THE PIMSNER-VOICULESCU SEQUENCE FOR
COACTIONS OF COMPACT
LIE GROUPS

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Abstract
The Pimsner-Voiculescu sequence is generalized to a Pimsner-Voiculescu tower describing the $KK$-category equivariant with respect to coactions of a compact Lie group satisfying the Hodgkin condition. A dual Pimsner-Voiculescu tower is used to show that coactions of a compact Hodgkin-Lie group satisfy the Baum-Connes property.

Introduction
When $G$ is a second countable, locally compact group and $A$ is a separable $C^*$-algebra with a continuous $G$-action, the Baum-Connes conjecture states that the $K$-theory of the reduced crossed product $A times_r G$ can be calculated by means of geometric and representation theoretical properties of $G$ and $A$, see more in [4]. To be more precise, the Baum-Connes conjecture states that the assembly mapping $\mu_A : K^G_*(\mathcal{E} G; A) \to K_*(A \rtimes_r G)$ is an isomorphism. The space $\mathcal{E} G$ is the universal proper $G$-space and $K^G_*(\mathcal{E} G; A)$ is the proper equivariant $K$-homology with coefficients in $A$. There are known counterexamples when $\mu_A$ is not an isomorphism, so it is more natural to speak of groups having the Baum-Connes property. In [10], the equivariant $K$-homology with coefficients in $A$ was proved to be the left derived functor of $F(A) = K_*(A \rtimes_r G)$ and the assembly mapping being the natural transformation from $LF$ to $F$. The approach to the Baum-Connes property using triangulated categories can be generalized to discrete quantum groups, see [9], which indicates that geometric techniques such as universal proper $G$-spaces can be generalized to discrete quantum groups.

The generalization of the Baum-Connes property to quantum groups has been studied in for instance [11] and [17]. The case studied in [11] is that of quantum group actions of the dual of a compact Lie group which correspond to coactions of the Lie group. In [11] duals of compact Lie groups were shown to satisfy the strong Baum-Connes property, i.e., the embedding of

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the triangulated category generated by proper coactions, the $C^*$-algebras that
are Baaj-Skandalis dual to trivial $G$-actions, into the $KK$-category equivariant
with respect to coactions is essentially surjective. In this paper we construct
an analogue of the Pimsner-Voiculescu sequence for coactions of a compact
Hodgkin-Lie group $G$ that describes how the $KK$-category equivariant with
respect to coactions of $G$ is built up from the $C^*$-algebras with coactions of $G$
which are proper in the sense of [11].

The starting point is to express the Pimsner-Voiculescu sequence for $\mathbb{Z}$-
actions in terms of a property of the representation ring of a rank one torus.
Using the Universal Coefficient Theorem, the Pimsner-Voiculescu sequence
can be constructed from a Koszul complex

$$0 \longrightarrow R(T) \xrightarrow{\alpha} R(T) \longrightarrow 0,$$

where $\alpha$ is defined as multiplication by $1 - t$ under the isomorphism $R(T) \cong
\mathbb{Z}[t, t^{-1}]$. When $A$ has a coaction of $T$, i.e., a $\mathbb{Z}$-action, the tensor
product over $R(T)$ between this Koszul complex and $K^*_n(A \rtimes_r \mathbb{Z})$
gives the Pimsner-Voiculescu sequence. In the generalization to higher rank, when $T$ is a torus
of rank $n$ we consider the Koszul complex

$$0 \longrightarrow \bigwedge^n R(T)^n \longrightarrow \bigwedge^{n-1} R(T)^n \longrightarrow \cdots$$

$$\longrightarrow \bigwedge^2 R(T)^n \longrightarrow R(T)^n \longrightarrow R(T) \longrightarrow 0.$$

The boundary mappings in this complex are defined from interior multiplication
with the element $\sum (1 - t_i)e_i^\tau \in \text{Hom}_{R(T)}(R(T)^n, R(T))$. If $G$ is a com-
 pact Hodgkin-Lie group with maximal torus $T$, the representation ring $R(T)$
is a free $R(G)$-module by [15], so the generalization from a torus to compact
Hodgkin-Lie groups goes in a straightforward fashion. Just as when the rank
is 1, the Koszul complex above can be used to produce sequence of distin-
guished triangles which is the analogue of a Pimsner-Voiculescu sequence for
the $K$-theory of crossed products by coactions of $G$.

We will give a geometric description of a sequence of distinguished triangles
in the $KK$-category equivariant with respect to coactions of $G$ that corresponds
to the above Koszul complex under the Universal Coefficient Theorem. As for
the Pimsner-Voiculescu sequence for $\mathbb{Z}$ we will obtain a projective resolu-
tion of the crossed product by a coaction in the sense of triangulated categories
rather than exact sequences. Using suitable tensor products we produce in The-
orem 3.4 a sequence of distinguished triangles in the $KK$-category equivariant
with respect to coactions of $G$ that we call the generalized Pimsner-Voiculescu
Here $t(A \rtimes_r \hat{G})$ denotes the $C^*$-algebra $A \rtimes_r \hat{G}$ equipped with the trivial $\hat{G}$-action and the terms $D_i(A)$ can be explicitly described as braided tensor products. Taking $K$-theory of the lower row will give a complex similar to the Koszul complex that in a sense forms a projective resolution of the $K$-theory of $A \rtimes \hat{G}$. The dual Pimsner-Voiculescu gives a more precise description of the results of [11] by a sequence of distinguished triangles in $KK^G$ that describes the crossed product $A \rtimes_r \hat{G}$ in terms of $G$-$C^*$-algebras with trivial $G$-action, thus giving a direct route to the strong Baum-Connes property of $\hat{G}$.

The paper is organized as follows; the first section consists of a review of $KK$-theory of actions and coactions. In particular we gather some known results about the braided tensor product and the Drinfeld double which plays a mayor role in constructing the dual Pimsner-Voiculescu tower. The main references of this section are [1], [2], [3], [7], [10], [12] and [16]. In the second section a geometric construction of the Pimsner-Voiculescu sequence for $\mathbb{Z}$-actions is presented and generalized to higher rank via a Koszul complex. In the third section the restriction functor for coactions is used to generalize the Pimsner-Voiculescu sequence to coactions of compact Hodgkin-Lie groups $G$. As an example of this we calculate the $K$-theory of some compact homogeneous spaces. By similar methods, a dual Pimsner-Voivulescu tower is constructed in $KK^G$, following the ideas of [10]. At the end of the paper we discuss some possible generalizations to duals of Woronowicz deformations.

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1. Actions and coactions of compact groups

The standard approach to equivariant $K$-theory is to introduce equivariant $KK$-theory. If $A$ and $B$ are two separable $C^*$-algebras with a continuous action of a locally compact group $G$, the equivariant $KK$-group $KK^G(A, B)$ is defined as the set of homotopy classes of $G$-equivariant $A - B$-Kasparov modules which forms an abelian group under direct sum. The $KK$-groups can be equipped with a product such that if $C$ is a third separable $C^*$-algebra with a continuous $G$-action there is an additive pairing called the Kasparov product

$$KK^G(A, B) \times KK^G(B, C) \rightarrow KK^G(A, C).$$

Following the standard construction, we let $KK^G$ denote the additive category of all separable $C^*$-algebras with a continuous $G$-action and a morphism in $KK^G$ from $A$ to $B$ is an element of $KK^G(A, B)$. The composition of two $KK^G$-morphisms is defined to be their Kasparov product. The group $KK^G(C, A)$ coincides with the equivariant $K$-theory of $A$. In particular, if $G$ is compact $KK^G(C, C) = R(G)$, the representation ring of $G$. The action of $R(G)$ on equivariant $K$-theory generalizes to an $R(G)$-module structure on the bivariant groups $KK^G(A, B)$.

The category $KK^G$ can be equipped with a triangulated structure with a mapping cone coming from the mapping cone construction of a $*$-homomorphism. The triangulated structure on $KK^G$ is universal in the sense that any homotopy invariant, stable, split-exact functor on the category of $C^*$-algebras with a continuous $G$-action defines a homological functor on $KK^G$. The construction of the triangulated structure and its universality are thoroughly explained in [10].

