha+\delta_i} g(t) = \partial_t^{\delta_i} \sum_{2\beta\le\alpha} \tfrac{(-2)^{|\alpha-\beta|}\alpha!|\alpha-\beta|!}{2^{|\beta|}\beta!(\alpha-2\beta)!}(t-\operatorname{Re} z)^{\alpha-2\beta}|t-z|^{-2|\alpha-\beta|-2}$$

$$= \sum_{2\beta\le\alpha} \tfrac{(-2)^{|\alpha-\beta|}\alpha!|\alpha-\beta|!}{2^{|\beta|}\beta!(\alpha-2\beta)!}(\partial_t^{\delta_i}(t-\operatorname{Re} z)^{\alpha-2\beta})|t-z|^{-2|\alpha-\beta|-2} \tag{3}$$

$$+ \sum_{2\beta\le\alpha} \tfrac{(-2)^{|\alpha-\beta|}\alpha!|\alpha-\beta|!}{2^{|\beta|}\beta!(\alpha-2\beta)!}(t-\operatorname{Re} z)^{\alpha-2\beta}(\partial_t^{\delta_i}|t-z|^{-2|\alpha-\beta|-2}). \tag{4}$$

For the sake of clarity, we will now consider each sum independently.

$$(3) = \sum_{2\beta\le\alpha} \tfrac{(-2)^{|\alpha-\beta|}\alpha!|\alpha-\beta|!}{2^{|\beta|}\beta!(\alpha-2\beta)!}(\alpha_i - 2\beta_i)(t - \operatorname{Re} z)^{\alpha-2\beta-\delta_i}|t - z|^{-2|\alpha-\beta|-2}$$

$$= \sum_{\substack{2\beta\le\alpha\\ 2\beta_i<\alpha_i}} 2(\beta_i + 1)\tfrac{(-2)^{|\alpha^i-\beta^i|}\alpha!|\alpha^i-\beta^i|!}{2^{|\beta^i|}\beta^i!(\alpha^i-2\beta^i)!}(t - \operatorname{Re} z)^{\alpha^i-2\beta^i}|t - z|^{-2|\alpha^i-\beta^i|-2}$$

$$(5) \quad = \sum_{2\beta\le\alpha+\delta_i} 2\beta_i \tfrac{(-2)^{|\alpha^i-\beta|}\alpha!|\alpha^i-\beta|!}{2^{|\beta|}\beta!(\alpha^i-2\beta)!}(t - \operatorname{Re} z)^{\alpha^i-2\beta}|t - z|^{-2|\alpha^i-\beta|-2}.$$

Using (2), we see that (4) equals

$$\sum_{2\beta\le\alpha} \tfrac{(-2)^{|\alpha-\beta|}\alpha!|\alpha-\beta|!}{2^{|\beta|}\beta!(\alpha-2\beta)!}(t-\operatorname{Re} z)^{\alpha-2\beta}(-2)(|\alpha-\beta| + 1)(t_i-\operatorname{Re} z_i)|t-z|^{-2|\alpha-\beta|-4}$$

$$= \sum_{2\beta\le\alpha}(\alpha_i + 1 - 2\beta_i)\tfrac{(-2)^{|\alpha^i-\beta|}\alpha!|\alpha^i-\beta|!}{2^{|\beta|}\beta!(\alpha^i-2\beta)!}(t - \operatorname{Re} z)^{\alpha^i-2\beta}|t - z|^{-2|\alpha^i-\beta|-2}$$

$$(6) \quad = \sum_{2\beta\le\alpha} \tfrac{(-2)^{|\alpha^i-\beta|}\alpha^i!|\alpha^i-\beta|!}{2^{|\beta|}\beta!(\alpha^i-2\beta)!}(t - \operatorname{Re} z)^{\alpha^i-2\beta}|t - z|^{-2|\alpha^i-\beta|-2}$$

$$(7) \quad - \sum_{2\beta\le\alpha} 2\beta_i \tfrac{(-2)^{|\alpha^i-\beta|}\alpha!|\alpha^i-\beta|!}{2^{|\beta|}\beta!(\alpha^i-2\beta)!}(t - \operatorname{Re} z)^{\alpha^i-2\beta}|t - z|^{-2|\alpha^i-\beta|-2}.$$