Let us just recall the basics of the construction of the triangulated structure on $KK^G$. The suspension $\Sigma A$ of a $G$-$C^*$-algebra is defined by $C_0(\mathbb{R}) \otimes A$. By Bott periodicity $\Sigma^2 \cong \text{id}$. A distinguished triangle in $KK^G$ is a triangle isomorphic to one of the form

$$C(f) \rightarrow A \rightarrow B,$$

where $C(f)$ is the mapping cone of the equivariant $*$-homomorphism $f : A \rightarrow B$. In particular, if $f : A \rightarrow B$ is a surjection and admits an equivariant completely positive splitting the natural mapping $\ker(f) \rightarrow C(f)$ defines an equivariant $KK$-isomorphism, so under suitable assumptions a distinguished triangle is isomorphic to a short exact sequence.

How to construct $KK$-theory of coactions of groups is easiest seen in the simpler case when $G$ is an abelian group. If $A$ is a $C^*$-algebra equipped with an
action $\alpha$ of the abelian group $G$, the crossed product $A \rtimes_r G$ carries a natural action of the Pontryagin dual $\hat{G}$. This action is called the dual action of $\hat{G}$. Since abelian groups are exact, the crossed product by an abelian group defines a triangulated functor $KK^G \to KK^{\hat{G}}$. The crossed product by the dual action is described by Takesaki-Takai duality which states that there is an equivariant isomorphism

$$\hat{A} \rtimes_r G \rtimes_r \hat{G} \cong A \otimes \mathcal{H}(L^2(G)),$$

where $A \rtimes_r G \rtimes_r \hat{G}$ is equipped with the dual action of $G$ and the $G$-action on $A \otimes \mathcal{H}(L^2(G))$ is defined as $\alpha \otimes \text{Ad}$. Takesaki-Takai duality implies that the crossed product defines a triangulated equivalence $KK^G \to KK^{\hat{G}}$.

An action $\alpha$ of a group $G$ on $A$ defines a $\ast$-homomorphism $\Delta_\alpha : A \to \mathcal{M}(A \otimes C_0(G))$ by letting $\Delta_\alpha(a) = \alpha_g(a)$, where $g \mapsto \alpha_g(a)$ is a function. When $G$ is abelian there is a natural isomorphism $C_0(\hat{G}) \cong C^*_r(G)$ and a $\hat{G}$-action corresponds to a non-degenerate $\ast$-homomorphism $\Delta : A \to \mathcal{M}(A \otimes_{\min} C^*_r(G))$ satisfying certain conditions. The first instance of a coaction of a group $G$ is on $C^*_r(G)$. Using the universal property of $C^*_r(G)$, one can construct a non-degenerate mapping $\Delta : C^*_r(G) \to \mathcal{M}(C^*_r(G) \otimes_{\min} C^*_r(G))$ called the comultiplication and is induced from the diagonal homomorphism $G \to G \times G$. Clearly, the mapping $\Delta$ satisfies:

$$(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta,$$

so we say that $\Delta$ is coassociative. Since $\Delta_2 = \Delta$ the comultiplication $\Delta$ is cocommutative, so if we interpret $C^*_r(G)$ as the functions on a reduced locally compact quantum group $\hat{G}$ then $\hat{G}$ can be thought of as abelian, see more in [7].

With the abelian setting as motivation, we say that a separable $C^*$-algebra $A$ has a coaction of the locally compact second countable group $G$ if there is non-degenerate $\ast$-homomorphism $\Delta_A : A \to \mathcal{M}(A \otimes_{\min} C^*_r(G))$ satisfying the two conditions that $\Delta_A(A) \cdot 1_A \otimes_{\min} C^*_r(G)$ is a dense subspace of $A \otimes_{\min} C^*_r(G)$ and that $\Delta_A$ is coassociative in the sense that

$$(\Delta_A \otimes \text{id}_{C^*_r(G)})\Delta_A = (\text{id}_A \otimes \Delta)\Delta_A.$$

A separable $C^*$-algebra equipped with a coaction of $G$ will be called a $\hat{G}$–$C^*$-algebra. An example of a coaction is the dual coaction on $C^*$-algebras of the form $A = B \rtimes_r G$, for some $G$–$C^*$-algebra $B$. When $G$ is discrete we can decompose $B \rtimes_r G$ by means of the dense subspace $\bigoplus_{g \in G} B \lambda_g$ and the dual coaction is defined by $\Delta_A(b \lambda_g) := b \lambda_g \otimes \lambda_g$. In the general setting, the construction of the dual coaction goes analogously and we refer the reader to [1].
Much of the theory for group actions also hold for group coactions, the crossed product will as for abelian groups be a stepping stone back and forth between actions and coactions. In [1], the $KK$-theory equivariant with respect to a bi-$C^*$-algebras and the corresponding Kasparov product was constructed. In [12] it was proved that the $KK$-theory equivariant with respect to a locally compact quantum group has a triangulated structure defined in the same fashion as for a group.

Let us explain the setting of [1] more explicitly in the case of coactions of a group. An $A - B$-$Hilbert$ $bimodule$ $E$ is called $\hat{G}$-equivariant if there is a coaction $\delta_E : E \to \mathcal{L}_{B \otimes_{\min} C^*_r(G)}(B \otimes_{\min} C^*_r(G), E \otimes C^*_r(G))$ satisfying a coassociativity condition similar to (1) and $\delta_E$ should commute with the $A$-action and $B$-action in the obvious ways. By Proposition 2.4 of [1], the coaction $\delta_E$ is uniquely determined by a unitary $V_E \in \mathcal{L}(E \otimes_{\Delta_{B}} B \otimes_{\min} C^*_r(G), E \otimes C^*_r(G))$ via the equation $\delta_E(x)y = V_E(x \otimes_{\Delta_{B}} y)$ for $x \in E$ and $y \in B \otimes_{\min} C^*_r(G)$. A $\hat{G}$-equivariant $A - B$-Kasparov module is an $A - B$-Kasparov module $(E, F)$ such that $E$ is a $\hat{G}$-equivariant $A - B$-$Hilbert$ module and the operator $F$ commutes with the unitary $V_E$ up to a compact operator. The group $KK^G(A, B)$ is defined as the homotopy classes of $\hat{G}$-equivariant $A - B$-Kasparov modules. The additive category $KK^G$ is defined by taking the objects to be separable $\hat{G}$--$C^*$-algebras and the group of morphisms from $A$ to $B$ is $KK^G(A, B)$. The composition in $KK^G$ is Kasparov product of $\hat{G}$-equivariant Kasparov modules.

To a closed subgroup $H$ of $G$, the restriction of a $G$-action to $H$ defines a restriction functor $Res^G_H : KK^G \to KK^H$ and its right adjoint is the induction functor $Ind^H_G : KK^H \to KK^G$. However the restriction goes in the other direction for coactions. When $H$ is a closed subgroup of $G$, there is a non-degenerate embedding $C^*(H) \subseteq \mathcal{M}(C^*(G))$ so a coaction of $H$ can be restricted to a coaction of $G$. This construction defines a triangulated functor $Res^\hat{G}_H : KK^\hat{G} \to KK^\hat{G}$.

The crossed product $B \mapsto B \rtimes_r G$ sends a $G$--$C^*$-algebra to a $\hat{G}$--$C^*$-algebra and if $G$ is exact the crossed product induces a triangulated functor $KK^G \to KK^\hat{G}$. In order to construct a duality similar to Takesaki-Takai duality one introduces the crossed product by a coaction. If $A$ is a $\hat{G}$--$C^*$-algebra we define

\[ A \rtimes_r \hat{G} := [\Delta_A(A) \cdot 1_{A} \otimes C_0(G)] \subseteq \mathcal{M}(A \otimes \mathcal{K}(L^2(G))). \]

It follows from Lemma 7.2 of [2] that $A \rtimes_r \hat{G}$ forms a $C^*$-algebra. For a thorough introduction to crossed products by coactions see [13]. The $C^*$-algebra $A \rtimes_r \hat{G}$ carries a continuous $G$-action defined in the dense subspace...
\( \Delta_A(A) \cdot 1_A \otimes C_0(G) \) by
\[
\gamma \cdot (\Delta_A(a) \cdot 1_A \otimes f) := \Delta_A(a) \cdot 1_A \otimes g \cdot f.
\]

Similarly to the abelian setting, Takesaki-Takai duality holds so there are equivariant isomorphisms \( B \rtimes_r G \cong B \otimes K(L^2(G)) \) and \( A \rtimes_r \hat{G} \rtimes_r G \cong A \otimes \mathcal{H}(L^2(G)) \) which ensures that the crossed product defines an equivalence of triangulated categories known as Baaj-Skandalis duality.

The tensor product on the category of \( G \)-\( C^* \)-algebras is well defined. If \( A \) and \( B \) have actions \( \alpha \) respectively \( \beta \) of \( G \) the tensor product \( A \otimes_{\min} B \) can be equipped with the action \( \alpha \otimes \beta : G \to \text{Aut}(A \otimes_{\min} B) \). However, for a non-abelian group \( G \) the construction of a tensor product of \( \hat{G} \)-\( C^* \)-algebras can not be done by just taking tensor products of the \( C^* \)-algebras. The tensor product relevant for \( \hat{G} \)-\( C^* \)-algebras is the braided tensor product over \( \hat{G} \) which requires one further structure. Suppose that \( A \) is a \( \hat{G} \)-algebra with a continuous \( G \)-action \( \alpha \). If the action \( \alpha \) satisfies that
\[
(2) \quad \Delta_A \circ \alpha_g = (\alpha_g \otimes \text{Ad}(g)) \Delta_A
\]
we say that \( A \) is a Yetter-Drinfeld algebra. An example of a Yetter-Drinfeld algebra is \( C^*_r(G) \) with \( G \)-action defined by the adjoint action \( G \to \text{Aut}(G) \). It is much easier to construct a Yetter-Drinfeld algebra from a \( G \)-\( C^* \)-algebra, if \( A \) is a \( G \)-\( C^* \)-algebra we can in a functorial way define a coaction of \( G \) on \( A \) by setting \( \Delta_A(a) := a \otimes 1 \). When \( A \) is a Yetter-Drinfeld algebra, the \( C^* \)-algebra \( A \rtimes_r \hat{G} \) is also a Yetter-Drinfeld algebra since the morphism \( \Delta_A \) is covariant with respect to the \( G \)-action and \( \Delta_A \) extends to a coaction of \( G \) on \( A \rtimes_r \hat{G} \), see more in [12]. This construction is functorial and the crossed product can be seen as a functor on the category of Yetter-Drinfeld algebras.

When \( A \) is a Yetter-Drinfeld algebra and \( B \) is a \( \hat{G} \)-\( C^* \)-algebra we define the mappings
\[
\begin{align*}
t_A : A &\to \mathcal{M}(A \otimes_{\min} B \otimes \mathcal{H}(L^2(G))), \quad t(a) := \Delta_A(a)_{13} \\
t_B : B &\to \mathcal{M}(A \otimes_{\min} B \otimes \mathcal{H}(L^2(G))), \quad t(b) := \Delta_B(b)_{23}.
\end{align*}
\]
Following [12], the braided tensor product \( A \boxtimes_{\hat{G}} B \) is defined as the closed linear span of \( t_A(A) \cdot t_B(B) \). By Proposition 8.3 of [16], \( A \boxtimes_{\hat{G}} B \) forms a \( * \)-subalgebra of \( \mathcal{M}(A \otimes_{\min} B \otimes \mathcal{H}(L^2(G))) \) so the braided tensor product is a \( C^* \)-algebra. The coaction of \( G \) on \( A \boxtimes_{\hat{G}} B \) is defined by
\[
\Delta_A \boxtimes_{\hat{G}} \Delta_B(t_A(a) \cdot t_B(b)) := (t_A \otimes \text{id})(\Delta_A(a)) \cdot (t_B \otimes \text{id})(\Delta_B(b)).
\]
Observe that since \( C^*_r(G) \) is cocommutative, the adjoint \( \hat{G} \)-action is trivial and a similar construction of a braided tensor product over \( G \) between \( G-\)
$C^*$-algebras with trivial $\hat{G}$-actions coincides with the usual tensor product. In general, the braided tensor product over $G$ does not need to coincide with the usual tensor product. By Lemma 3.5 of [12] there is a $G$-equivariant isomorphism

$$ (A \boxtimes_{\hat{G}} B) \rtimes_{r} \hat{G} \cong (A \rtimes_{r} \hat{G}) \boxtimes_{\hat{G}} B $$

where the $G$-coaction on the right hand side is the trivial one on $B$. More generally, this identity holds for any quantum group and in particular also for braided tensor products over $G$. We will prove this statement in special case of braided tensor products over a compact group $G$ with $C(G)$ below in Lemma 3.3.

If we interpret the structure of a Yetter-Drinfeld algebra as two actions of the quantum groups $G$ and $\hat{G}$ satisfying a certain cocycle relation, the cocycle defines a quantum group by means of a double crossed product such that Yetter-Drinfeld algebras are precisely the $C^*$-algebras with an action of this double crossed product. The right quantum group to look at is the Drinfeld double $D(G)$. Using the notations of quantum groups, the algebra of functions on $D(G)$ is $C_0(G) \otimes C^*_r(G)$ with the obvious action and coaction of $G$. The action and coaction define a comultiplication

$$ \Delta_{D(G)} : C_0(D(G)) \longrightarrow M(C_0(D(G)) \otimes C_0(D(G))) $$

by $\Delta_{D(G)} := \sigma_{23} \text{Ad}(W_{23})(\Delta_{C_0(G)} \otimes \Delta_{C^*_r(G)})$ where $W \in \mathcal{B}(L^2(G) \otimes L^2(G))$ is the multiplicative unitary of $G$ defined by $Wf(g_1, g_2) = f(g_1, g_1g_2)$. The comultiplication $\Delta_{D(G)}$ makes $D(G)$ into a quantum group by Theorem 5.3 of [3]. A Yetter-Drinfeld algebra $A$ with the action $\alpha$ and coaction $\Delta_A$ correspond to a $D(G) - C^*$-algebra by defining the $D(G)$-coaction

$$ \Delta_A^{D(G)} := (\Delta_{\alpha} \otimes \text{id}) \Delta_A : A \longrightarrow M(A \otimes_{\text{min}} C_0(D(G))), $$

see more in Proposition 3.2 of [12]. Therefore we can consider the braided tensor product as a tensor product between $D(G) - C^*$-algebras and $\hat{G} - C^*$-algebras. The braided tensor product induces a biadditive functor

$$ \boxtimes_{\hat{G}} : KK^{D(G)} \times KK^{\hat{G}} \longrightarrow KK^{\hat{G}}. $$

Much of the theory of coactions can be done without introducing any quantum groups, but in order to construct the Pimsner-Voiculescu sequence for coactions of compact Hodgkin-Lie groups we will need the braided tensor product as a biadditive functor between $KK$-categories.
2. The Pimsner-Voiculescu sequence from the viewpoint of representation rings

In this section we will study the Pimsner-Voiculescu sequence for $\mathbb{Z}$ and generalize to a Pimsner-Voiculescu tower for $\mathbb{Z}^n$. We will use representation theory to calculate all the mappings explicitly. These calculations will in a surprisingly straightforward way give a natural route to a Pimsner-Voiculescu tower for coactions of compact Lie groups.

Consider the evaluation mapping $l : C_0(\mathbb{R}) \to C_0(\mathbb{Z})$. This mapping fits into a $\mathbb{Z}$-equivariant short exact sequence

$$0 \to \Sigma C_0(\mathbb{Z}) \to C_0(\mathbb{R}) \overset{l}{\to} C_0(\mathbb{Z}) \to 0.$$  

The $\mathbb{Z}$-equivariant Dirac operator $D$ on $\mathbb{R}$ defines a $\mathbb{Z}$-equivariant odd unbounded $K$-homology class, thus an element $[D] \in KK^Z(C_0(\mathbb{R}), \Sigma \mathbb{C})$. While $\mathbb{R}$ is the universal proper $\mathbb{Z}$-space the element $[D]$ is the Dirac element of $\mathbb{Z}$ and the strong Baum-Connes property of $\mathbb{Z}$ implies that $[D]$ is a $KK^Z$-isomorphism.