Now (7) cancels (5) except for possible terms with $2\beta = \alpha + \delta_i$:

$$(8) \quad (5) + (7) = \sum_{2\beta=\alpha+\delta_i} \tfrac{(-2)^{|\alpha^i-\beta|}\alpha^i!|\alpha^i-\beta|!}{2^{|\beta|}\beta!(\alpha^i-2\beta)!}(t - \operatorname{Re} z)^{\alpha^i-2\beta}|t - z|^{-2|\alpha^i-\beta|-2}.$$

Adding (6) and (8) finishes the induction.

LEMMA 7. *Let $B \in C^{n_0}(A)$ for some $n_0 \ge 1$ and let $n \in \mathsf{N}_0$ and α_0 be a multi-index satisfying $|\alpha_0| + n + 1 \le n_0$. Then*

$$(9) \quad [\operatorname{ad}_A^{\alpha_0}(B), g(A)] = \sum_{|\alpha|=1}^{n} \frac{1}{\alpha!}\partial^\alpha g(A)\operatorname{ad}_A^{\alpha_0+\alpha}(B) + R_n^g(A, \operatorname{ad}_A^{\alpha_0}(B)),$$

where

$$R_n^g(A, \operatorname{ad}_A^{\alpha_0}(B))$$

$$(10) \quad = \sum_{|\alpha|=n-1}\sum_{2\beta\le\alpha}\sum_{i=1}^{\nu} \tfrac{\beta_i+1}{|\alpha+\delta_i-\beta|}T_{\alpha+2\delta_i}^{\beta+\delta_i}(A, z)\operatorname{ad}_A^{\alpha_0+\alpha+2\delta_i}(B)|A - z|^{-2}$$

$$(11)\qquad + \sum_{|\alpha|=n} \sum_{2\beta\le\alpha} \sum_{i=1}^{\nu} \tfrac{\beta_i+1}{|\alpha+\delta_i-\beta|} T^{\beta+\delta_i}_{\alpha+2\delta_i}(A,z)(A_i-\bar{z}_i)\,\mathrm{ad}_A^{\alpha_0+\alpha+\delta_i}(B)|A-z|^{-2}$$

$$(12)\qquad + \sum_{|\alpha|=n} \sum_{2\beta\le\alpha} \sum_{i=1}^{\nu} \tfrac{\beta_i+1}{|\alpha+\delta_i-\beta|} T^{\beta+\delta_i}_{\alpha+2\delta_i}(A,z)\,\mathrm{ad}_A^{\alpha_0+\alpha+\delta_i}(B)(A_i-z_i)|A-z|^{-2}.$$

PROOF. The proof goes by induction. One may check by inspection of the following identity that the statement is true for $n=0$.

$$(13)\qquad \begin{aligned}&[\mathrm{ad}_A^{\alpha_0}(B), |A-z|^{-2}]\\ &\quad= -\sum_{i=1}^{\nu} |A-z|^{-2}(A_i-\bar{z}_i)\,\mathrm{ad}_A^{\alpha_0+\delta_i}(B)|A-z|^{-2}\\ &\qquad -\sum_{i=1}^{\nu} |A-z|^{-2}\,\mathrm{ad}_A^{\alpha_0+\delta_i}(B)(A_i-z_i)|A-z|^{-2}.\end{aligned}$$

Now assume that we have proven the formula for $k\le n$, $|\alpha_0|+n+2\le n_0$. We will now show that this implies that the formula holds for $k=n+1$. We begin by noting two useful identities.

$$(14)\qquad T^{\beta}_{\alpha}(t,z)|t-z|^{-2} = -\tfrac{\beta_j+1}{|\alpha+\delta_j-\beta|} T^{\beta+\delta_j}_{\alpha+2\delta_j}(t,z), \qquad \forall j.$$

$$(15)\qquad (\beta_i+1)T^{\beta+\delta_i}_{\alpha+2\delta_i}(t,z)2(t_i-\mathrm{Re}\,z_i) = (\alpha_i+1-2\beta_i)T^{\beta}_{\alpha+\delta_i}(t,z).$$

Now using (13) and (14) we see that

$$(16)\qquad (10) = \sum_{|\alpha|=n-1} \sum_{2\beta\le\alpha} \sum_{i=1}^{\nu} \tfrac{\beta_i+1}{|\alpha+\delta_i-\beta|} T^{\beta+\delta_i}_{\alpha+2\delta_i}(A,z)|A-z|^{-2}\,\mathrm{ad}_A^{\alpha_0+\alpha+2\delta_i}(B)$$

$$(17)\qquad \begin{aligned}&+ \sum_{|\alpha|=n-1} \sum_{2\beta\le\alpha} \sum_{i=1}^{\nu}\sum_{j=1}^{\nu} \tfrac{\beta_i+1}{|\alpha+\delta_i-\beta|}\,\tfrac{\beta_j+\delta_{ij}+1}{|\alpha+\delta_i+\delta_j-\beta|} T^{\beta+\delta_i+\delta_j}_{\alpha+2\delta_i+2\delta_j}(A,z)\\ &\qquad\times (A_j-\bar{z}_j)\,\mathrm{ad}_A^{\alpha_0+\alpha+2\delta_i+\delta_j}(B)|A-z|^{-2}\end{aligned}$$

$$(18)\qquad \begin{aligned}&+ \sum_{|\alpha|=n-1} \sum_{2\beta\le\alpha} \sum_{i=1}^{\nu}\sum_{j=1}^{\nu} \tfrac{\beta_i+1}{|\alpha+\delta_i-\beta|}\,\tfrac{\beta_j+\delta_{ij}+1}{|\alpha+\delta_i+\delta_j-\beta|} T^{\beta+\delta_i+\delta_j}_{\alpha+2\delta_i+2\delta_j}(A,z)\\ &\qquad\times \mathrm{ad}_A^{\alpha_0+\alpha+2\delta_i+\delta_j}(B)(A_j-z_j)|A-z|^{-2},\end{aligned}$$

and by reordering and reindexing the sum in (16), (17) and (18), we get

$$(16) = \sum_{i=1}^{\nu} \sum_{\substack{|\alpha|=n+1 \\ \alpha_i \geq 2}} \sum_{\substack{2\beta\leq\alpha \\ \beta_i \geq 1}} \tfrac{\beta_i}{|\alpha-\beta|} T_\alpha^\beta(A,z)|A-z|^{-2} \operatorname{ad}_A^{\alpha_0+\alpha}(B), \tag{19}$$

and (17) equals

$$\begin{aligned}\sum_{i=1}^{\nu} \sum_{\substack{|\alpha|=n+1 \\ \alpha_i \geq 2}} \sum_{\substack{2\beta\leq\alpha \\ \beta_i \geq 1}} \sum_{j=1}^{\nu} \tfrac{\beta_i}{|\alpha-\beta|} \tfrac{\beta_j+1}{|\alpha+\delta_j-\beta|} T_{\alpha+2\delta_j}^{\beta+\delta_j}(A,z) \\ \times (A_j-\bar{z}_j) \operatorname{ad}_A^{\alpha_0+\alpha+\delta_j}(B)|A-z|^{-2}\end{aligned} \tag{20}$$

and similarly for (18) with $(A_j - \bar{z}_j) \operatorname{ad}_A^{\alpha_0+\alpha+\delta_j}(B)$ replaced by $\operatorname{ad}_A^{\alpha_0+\alpha+\delta_j}(B)(A_j - z_j)$. Note that we may relax the extra conditions on α and β in the above statements, as a term with $\beta_i = 0$ contributes nothing.