The exact sequence (4) induces a distinguished triangle in $KK^Z$ which after using the isomorphism $C_0(\mathbb{R}) \cong \Sigma \mathbb{C}$ and rotation 4 steps to the left becomes

$$C_0(\mathbb{Z}) \longrightarrow C_0(\mathbb{Z}) \longrightarrow C_0(\mathbb{Z}) \longrightarrow C_0(\mathbb{R}).$$

In a certain sense, the distinguished triangle (5) captures the entire behavior of the Pimsner-Voiculescu sequence. If $A$ is a $\mathbb{Z} - C^*$-algebra we can apply Baaj-Skandalis duality to (5) and tensor with $A \rtimes_r \mathbb{Z}$. If we apply Baaj-Skandalis duality again, we obtain a distinguished triangle in $KK^Z$

$$A \longrightarrow A \longrightarrow A \longrightarrow A \rtimes_r \mathbb{Z},$$

where $A \rtimes_r \mathbb{Z}$ is given the trivial $\mathbb{Z}$-action. Taking $K$-theory of this distinguished triangle gives back the classical Pimsner-Voiculescu sequence due to the following proposition:

**Proposition 2.1.** When $T$ is a torus of rank 1 and the element $\kappa \in KK^T(\mathbb{C}, \mathbb{C})$ is defined using the isomorphisms $KK^T(\mathbb{C}, \mathbb{C}) \cong \text{Hom}_{R(T)}(R(T), R(T))$ and $R(T) \cong \mathbb{Z}[t, t^{-1}]$ as

$$\kappa f(t, t^{-1}) = (1 - t)f(t, t^{-1}),$$
the KK-morphism $\kappa$ is Baaj-Skandalis dual to the KK-morphism $C_0(\mathbb{Z}) \to C_0(\mathbb{Z})$ defined by (4).

Observe that the $K$-theory of the exact sequence (4) is described from the exact sequence:

\[ 0 \to R(T) \xrightarrow{1-t} R(T) \to \mathbb{Z} \to 0, \]

by Proposition 2.1. The first terms in this exact sequence is the Koszul complex defined by $1-t \in \text{Hom}_{R(T)}(R(T), R(T))$ and $\mathbb{Z}$ is the cohomology of the Koszul complex.

**Proof.** Let $\kappa_0 \in \text{Hom}_{R(T)}(R(T), R(T))$ denote the Baaj-Skandalis dual of the KK-morphism induced from (4). It follows directly from the construction that the mapping $R(T) \to \mathbb{Z}$ induced from $\Sigma C_0(\mathbb{Z}) \to C_0(\mathbb{R})$ is the augmentation mapping $[t, t^{-1}] \to \mathbb{Z}$ onto the generator of $K_1(C_0(\mathbb{R}))$. Therefore the image of $\kappa_0$ is the ideal generated by either $1+t$ or $1-t$ so $\kappa_0$ is of the form $u \cdot (1 \pm t)$ for some unit $u \in \mathbb{Z}[t, t^{-1}]$. The sign and $u=1$ is found by either a direct calculation or by considering the Pimsner-Voiculescu sequence for $C_0(\mathbb{Z})$.

We will return to the Koszul complexes later on. First we will construct a geometric interpretation of the higher rank situation. Assume that $T$ is a torus of rank $n$ and consider the semi-open unit cube $I = [0, 1[^n \subseteq \mathbb{R}^n$. For $i = 1, \ldots, n$ we define $\tilde{X}_i$ as the set of open $i-1$-dimensional faces of $I$. The union satisfies

\[ \bigcup_{i=1}^{n} \tilde{X}_i = \partial I \cap I. \]

We let $k_i$, for $i = 1, 2, \ldots, n$, denote the integers $k_i := \binom{n}{i-1}$. The set $\tilde{X}_i$ has $k_i$ connected components so if we choose a homeomorphism $]0, 1[ \cong \mathbb{R}$ there are homeomorphisms

\[ \tilde{X}_i \cong \bigsqcup_{j=1}^{k_i} \mathbb{R}^{i-1} \quad \text{for} \quad i = 1, 2, \ldots, n, \]

where we interpret $\mathbb{R}^0$ as the one-point space. We take $X_i$ to be the $\mathbb{Z}^n$-translates of $\bigcup_{j \leq i} \tilde{X}_j$ and define $Y_i := \mathbb{R}^n \setminus X_i$ for $i = 1, 2, \ldots, n$ and $Y_0 := \mathbb{R}^n$.

**Proposition 2.2.** For $i = 1, 2, \ldots, n$ there are $\mathbb{Z}^n$-equivariant isomorphisms

\[ C_0(Y_{i-1})/C_0(Y_i) \cong \mathbb{C}^{k_i} \otimes \Sigma^{i-1}C_0(\mathbb{Z}^n). \]
PROOF. By equation (6) there is a $\mathbb{Z}^n$-equivariant homeomorphism
\[
Y_{i-1} \setminus Y_i \cong \bigsqcup_{m \in \mathbb{Z}^n} \left( \bigsqcup_{j=1}^{k_i} \mathbb{R}^{i-1} \right),
\]
where $\mathbb{Z}^n$ acts by translation on the first disjoint union. Therefore
\[
C_0(Y_{i-1})/C_0(Y_i) \cong C_0(Y_{i-1} \setminus Y_i) \cong C_0 \left( \bigsqcup_{m \in \mathbb{Z}^n} \left( \bigsqcup_{j=1}^{k_i} \mathbb{R}^{i-1} \right) \right) \cong C_{k_i} \otimes C_0(\mathbb{Z}^n \times \mathbb{R}^{i-1}) \cong C_{k_i} \otimes \Sigma^{i-1}C_0(\mathbb{Z}^n).
\]

Consider the classifying space $\mathbb{R}^n$ for proper actions of $\mathbb{Z}^n$. Since $\mathbb{Z}^n$ has the strong Baum-Connes property, the Dirac element $[\mathcal{D}]$ induces a $KK\mathbb{Z}^n$-isomorphism $C_0(\mathbb{R}^n) \cong \Sigma^nC$. An alternative approach to constructing this isomorphism is the Julg theorem which implies that for any $T - C^*$-algebra $A$ there is an isomorphism $K^*_s(T) \cong K^*_s(A \rtimes_r T)$. Therefore $K^*_s(\Sigma^nC \rtimes \mathbb{Z}^n) \cong K^*_s(C_0(\mathbb{R}^n) \rtimes \mathbb{Z}^n)$ and the statement follows from the Universal Coefficient Theorem for the compact Hodgkin-Lie group $T$, see more in [14].

For $i = 1, 2, \ldots, n$, Proposition 2.2 implies that there is a $\mathbb{Z}^n$-equivariant short exact sequence
\[
0 \rightarrow C_0(Y_i) \rightarrow C_0(Y_{i-1}) \rightarrow C_{k_i} \otimes \Sigma^{i-1}C_0(\mathbb{Z}^n) \rightarrow 0.
\]

We will by $\kappa_i \in KK\mathbb{Z}^n(C_{k_i} \otimes C_0(\mathbb{Z}^n), C_{k_{i+1}} \otimes C_0(\mathbb{Z}^n))$ denote the $\mathbb{Z}^n$-equivariant $KK$-morphism defined in such a way that the extension class defined by (7) composed with the restriction mapping $C_0(Y_i) \rightarrow C_{k_{i+1}} \otimes \Sigma^iC_0(\mathbb{Z}^n)$ coincides with $\Sigma^{i-1}\kappa_i$. Notice that $Y_n = \mathbb{Z}^n \times \{0, 1\}^n$ and $Y_0 = \mathbb{R}^n$ so we have that $C_0(Y_n) = \Sigma^nC_0(\mathbb{Z}^n)$ and $C_0(Y_0) = C_0(\mathbb{R}^n)$, the latter being $KK\mathbb{Z}^n$-isomorphic to $\Sigma^nC$. Thus we get a sequence of distinguished triangles in $KK\mathbb{Z}^n$:

\[
\begin{align*}
\Sigma^nC_0(\mathbb{Z}^n) & \rightarrow C_0(Y_{n-1}) & \rightarrow \cdots & \leftarrow \Sigma^n\kappa_n \rightarrow \Sigma^{n-1}C_0(\mathbb{Z}^n) & \leftarrow \cdots \\
C_0(\mathbb{Z}^n) & \leftarrow C_{k_{i+1}} \otimes \Sigma^iC_0(\mathbb{Z}^n) & \leftarrow \cdots & \leftarrow C_0(Y_2) & \rightarrow \cdots & \rightarrow C_0(Y_1) & \rightarrow \Sigma^nC
\end{align*}
\]
A sequence of distinguished triangles of this type will be called a tower. The tower (8) in $KK^Z_n$ is the higher rank analogue of the distinguished triangle (5). The tower (8) can be generalized to contain any coefficient ring.