Instead of continuing in the same fashion with (11) and (12), we note using (15) that

$$(11)+(12) = \sum_{|\alpha|=n} \sum_{2\beta\leq\alpha} \sum_{i=1}^{\nu} \tfrac{\beta_i+1}{|\alpha+\delta_i-\beta|} T_{\alpha+2\delta_i}^{\beta+\delta_i}(A,z) \operatorname{ad}_A^{\alpha_0+\alpha+2\delta_i}(B)|A-z|^{-2} \tag{21}$$

$$+ \sum_{|\alpha|=n} \sum_{2\beta\leq\alpha} \sum_{i=1}^{\nu} \tfrac{\alpha_i+1-2\beta_i}{|\alpha+\delta_i-\beta|} T_{\alpha+\delta_i}^{\beta}(A,z) \operatorname{ad}_A^{\alpha_0+\alpha+\delta_i}(B)|A-z|^{-2}, \tag{22}$$

so we may focus our attention on (22):

$$(22) = \sum_{i=1}^{\nu} \sum_{\substack{|\alpha|=n+1 \\ \alpha_i \geq 1}} \sum_{\substack{2\beta\leq\alpha \\ 2\beta_i < \alpha_i}} \tfrac{\alpha_i-2\beta_i}{|\alpha-\beta|} T_\alpha^\beta(A,z)|A-z|^{-2} \operatorname{ad}_A^{\alpha_0+\alpha}(B) \tag{23}$$

$$\begin{aligned}+\sum_{i=1}^{\nu} \sum_{\substack{|\alpha|=n+1 \\ \alpha_i \geq 1}} \sum_{\substack{2\beta\leq\alpha \\ 2\beta_i < \alpha_i}} \sum_{j=1}^{\nu} \tfrac{\alpha_i-2\beta_i}{|\alpha-\beta|} \tfrac{\beta_j+1}{|\alpha+\delta_j-\beta|} T_{\alpha+2\delta_j}^{\beta+\delta_j}(A,z) \\ \times (A_j-\bar{z}_j) \operatorname{ad}_A^{\alpha_0+\alpha+\delta_j}(B)|A-z|^{-2}.\end{aligned} \tag{24}$$

$$\begin{aligned}+\sum_{i=1}^{\nu} \sum_{\substack{|\alpha|=n+1 \\ \alpha_i \geq 1}} \sum_{\substack{2\beta\leq\alpha \\ 2\beta_i < \alpha_i}} \sum_{j=1}^{\nu} \tfrac{\alpha_i-2\beta_i}{|\alpha-\beta|} \tfrac{\beta_j+1}{|\alpha+\delta_j-\beta|} T_{\alpha+2\delta_j}^{\beta+\delta_j}(A,z) \\ \times \operatorname{ad}_A^{\alpha_0+\alpha+\delta_j}(B)(A_j-z_j)|A-z|^{-2}\end{aligned} \tag{25}$$

We note again that the additional conditions on α and β are superfluous.

We may now recollect the terms. First we see using Lemma 6:

$$\sum_{|\alpha|=1}^{n} \frac{1}{\alpha!}\partial^\alpha g(A)\,\mathrm{ad}_A^{\alpha_0+\alpha}(B)+(19)+(23)=\sum_{|\alpha|=1}^{n+1}\frac{1}{\alpha!}\partial^\alpha g(A)\,\mathrm{ad}_A^{\alpha_0+\alpha}(B), \tag{26}$$

then

$$\begin{aligned}&(20)+(24)\\ &=\sum_{|\alpha|=n+1}\sum_{2\beta\le\alpha}\sum_{j=1}^{\nu}\tfrac{\beta_j+1}{|\alpha+\delta_j-\beta|}T^{\beta+\delta_j}_{\alpha+2\delta_j}(A,z)(A_j-\bar z_j)\,\mathrm{ad}_A^{\alpha_0+\alpha+\delta_j}(B)|A-z|^{-2},\end{aligned} \tag{27}$$

and

$$\begin{aligned}&(18)+(25)\\ &=\sum_{|\alpha|=n+1}\sum_{2\beta\le\alpha}\sum_{j=1}^{\nu}\tfrac{\beta_j+1}{|\alpha+\delta_j-\beta|}T^{\beta+\delta_j}_{\alpha+2\delta_j}(A,z)\,\mathrm{ad}_A^{\alpha_0+\alpha+\delta_j}(B)(A_j-z_j)|A-z|^{-2},\end{aligned} \tag{28}$$

so adding up, we have proved that (9) equals the sum of (26), (21), (27) and (28) as stated.

The following lemma plays the same role for g_ℓ as Lemma 7 plays for g, but contrary to Lemma 7, the proof is trivial.

LEMMA 8. *For $n \in \mathsf{N}_0$ and $|\alpha_0|+1 \le n_0$*

$$[\mathrm{ad}_A^{\alpha_0}(B), g_\ell(A)] = \sum_{|\alpha|=1}^{n}\frac{1}{\alpha!}\partial^\alpha g_\ell(A)\,\mathrm{ad}_A^{\alpha_0+\alpha}(B) + R_n^{g_\ell}(A, \mathrm{ad}_A^{\alpha_0}(B)),$$

where $R_n^{g_\ell}(A, \mathrm{ad}_A^{\alpha_0}(B)) = 0$ for $n \ge 1$, $R_0^{g_\ell}(A, \mathrm{ad}_A^{\alpha_0}(B)) = \mathrm{ad}_A^{\alpha_0+\delta_\ell}(B)$.

The following lemma also follows by induction.

LEMMA 9. *Assume $B \in C^{n_0}(A)$ and that $h_i \in C^\infty(\mathsf{R}^\nu)$, $1 \le i \le k$, satisfies*

$$[\mathrm{ad}_A^{\alpha_0}(B), h_i(A)] = \sum_{|\alpha|=1}^{n}\frac{1}{\alpha!}\partial^\alpha h_i(A)\,\mathrm{ad}_A^{\alpha_0+\alpha}(B) + R_n^{h_i}(A, \mathrm{ad}_A^{\alpha_0}(B)),$$

where $R_n^{h_i}(A, \mathrm{ad}_A^{\alpha_0}(B))$ is bounded for all $n + |\alpha_0| \le n_0$ and $\partial^\alpha h_i(A)$

is bounded for all $0 \le |\alpha| + 1 \le n_0$. *Then*

$$\Big[B, \prod_{i=1}^{k} h_i(A)\Big] = \sum_{|\alpha|=1}^{n} \frac{1}{\alpha!}\partial^\alpha\Big(\prod_{i=1}^{k} h_i\Big)(A)\,\mathrm{ad}_A^\alpha(B) + \sum_{j=1}^{k}\sum_{|\alpha|=0}^{n} \frac{1}{\alpha!}\partial^\alpha\Big(\prod_{i=1}^{j-1} h_i\Big)(A) R^{h_j}_{n-|\alpha|}(A, \mathrm{ad}_A^\alpha(B)) \prod_{i=j+1}^{k} h_i(A).$$

Let $n+1 \le n_0$. If we put $k = \nu + 1$, $h_i = g$ for $i \ne \nu$, $h_\nu = g_\ell$ and apply Lemmas 7, 8 and 9 we see that

$$[B, |A-z|^{-2\nu}(A_\ell - \bar z_\ell)] = \sum_{|\alpha|=1}^{n} \frac{1}{\alpha!}\partial^\alpha\big(|\cdot - z|^{-2\nu}(\cdot_\ell - \bar z_\ell)\big)(A)\,\mathrm{ad}_A^\alpha(B) + R_{\ell,n}(A,B), \tag{29}$$

where

$$R_{\ell,n}(A,B) = \sum_{j=1}^{\nu-1}\sum_{|\alpha|=0}^{n} \frac{1}{\alpha!}\partial^\alpha(g^{j-1})(A) R^g_{n-|\alpha|}(A, \mathrm{ad}_A^\alpha(B))|A-z|^{-2(\nu-j)}(A_\ell - \bar z_\ell) \tag{30}$$

$$+ \sum_{|\alpha|=n} \frac{1}{\alpha!}\partial^\alpha(g^{\nu-1})(A)\,\mathrm{ad}_A^{\alpha+\delta_\ell}(B)|A-z|^{-2} \tag{31}$$

$$+ \sum_{|\alpha|=0}^{n} \frac{1}{\alpha!}\partial^\alpha(g^{\nu-1}g_\ell)(A) R^g_{n-|\alpha|}(A, \mathrm{ad}_A^\alpha(B)) \tag{32}$$