To find a better description of the morphisms $\kappa_i$ let us recall the notion of a Koszul complex. Let $R$ denote a commutative ring and $E$ an $R$-module. For simplicity we will assume that $E$ is free and finitely generated, let us say of rank $N$. For an element $v \in \text{Hom}_R(E, R)$, the Koszul complex of $E$ with respect to $v$ is the complex

$$
0 \longrightarrow \wedge^N E \xrightarrow{\partial_1} \wedge^{N-1} E \xrightarrow{\partial_2} \cdots \xrightarrow{\partial_{N-2}} \wedge^2 E \xrightarrow{\partial_{N-1}} E \xrightarrow{v} R \longrightarrow 0,
$$

where each $\partial_k$ is defined as interior multiplication by $v$. Since we have assumed $E$ to be free, we may write $v = \sum_{i=1}^{N} \nu_i e_i^*$ for some $\nu_1, \nu_2, \ldots, \nu_N \in R$ and the dual basis $e_i^*$ of a basis $e_i, i = 1, 2, \ldots, N$ of $E$. If the sequence $\nu_1, \nu_2, \ldots, \nu_N$ is a regular sequence the Koszul complex is exact except at $R$. The cohomology of the Koszul complex is in this case $R/v(E)$ at $R$. See more in [5].

The Koszul complex of interest to us is constructed from the module $E := R(T)^n$ over the representation ring of the torus $T$ which has the following form:

$$
R(T) \cong \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}].
$$

Observe that Baaj-Skandalis duality and the Universal Coefficient Theorem implies that

$$
KK^Z_n(C^k \otimes C_0(Z^n), C^{k+1} \otimes C_0(Z^n)) \cong KK^T_n(C^k, C^{k+1}) \cong \text{Hom}_{R(T)}(R(T)^k, R(T)^{k+1}).
$$

We have that $R(T)^k_i \cong \wedge^{n-i+1} E$ so the lower row in (8) have the right ranks for coinciding with a Koszul complex. Let $f_i \in \text{Hom}_{R(T)}(\wedge^{n-i+1} E, \wedge^{n-i} E)$ denote the image of $\kappa_i$ under the isomorphisms above. To simplify notations, we will by $(e_i)_{i=1}^n$ denote the $R(T)$-basis of $E$ coming from the isomorphism $E \cong R(T) \otimes \mathbb{Z} Z^n$ and by $(e_i^*)_{i=1}^n$ denote the dual basis.

**Theorem 2.3.** Under the isomorphisms $R(T)^k_i \cong \wedge^{n-i+1} E$ the mappings $f_i$ coincide with interior multiplication by the element $v := \sum_{i=1}^n (1 - t_i)e_i^*$. Therefore the sequence

$$
0 \longrightarrow \wedge^n E \xrightarrow{f_1} \wedge^{n-1} E \xrightarrow{f_2} \cdots \xrightarrow{f_{n-2}} \wedge^2 E \xrightarrow{f_{n-1}} E \xrightarrow{f_n} R(T) \longrightarrow 0
$$

defines a complex isomorphic to the Koszul complex of $E$ whose cohomology at $R(T)$ is $\mathbb{Z}$. 
Proof. While both $f_i$ and the mapping defined by interior multiplication by $v$ are $R(T)$-linear it is sufficient to prove that $f_i(u) = v - u$ for elements of the form $u = e_{m_1} \wedge \cdots \wedge e_{m_{n-i+1}} \in \wedge^{n-i+1} E$, where $m_1, \ldots, m_{n-i+1} \in \{1, 2, \ldots, n\}$. Let $(m_p)_p^{n-i+1}$ be an enumeration of all $j = 1, 2, \ldots, n$ such that $j \notin (m_p)_p^{n-i+1}$. If we view $\mathbb{Z}^n$ as a subset of $\mathbb{R}^n$ we can define $X_u \subseteq \tilde{X}_i$ as the open face in $\mathbb{R}^n$ spanned by the vectors $e_{m_1}, e_{m_{n-i+1}}, \ldots, e_{m_n}$. Under the isomorphism $\wedge^{n-i+1} E \cong \mathcal{K}_i(\mathcal{C}_k^0(\mathbb{Z}^n))$ the element $u$ corresponds to a $\mathbb{K}$-theory class on $\tilde{X}_i$ which is trivial except on the face $X_u$. Therefore there exists sequences of numbers $(a_j)_{j=1}^{n-i+1}, (b_j)_{j=1}^{n-i+1} \subseteq \mathbb{Z}$ such that

$$f_i(u) = \sum_{j=1}^{n-i+1} (a_j + b_j t_j) e_{m_j} - u.$$ 

If $j = 1, 2, \ldots, n-i+1$, we will let $X_{u,j}$ denote the open face spanned by the vectors $e_{m_1}, e_{m_{n-i+1}}, \ldots, e_{m_n}$. It follows from restricting to $X_{u,j}$ that $a_j = 1$ since Bott periodicity implies that the index mapping $K_{i-1}(\mathcal{C}_0(\mathbb{X}_u)) \to K_i(\mathcal{C}_0(\mathbb{X}_{u,j}))$ is an isomorphism. In a similar fashion it follows that $b_j = -1$.

While $v(E)$ is the ideal generated by the regular sequence $1 - t_1, 1 - t_2, \ldots, 1 - t_n$, the cohomology of the Koszul complex is $R(T)/v(E) = \mathbb{Z}$ and the quotient mapping $R(T) \to \mathbb{Z}$ coincides with the augmentation mapping.

Consider the tower Baaj-Skandalis dual to (8). Given $A, B \in KK^T$ we can apply the homological functor $KK^T(A, - \otimes_{\text{min}} B)$ to this tower. This functor is only homological on the bootstrap category if $B$ is not exact, but all objects in the tower Baaj-Skandalis dual to (8) are in the bootstrap category. The lowest row of the corresponding tower in the category of $R(T)$-modules is a Koszul complex:

$$0 \to \wedge^n \mathbb{Z}^n \otimes KK^T_*(A, B) \xrightarrow{\nu_A} \wedge^{n-1} \mathbb{Z}^n \otimes KK^T_*(A, B) \xrightarrow{\nu_A} \ldots$$

where

$$\nu_A := \sum_{i=1}^n (1 - \beta_i) e_i^* \in \text{Hom}_{R(T)}(KK^T_*(A, B)^n, KK^T_*(A, B))$$

and $(\beta_i)_{i=1}^n$ are the commuting equivariant automorphisms of $A$ that are Baaj-Skandalis to the $\mathbb{Z}^n$-action on $B \rtimes_r T$. The cohomology of this Koszul complex can be calculated from $KK^T_*(A, B)$. We will return to this subject in the next section in the more general case of Hodgkin-Lie groups and explain this procedure further.
3. The generalized Pimsner-Voiculescu-towers

As mentioned in the introduction, the representation ring $R(T)$ is free over $R(G)$ when $G$ is a Hodgkin-Lie group, so the step to coactions of a compact Hodgkin-Lie group will not be too large. We will throughout this section assume that $G$ is a compact Hodgkin-Lie group of rank $n$ with maximal torus $T$. Recall that a group satisfies the Hodgkin condition if it is connected and the fundamental group is torsion-free.