In the following, we will refer to the terms of $R_{\ell,n}(A,B)$ as the remainder terms. Let $0 \le t_1 \le n+1$ and $0 \le t_2 \le 1$. By Hadamard's three-line lemma and using (10–12), (30–32), Lemma 6 and the identity

$$\partial^\alpha\Big(\prod_{i=1}^{j} f_i\Big) = \sum_{\sum \alpha_i = \alpha} \frac{\alpha!}{\prod_{i=1}^{j}\alpha_i!} \prod_{i=1}^{j} \partial^{\alpha_i} f_i,$$

we may inspect that each remainder term (with $R_{\ell,n}(A,B)$ replaced by the remainder term) and hence $R_{\ell,n}(A,B)$ satisfies the inequality

$$\|\langle A\rangle^{t_1} R_{\ell,n}(A,B)\langle A\rangle^{t_2}\| \le C\langle z\rangle^{t_1+t_2}|\operatorname{Im} z|^{-n-2\nu}. \tag{33}$$

We will now use the functional calculus of almost analytic extensions. See [3] for details. In the following, we write $\bar\partial = (\bar\partial_1, \dots, \bar\partial_\nu)$ where $\bar\partial_j = \frac{1}{2}(\partial_{u_j} + i\partial_{v_j})$

and u_j and v_j are real and imaginary parts of $z_j \in \mathsf{C}$, $z = (z_1, \dots, z_n) \in \mathsf{C}^\nu$. The following proposition is inspired by [8, Chap. X.2] and [4].

PROPOSITION 10. *Let* $s \in \mathsf{R}$ *and* $\{f_\lambda\}_{\lambda\in I} \subset C^\infty(\mathsf{R}^\nu)$ *satisfy*

$$\forall\alpha\ \exists C_\alpha\colon |\partial^\alpha f_\lambda(x)| \le C_\alpha \langle x\rangle^{s-|\alpha|}. \tag{34}$$

There exists a family of almost analytic extensions $\{\tilde f_\lambda\}_{\lambda\in I} \subset C^\infty(\mathsf{C}^\nu)$ *satisfying*

(i) $\operatorname{supp}(\tilde f_\lambda) \subset \{u + iv \mid u \in \operatorname{supp}(f_\lambda), |v| \le C\langle u\rangle\}$.
(ii) $\forall\ell \ge 0\ \exists C_\ell\colon |\bar\partial \tilde f_\lambda(z)| \le C_\ell \langle z\rangle^{s-\ell-1} |\operatorname{Im} z|^\ell$.

PROOF. We define a mapping $C^\infty(\mathsf{R}^\nu) \ni f \mapsto \tilde f \in C^\infty(\mathsf{C}^\nu)$ in the following way. Choose a function $\kappa \in C_0^\infty(\mathsf{R})$ which equals 1 in a neighbourhood of 0 and put $\lambda_0 = C_0$, $\lambda_k = \max\{\max_{|\alpha|=k} C_\alpha, \lambda_{k-1} + 1\}$ for $k \ge 1$. Writing $z = u + iv \in \mathsf{R}^\nu \oplus i\mathsf{R}^\nu$, we now define

$$\tilde f(z) = \sum_\alpha \frac{\partial^\alpha f(u)}{\alpha!} (iv)^\alpha \prod_{j=1}^{\nu} \kappa\left(\frac{\lambda_{|\alpha|} v_j}{\langle u\rangle}\right).$$

One can now check that the properties hold.

REMARK 11. Note that if we define for a $\chi \in C_0^\infty(\mathsf{R}^\nu; [0, 1])$ with $\chi(0) = 1$ a sequence of functions by $f_{k,\lambda}(x) = \chi(\frac{x}{k}) f_\lambda(x)$, then

$$[B, f_\lambda(A)] = \lim_{k\to\infty} [B, f_{k,\lambda}(A)]$$

as a form identity on $\mathscr{D}(\langle A\rangle^s)$ and we have the dominated pointwise convergence

$$\bar\partial \tilde f_{k,\lambda}(x) \to \bar\partial \tilde f_\lambda(x) \qquad \text{for} \quad k \to \infty.$$