The embedding $T \subseteq G$ induces a restriction functor $KK_{\hat{T}} \rightarrow KK_{\hat{G}}$. Using the isomorphism $\hat{T} \cong \mathbb{Z}^n$, the tower (8) can be restricted to a $KK_{\hat{G}}$-tower:

In order to work with this $KK_{\hat{G}}$-tower we need to describe the terms $C^*(T)$ in the second row.

**Lemma 3.1.** If $G$ is a compact Hodgkin-Lie group with Weyl group of order $w$ there is an isomorphism

$$C^*(T) \cong C^w \otimes C^*(G) \text{ in } KK_{\hat{G}}.$$

Observe that the condition on $G$ to be a Hodgkin group is equivalent to $\hat{G}$ being a torsion-free quantum group in the sense of Meyer, see [9]. The torsion-free quantum groups are the only non-classical discrete quantum groups for which there is a general formulation of the Baum-Connes property in terms of triangulated categories. In [11], coactions of compact non-Hodgkin Lie groups were considered and the “torsion” turned out to be the torsion elements of $H^2(G, S^1)$. The less precise statement $C(G/T) \cong C^k$ in $KK_{\hat{G}}$ for some $k$ is stated and proved in [11]. An explicit calculation that $k = |W|$ can be found in [15]. We will review the conceptually important part of the proof of a Proposition in [11] which proves Lemma 3.1 aside from the calculation of $k$.

**Proof.** By [15], the representation ring $R(T)$ is free of rank $w$ over the representation ring $R(G)$ if $\pi_1(G)$ is torsion-free. If we let $\mathcal{S}$ denote the
localizing subcategory of $KK^G$ generated by $\mathcal{C}$ and $C(G/T)$, Lemma 11 of [10] states that for $A \in \mathcal{S}$ the natural homomorphism

$$R(T) \otimes_{R(G)} KK^G(A, \mathcal{C}) \longrightarrow KK^T(A, \mathcal{C})$$

is an isomorphism. Thus the representable functor on $\mathcal{S}$

$$A \longrightarrow KK^G(A, C^w) \cong R(T) \otimes_{R(G)} KK^G(A, \mathcal{C})$$

coincides with the representable functor

$$A \longrightarrow KK^G(A, C(G/T)) \cong KK^T(A, \mathcal{C}).$$

The last isomorphism exists as a consequence of the fact that the induction functor $\text{Ind}_T^G$ is the right adjoint of the restriction functor from $G$ to $T$. So the Yoneda lemma implies that $C(G/T) \cong C^w$ in $\mathcal{S}$ and therefore in $KK^G$. Applying Baaj-Skandalis duality it follows that there is an equivariant $KK$-isomorphism $C^*(T) \cong C^w \otimes C^*(G)$.

Using Lemma 3.1 the tower (8) takes the form:

\[ \Sigma^n C^w \otimes C^*(G) \longrightarrow C_0(Y_{n-1}) \longrightarrow \cdots \]

\[ \Sigma^{n-1} C^w \otimes C^*(G) \longrightarrow C_0(Y_n) \longrightarrow \Sigma^n \mathcal{C} \]

\[ \cdots \longrightarrow C_0(Y_2) \longrightarrow C_0(Y_1) \longrightarrow \Sigma^n \mathcal{C} \]

\[ \cdots \longrightarrow \Sigma C^w \otimes C^*(G) \longrightarrow C^w \otimes C^*(G) \]

We will call this $KK^G$-tower the fundamental $G$–PV-tower. The dual fundamental $G$–PV-tower is defined to be the $KK^G$-tower which is Baaj-Skandalis dual to the fundamental $G$–PV-tower:

\[ \Sigma^n C^w \longrightarrow D_{n-1} \longrightarrow \cdots \]

\[ \Sigma^{n-1} C^w \longrightarrow \Sigma^{n-2} C^{w_{k-1}} \longrightarrow \cdots \]

\[ \cdots \longrightarrow D_2 \longrightarrow D_1 \longrightarrow \Sigma^n C(G) \]

\[ \cdots \longrightarrow \Sigma C^w \longrightarrow C^w \]

where $D_i := C_0(Y_i) \rtimes_r \hat{G}$. 
As mentioned above, if $A$ is a $G$-$C^*$-algebra, the trivial coaction of $G$ on $A$ makes $A$ into a Yetter-Drinfeld algebra. This follows from that $C(G)$ is commutative so we can extend a $G$-action via the $D(G)$-equivariant $*$-monomorphism $C(G) \to \mathcal{M}(C_0(D(G)))$. Clearly, a $G$-equivariant mapping is equivariant in this new $D(G)$-action. Furthermore, since mapping cones does not depend on the action, the trivial extension of a $G$-action to a $D(G)$-action is functorial and respects mapping cones. The following proposition follows from universality.

**Proposition 3.2.** If $G$ is a locally compact group, the functor mapping a $G$-$C^*$-algebra to a $G$-Yetter-Drinfeld algebra with trivial $\hat{G}$-action defines a triangulated functor $KK^G \to KK^{D(G)}$.

Using the triangulated functor of Proposition 3.2, we may consider the tower (11) as a tower in $KK^{D(G)}$. Applying a crossed product by $G$ we obtain that also the tower (10) is a tower in $KK^{D(G)}$. For a $C^*$-algebra $B$ we will use the notation $t(B)$ for the $\hat{G}$-$C^*$-algebra with trivial coaction, or in the context of $G$-$C^*$-algebras $t(B)$ will denote the $G$-$C^*$-algebra with trivial action. Let us state and prove the corresponding version of (3) in a simple case of a braided tensor product over $G$ with $C(G)$, a more general proof can be found in [12].

**Lemma 3.3.** When $B$ has a continuous $G$-action, there is a $\hat{G}$-equivariant Morita equivalence

$$(C(G) \otimes B) \rtimes_r G \sim_M t(B).$$

**Proof.** By Baaj-Skandalis duality, it suffices to prove that there is a $\hat{G}$-equivariant isomorphism $(C(G) \otimes B) \rtimes_r G \cong (C(G) \rtimes_r G) \otimes t(B)$. Denote the $G$-action on $B$ by $\beta$ and define the equivariant mapping $\varphi_0 : L^1(G, C(G, B)) \to (C(G) \rtimes_r G) \otimes t(B)$ by setting

$$\varphi_0(f)(g_1, g_2) := \beta_{g_1}^{-1} f(g_1, g_2).$$

The linear mapping $\varphi_0$ is a $*$-homomorphism when $L^1(G, C(G, B))$ is equipped with the convolution twisted by the $G$-action on $C(G) \otimes B$. It is straightforward to verify that $\varphi_0$ is bounded in $C^*$-norm so we can define $\varphi : (C(G) \otimes B) \rtimes_r G \to (C(G) \rtimes_r G) \otimes B$ by continuity. The $*$-homomorphism $\varphi$ is an equivariant isomorphism since an inverse can be constructed by extending

$$\varphi^{-1}(f \otimes b)(g_1, g_2) := f(g_1, g_2)\beta_{g_1}(b)$$

to a $*$-homomorphism $\varphi^{-1} : (C(G) \rtimes_r G) \otimes t(B) \to (C(G) \otimes B) \rtimes_r G$. 
Theorem 3.4 (The Pimsner-Voiculescu tower). Let $G$ be a compact Hodgkin-Lie group of rank $n$ and Weyl group of order $w$. For any separable $\hat{G}$-C*-algebra $A$ there is a $KK^{\hat{G}}$-tower

$$
\begin{array}{cccccc}
\Sigma C^w \otimes A & \longrightarrow & \Sigma^n D_{n-1}(A) & \longrightarrow & \Sigma^n D_{n-2}(A) & \longrightarrow & \cdots \\
\Sigma^n \otimes A & \longleftarrow & \Sigma^2 C^{w_{k-1}} \otimes A & \longleftarrow & \cdots \\
\cdots & \longrightarrow & \Sigma^n D_2(A) & \longrightarrow & \Sigma^n D_1(A) & \longrightarrow & t(A \ltimes_r \hat{G}) \\
\cdots & \longleftarrow & \Sigma^{n-1} C^{w_{k+2}} \otimes A & \longleftarrow & \Sigma^n C^w \otimes A
\end{array}
$$

where $D_i(A) := (C_0(Y_i) \otimes \mathcal{H}(L^2(G))) \boxtimes G (A \ltimes_r \hat{G})$ and is equipped with the $\hat{G}$-action induced from the diagonal $\hat{G}$-action on $C_0(Y_i) \otimes \mathcal{H}(L^2(G))$.