Let $\{f_\lambda\}_{\lambda\in I}$ satisfy the assumption of Proposition 10 with $s < 0$. Then the almost analytic extensions provide a functional calculus via the formula

$$f_\lambda(A) = C_\nu \sum_{\ell=1}^{\nu} \int_{\mathsf{C}^\nu} \bar\partial_\ell \tilde f_\lambda(z)(A_\ell - \bar z_\ell)|A - z|^{-2\nu} dz, \tag{35}$$

where C_ν is a positive constant (see [3, formula (8.18)] for details). Note that the integrals are absolutely convergent by Proposition 10 (ii).

Multiplying $\langle A\rangle^{t_1} R_{\ell,n}(A, B)\langle A\rangle^{t_2}$ with $\bar\partial \tilde f_\lambda(z)$, we get from Proposition 10 (ii) and (33) that

$$\|\langle A\rangle^{t_1} \bar\partial \tilde f_\lambda(z) R_{\ell,n}(A, B)\langle A\rangle^{t_2}\| \le C\langle z\rangle^{t_1+t_2+s-n-1-2\nu}. \tag{36}$$

Hence, if $t_1 + t_2 + s < n + 1$, $\langle A\rangle^{t_1}\bar\partial\tilde f_\lambda(z)R_{\ell,n}(A, B)\langle A\rangle^{t_2}$ is integrable over C^ν. Using (29), (35) and (36), we see that

$$
\begin{aligned}
&[B, f_\lambda(A)] \\
&\quad= C_\nu \sum_{\ell=1}^{\nu} \int_{\mathsf{C}^\nu} \bar\partial_\ell \tilde f_\lambda(z)[B, (A_\ell - \bar z_\ell)|A - z|^{-2\nu}]\, dz \\
&\quad= C_\nu \sum_{\ell=1}^{\nu} \int_{\mathsf{C}^\nu} \bar\partial_\ell \tilde f_\lambda(z) \sum_{|\alpha|=1}^{n} \frac{1}{\alpha!}\partial^\alpha\big(|\cdot - z|^{-2\nu}(\cdot_\ell - \bar z_\ell)\big)(A)\, dz\ \mathrm{ad}_A^\alpha(B) \\
&\qquad + C_\nu \sum_{\ell=1}^{\nu} \int_{\mathsf{C}^\nu} \bar\partial_\ell \tilde f_\lambda(z) R_{\ell,n}(A, B)\, dz.
\end{aligned} \tag{37}
$$

We denote (37) by $R_{\lambda,n}(A, B)$. Note that

$$
\begin{aligned}
&C_\nu \sum_{\ell=1}^{\nu} \int_{\mathsf{C}^\nu} \bar\partial_\ell \tilde f_\lambda(z) \frac{1}{\alpha!}\partial_t^\alpha\big(|t - z|^{-2\nu}(t_\ell - \bar z_\ell)\big)\, dz \\
&\qquad = \frac{C_\nu}{\alpha!}\partial_t^\alpha \sum_{\ell=1}^{\nu} \int_{\mathsf{C}^\nu} \bar\partial_\ell \tilde f_\lambda(z)|t - z|^{-2\nu}(t_\ell - \bar z_\ell)\, dz = \frac{1}{\alpha!}\partial^\alpha f_\lambda(t),
\end{aligned}
$$

which implies

$$
[B, f_\lambda(A)] = \sum_{|\alpha|=1}^{n} \frac{1}{\alpha!}\partial^\alpha f_\lambda(A)\ \mathrm{ad}_A^\alpha(B) + R_{\lambda,n}(A, B).
$$

We have now proved Theorem 3 in the case $s < 0$. For the general case, we use Remark 11 to see that $[B, f_\lambda(A)] = \lim_{k\to\infty}[B, f_{k,\lambda}(A)]$ and clearly, $f_{k,\lambda}$ satisfies the assumption of Proposition 10 with the same s, so the estimate corresponding to (36) is now uniform in k and λ. The pointwise convergence and Lebesgue's theorem on dominated convergence now finishes the argument.

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