Observe that the $D(G)$-actions on the C*-algebras $C_0(Y_i) \otimes \mathcal{H}(L^2(G))$ is defined to come from those on their Baaj-Skandalis duals $C_0(Y_i) \ltimes_r \hat{G}$, which are $D(G) - C^*$-algebras in the dual $G$-actions on the crossed products and the trivial $\hat{G}$-actions. So in general, $D_i(A)$ is not the tensor product of $C_0(Y_i) \otimes \mathcal{H}(L^2(G))$ and $A \ltimes_r \hat{G}$.

Proof. By Lemma 3.3 the $\hat{G}$-C*-algebra $A$ admits the equivariant Morita equivalence:

$$
(C(G) \otimes (A \ltimes_r \hat{G})) \ltimes_r G \sim_M t(A \ltimes_r \hat{G}).
$$

Furthermore, the isomorphism of equation (3) holds for braided tensor products over $G$ so while the $\hat{G}$-actions on $D_i = C_0(Y_i) \ltimes_r \hat{G}$ are trivial there are equivariant isomorphisms

$$
(D_i \otimes (A \ltimes_r \hat{G})) \ltimes_r G \cong ((C_0(Y_i) \ltimes_r \hat{G}) \boxtimes G (A \ltimes_r \hat{G})) \ltimes_r G
$$

$$
\cong (C_0(Y_i) \otimes \mathcal{H}(L^2(G))) \boxtimes G (A \ltimes_r \hat{G}).
$$

Thus if we tensor the dual fundamental $G$–PV-tower (11) by the $G$–C*-algebra $A \ltimes_r \hat{G}$ we obtain a new $KK^{\hat{G}}$-tower which becomes the Pimsner-Voiculescu tower of $A$ after applying Baaj-Skandalis duality, using the Morita equivalence (13) and the isomorphisms (14).

The Pimsner-Voiculescu tower (12) is the generalization of the resolution in (9) to compact Hodgkin-Lie groups. Applying the cohomological functor $KK(-,B)$ to the Pimsner-Voiculescu tower we obtain a similar resolution of
\( KK^*_*(A \rtimes_r \hat{G}, B) \) in terms of \( KK^*_*(A, B) \) as in (9). Similarly, the homological functor \( KK(B, -) \) applied to the Pimsner-Voiculescu tower gives a resolution of \( KK(B, A \rtimes \hat{G}) \) in terms of \( KK(B, A) \). Observe that since \( A \) has a \( \hat{G} \)-action, the groups \( KK(C^w \otimes A, B) \) and \( KK(B, C^w \otimes A) \) will always have an \( R(G) \)-module structure and since \( R(T) \) is free over \( R(G) \) also an \( R(T) \)-module structure.

As an example of this, we will use the Pimsner-Voiculescu tower to calculate the \( K \)-theory of the homogeneous space \( G/H \) when \( H \subseteq G \) is a Lie subgroup. More generally, this technique can be used to calculate \( K^*_*(A \rtimes_r \hat{G}) \) for any \( \hat{G} \)-C*-algebra \( A \) when one knows \( K^*_*(A) \) and its \( R(G) \)-module structure coming from the Julg isomorphism \( K^*_*(A) \cong K^*_G(A \rtimes_r \hat{G}) \).

We use \( \Sigma \) to denote degree shift in the category of \( \mathbb{Z}/2\mathbb{Z} \)-graded abelian groups. Here we have used that \( R(T) \) is a free \( R(G) \)-module of rank \( w \) so \( K^*_*(C^w \otimes C^* (H)) \cong R(T) \otimes_{R(G)} R(H). \) Thus the lowest row is the tensor product of \( R(H) \) with the Koszul complex of \( R(T) \) that is associated with the regular sequence \( 1 - t_1, 1 - t_2, \ldots, 1 - t_n \) under the isomorphism \( R(T) \cong \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}] \).

If we restrict our attention to simple compact Lie groups we can perform an explicit calculation of all the groups in (15). Assume that \( G = G_n \) is a simple compact Hodgkin-Lie group in the classical \( A, B, C \)- or \( D \)-series of rank \( n \) and assume that \( H = G_k \subseteq G_n \) is a simple simply connected compact Lie group in the same classical series being of rank \( k < n \). We may take a maximal torus \( T_n \subseteq G_n \) such that \( T_k := T_n \cap G_k \) is a maximal torus in \( G_k \). In this case we may consider \( R(T_k) \) as an ideal in \( R(T_n) \) and \( R(T_n) \otimes_{R(G_k)} R(G_k) \cong R(T_k) \) as \( R(T_n) \)-modules. Under the isomorphisms \( R(T_k) \cong \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_k^{\pm 1}] \) and...
$R(T_n) \cong \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]$, the Koszul vector $v$ is identified with $\sum_{i=1}^k (1 - t_i)e_i^* \in \text{Hom}(R(\hat{T}_k)^n, R(T_k))$. Thus we arrive at the tower

\[
\begin{array}{cccccc}
R(T_k) & \to & K_{* - n}(D_{n-1}(C^*(G_k))) & \to & \cdots \\
\Sigma \partial_n & & & & \\
\Sigma \mathbb{Z} \otimes Z R(T_k) & \to & \Sigma \mathbb{Z} \otimes Z R(T_k) & \leftarrow & \cdots \\
\cdots & \to & K_{* - n}(D_1(C^*(G_k))) & \to & K^*(G_n/G_k) \\
\cdots & & & & & & \Sigma \partial_1 \otimes \Sigma \mathbb{Z} \otimes Z R(T_k) \\
\end{array}
\]

Let us use the notation $E^*$ for the complex $\wedge^{n-*} \mathbb{Z} \otimes R(T_k)$ equipped with the Koszul differential from the vector $\sum_{i=1}^k (1 - t_i)e_i^*$ which we as above denote by $\partial_l : E^{l-1} \to E^l$. After some simpler calculations in this Koszul complex we arrive at the conclusion that

\[
K_{* - n}(D_l(C^*(G_k))) \cong \ker(\partial_{l+1}) \oplus \bigoplus_{j=l+2}^{n+1} \Sigma^{n-j} H^j(E^*).
\]

Hence we obtain the isomorphism $K^*(G_n/G_k) \cong \bigoplus_{j=1}^{n+1} \Sigma^{n-j} H^j(E^*)$. These cohomology groups are calculated in Corollary 17.10 of [5] and $H^j(E^*)$ is a free group of rank $k(j) := (n - k)!/(n - j)! (n - j - k)!$ if $0 \leq j \leq n - k$ and 0 otherwise. Therefore

\[
K^*(G_n/G_k) \cong \bigoplus_{j=0}^{n-k} \Sigma^{n-j} Z^{k(j)} = \bigoplus_{j=0}^{n-k-1} Z^{2^{n-k-1}} \oplus \bigoplus_{j=n-k}^{n-k+1} Z^{2^{n-k-1}}.
\]

**Theorem 3.5 (The dual Pimsner-Voiculescu tower).** Under the assumptions of Theorem 3.4 there is a KK$^G$-tower

\[
\begin{array}{cccccccc}
\mathcal{C}^w \otimes t(A) & \to & \Sigma^n \tilde{D}_{n-1}(A) & \to & \cdots \\
\Sigma \mathcal{C}^{w,n} \otimes t(A) & \leftarrow & \cdots & \Sigma^2 \mathcal{C}^{w,k-1} \otimes t(A) & \leftarrow & \cdots \\
\cdots & \to & \Sigma^n \tilde{D}_2(A) & \to & \Sigma^n \tilde{D}_1(A) & \to & A \rtimes_r \hat{G} \\
\cdots & & & & & & \Sigma^{n-1} \mathcal{C}^{w,k} \otimes t(A) & \leftarrow & \cdots & \Sigma^n \mathcal{C}^w \otimes t(A) \\
\end{array}
\]

where $\tilde{D}_i(A) := D_i \otimes_{\hat{G}} A$. 

For a homological functor $F : KK^\hat{G} \to Ab$, the dual Pimsner-Voiculescu tower of $A$ allows us to calculate $F(A)$ in terms of the objects $F(C_r^*(G) \otimes t(A))$. As we shall see below, $\hat{G}$-$C^*$-algebras of the form $C_r^*(G) \otimes t(A)$ behaves similarly to proper actions. Compare this result to Theorem 4.4 of [8].

**Proof.** Consider the braided tensor product by $\Sigma^n A$ and the tower (10):

\[
\begin{align*}
&\Sigma^n C^w \otimes C^*(G) \hat{\otimes} A \\
&\longrightarrow \Sigma^n C_0(Y_{n-1}) \hat{\otimes} A \\
&\longrightarrow \cdots
\end{align*}
\]

Taking crossed product between this tower and $\hat{G}$ implies the Theorem since the following equivariant Morita equivalences follows from (3)

\[
(C^*(G) \hat{\otimes} A) \rtimes_r \hat{G} \sim_M t(A)
\]

and

\[
(C_0(Y_i) \hat{\otimes} A) \rtimes_r \hat{G} \sim_M (C_0(Y_i) \rtimes_r \hat{G}) \hat{\otimes} A = D_i \hat{\otimes} A.
\]

One of the main motivations behind this paper was to give a precise description of the Baum-Connes property of duals of Hodgkin-Lie groups. The Baum-Connes property for coactions of compact Lie groups was given meaning to and was proved to hold in [11]. More generally, this fits into the program of generalizing the Baum-Connes property to quantum groups. So far, it is not known what a suitable property the Baum-Connes property should be for a general locally compact quantum group. For discrete quantum groups which are torsion-free, in the sense of [9], there is a formulation and as mentioned above duals of compact Hodgkin-Lie groups are torsion-free.

The problem that arises when one tries to define the Baum-Connes assembly mapping for a quantum group is that there is no natural notion of a proper action and there are in general too many quantum homogeneous spaces. It is much easier to generalize certain notions of free actions than proper actions of a quantum group by just saying that an action of a discrete quantum group $\Gamma$ on a $C^*$-algebra $A$ is truly free if there is a $C^*$-algebra $A_0$ and an equivariant
$\ast$-isomorphism $A \cong A_0 \otimes_{\text{min}} C_0(\Gamma)$ with $\Gamma$ only acting on the second leg. In the case of a group, there are many free actions that are not truly free but this stronger notion of a free action will suffice for our purposes.

Restricting one’s attention to generalizing the Baum-Connes property of the simpler class of torsion-free discrete groups to the quantum setting, when proper actions are free, Meyer introduced a class of quantum groups known as torsion-free in [9]. Following [9], we say that a discrete quantum group $\Gamma$ is torsion-free if every coaction of the compact quantum group $\hat{\Gamma}$ on a finite-dimensional $C^*$-algebra is Morita equivalent to a trivial coaction on a direct sum of C:s. This fact implies that any finite-dimensional projective representation of the dual compact quantum group is equivalent to a representation. If $\Gamma$ is a discrete group, coactions of the dual compact quantum group on finite-dimensional $C^*$-algebras that are not Morita equivalent to a trivial coaction on a direct sum of C:s correspond to finite subgroups so a discrete group is torsion-free if and only if it is torsion-free in the sense of [9].

For a torsion-free quantum group a proper action should correspond to a free action. Under Baaj-Skandalis duality, a truly free $\Gamma - C^*$-algebra corresponds to a trivial $\hat{\Gamma}$-action. Let $\mathcal{C}\mathcal{I}_{\hat{\Gamma}}$ denote the image of $t : KK \to KK^{\hat{\Gamma}}$. The triangulated category $\langle \mathcal{C}\mathcal{I}_{\hat{\Gamma}} \rangle$ is defined as the localizing subcategory generated by $\mathcal{C}\mathcal{I}_{\hat{\Gamma}}$. Following the formulation of [9], $\Gamma$ is said to satisfy the strong Baum-Connes property if the embedding of triangulated categories $\langle \mathcal{C}\mathcal{I}_{\hat{\Gamma}} \rangle \to KK^{\hat{\Gamma}}$ is essentially surjective. The strong Baum-Connes property of $\Gamma$ is equivalent to that any $\Gamma - C^*$-algebra is in the localizing category generated by all truly free actions. So regardless of what notion of a proper action we choose, the strong Baum-Connes conjecture will imply that the localizing category generated by all such proper actions will be $KK^{\hat{\Gamma}}$. The quantum group is said to satisfy the Baum-Connes property if the same statement holds after localizing with respect to the kernel of equivariant $K$-theory.

In [11] the Baum-Connes property was formulated in the slightly more general setting of duals of compact Lie groups. The finite-dimensional projective representations of a compact Lie group $G$ correspond to the torsion classes of $H^2(G, S^1)$, which can be thought of as the torsion of $\hat{G}$. When $G$ is Hodgkin, $H^2(G, S^1)$ is torsion-free so $\hat{G}$ is torsion-free. In this case a “proper” action is an object of the additive category generated by $\hat{G}$-algebras that are Baaj-Skandalis dual to $A_0 \otimes C_\omega$, with $C_\omega$ denoting the endomorphisms of a projective representation $\omega$ and $A_0$ having trivial $G$-action. So the substitute in the setting of [11] for proper actions is the category of tensor products between Baaj-Skandalis duals of coactions on finite-dimensional $C^*$-algebras and trivial actions, just as the truly free actions form a substitute for proper actions of torsion-free quantum groups. The Baum-Connes property of coactions of a compact Hodgkin-Lie group is a direct consequence of Theorem 3.5.
The method of proof of Proposition 2.1 of [11] can be used to generalize both Theorem 3.4 and Theorem 3.5 to arbitrary compact Lie group.

Finally, let us mention a promising generalization of Theorem 3.5 to Woronowicz deformations. It was proved in [12] that the compact quantum group $SU_q(2)$ satisfies that $C(SU_q(2)/T)$ is $KK^{D(SU_q(2))}$-isomorphic to $C^2$ for $q \in ]0, 1[$. So if we apply the induction functor $\text{Ind}^{SU_q(2)}_T : KK^T \to KK^{SU_q(2)}$ to the distinguished triangle Baaj-Skandalis dual to (5) and use the isomorphism of Nest-Voigt we arrive at the distinguished triangle in $KK^{D(SU_q(2))}$:

$$
\begin{array}{ccc}
C^2 & \longrightarrow & C^2 \\
& \searrow & \\
& C(SU_q(2)) & \\
\end{array}
$$

Using the technique from the proof of Theorem 3.5 any $A \in KK^{SU_q(2)}$ fits into a distinguished triangle

$$
\begin{array}{ccc}
C^2 \otimes t(A) & \longrightarrow & C^2 \otimes t(A) \\
& \swarrow & \\
& A \rtimes SU_q(2) & \\
\end{array}
$$

This distinguished triangle gives an alternative proof of the strong Baum-Connes property for $SU_q(2)$, a result first proved in [17]. The interesting part about this proof is that it only relies on the isomorphism $C(G_q/T) \cong C^w$ in $KK^{D(G_q)}$. So if such an isomorphism exists for a simply connected semi-simple compact Lie group $G$, the strong Baum-Connes conjecture holds for $\hat{G}_q$, the quantum dual of the Woronowicz deformation of $G$. To formulate the Baum-Connes property for $\hat{G}_q$ we must of course know that it is torsion-free, a statement proved in [17] for $G = SU(2)$ and the general case was proved in [6]. Another striking application of such an isomorphism is that the method above for calculating $K$-theory of homogeneous spaces can be generalized to classical quantum homogeneous spaces of the Woronowicz deformations.

REFERENCES


