Abstract
Let \( A \) be an \( N \times N \) irreducible matrix with entries in \( \{0, 1\} \). We define the topological Markov Dyck shift \( D_A \) to be a nonsofic subshift consisting of bi-infinite sequences of the \( 2N \) brackets \((1, \ldots, (N, 1), \ldots)\) with both standard bracket rule and Markov chain rule coming from \( A \). It is regarded as a subshift defined by the canonical generators \( S_1^*, \ldots, S_N^*, S_1, \ldots, S_N \) of the Cuntz-Krieger algebra \( \mathcal{O}_A \). We construct an irreducible \( \lambda \)-graph system \( \mathcal{G}_{\text{Ch}}(D_A) \) that presents the subshift \( D_A \) so that we have an associated simple purely infinite \( C^* \)-algebra \( \mathcal{O}_{\mathcal{G}_{\text{Ch}}(D_A)} \). We prove that \( \mathcal{O}_{\mathcal{G}_{\text{Ch}}(D_A)} \) is a universal unique \( C^* \)-algebra subject to some operator relations among \( 2N \) generating partial isometries.

1. Introduction
Let \( \Sigma \) be a finite set with its discrete topology, that is called an alphabet. Each element of \( \Sigma \) is called a symbol. Let \( \Sigma^\mathbb{Z} \) be the infinite product space \( \prod_{i=-\infty}^{\infty} \Sigma_i \), where \( \Sigma_i = \Sigma \), endowed with the product topology. The transformation \( \sigma \) on \( \Sigma^\mathbb{Z} \) given by \( \sigma((x_i)_{i \in \mathbb{Z}}) = (x_{i+1})_{i \in \mathbb{Z}} \) is called the full shift over \( \Sigma \). Let \( \Lambda \) be a closed subset of \( \Sigma^\mathbb{Z} \) such that \( \sigma(\Lambda) = \Lambda \). The topological dynamical system \( (\Lambda, \sigma|_{\Lambda}) \) is called a subshift or a symbolic dynamical system. It is written as \( \Lambda \) for brevity. There is a class of subshifts called sofic shifts, that contains the class of topological Markov shifts. Sofic shifts are presented by finite labeled graphs, called \( \lambda \)-graphs. In [19], the author has introduced a notion of \( \lambda \)-graph system as a generalization of \( \lambda \)-graph. A \( \lambda \)-graph system \( \mathcal{U} = (V, E, \lambda, \iota) \) over \( \Sigma \) consists of a vertex set \( V = V_0 \cup V_1 \cup V_2 \cup \ldots \), an edge set \( E = E_{0,1} \cup E_{1,2} \cup E_{2,3} \cup \ldots \), a labeling map \( \lambda : E \to \Sigma \) and a surjective map \( \iota_{l,l+1} : V_{l+1} \to V_l \) for each \( l \in \mathbb{Z}_+ \), where \( \mathbb{Z}_+ \) denotes the set of all nonnegative integers. An edge \( e \in E_{l,l+1} \) has its source vertex \( s(e) \) in \( V_l \), its terminal vertex \( t(e) \) in \( V_{l+1} \) and its label \( \lambda(e) \) in \( \Sigma \).

The theory of symbolic dynamical system has a close relationship with formal language theory. In the theory of formal language, there is a class of universal languages due to W. Dyck. The symbolic dynamics generated by the
languages are called the Dyck shifts $D_N$ (cf. [3], [10], [11], [12]). They are non-sofic subshifts. Its alphabet consists of the $2N$ brackets: $(1, \ldots, (N, )_1, \ldots, )_N$. The forbidden words consist of words that do not obey the standard bracket rules. In [14], a $\lambda$-graph system $\mathcal{Ch}^{D_N}$ that presents the subshift $D_N$ has been introduced. The $\lambda$-graph system is called the Cantor horizon $\lambda$-graph system for the Dyck shift $D_N$. The K-groups for $\mathcal{Ch}^{D_N}$, that are invariant under topological conjugacy of the subshift $D_N$, have been computed ([14]).

In [20], a nuclear $C^*$-algebra $\mathcal{O}_\lambda$ associated with a $\lambda$-graph system $\mathcal{Ch}^{D_N}$ has been introduced. The class of the $C^*$-algebras contain the class of the Cuntz-Krieger algebras. They are universal unique concrete $C^*$-algebras generated by finite families of partial isometries and sequences of projections subject to certain operator relations encoded by structure of the $\lambda$-graph systems. Its K-groups $K_i(\mathcal{O}_\lambda)$, $i = 0, 1$ are realized as the K-groups of the $\lambda$-graph system $\mathcal{Ch}^{D_N}$. The results of [14] imply that the $C^*$-algebras $\mathcal{O}_\lambda$ for $N = 2, 3, \ldots$ are unital, simple and purely infinite whose K-groups are

$$K_0(\mathcal{O}_\lambda) \cong \mathbb{Z} / N \mathbb{Z} \oplus C(\mathfrak{s} \lambda, \mathbb{Z}), \quad K_1(\mathcal{O}_\lambda) \cong 0$$

where $C(\mathfrak{s} \lambda, \mathbb{Z})$ denotes the abelian group of all integer valued continuous functions on a Cantor discontinuum $\mathfrak{s} \lambda$. Let $u_1, \ldots, u_N$ be the canonical generating isometries of the Cuntz algebra $\mathcal{O}_N$ that satisfy the relations: $\sum_{j=1}^N u_j u_j^* = 1$, $u_i^* u_i = 1$ for $i = 1, \ldots, N$. Then the bracket rule of the symbols $(1, \ldots, (N, )_1, \ldots, )_N$ of the Dyck shift $D_N$ may be interpreted as the relations

$$u_i^* u_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases}$$

for $i, j = 1, \ldots, N$ in $\mathcal{O}_N$ through the correspondence $(i \rightarrow u_i^*, )_i \rightarrow u_i$ (cf. (2.1)).

In the present paper, we consider a generalization of the Dyck shifts $D_N$ by using the canonical generating partial isometries of the Cuntz-Krieger algebras $\mathcal{O}_A$ for $N \times N$ matrices $A$ with entries in $\{0, 1\}$. The generalized Dyck shift is denoted by $D_A$ and called the topological Markov Dyck shift for $A$ (cf. [7], [11], [15], [16]). Let $\alpha_1, \ldots, \alpha_N, \beta_1, \ldots, \beta_N$ be the alphabet of $D_A$, corresponding to the brackets $(1, \ldots, (N, )_1, \ldots, )_N$ respectively. Let $t_1, \ldots, t_N$ be the canonical generating partial isometries of $\mathcal{O}_A$ satisfying the relations

$$\sum_{j=1}^N t_j t_j^* = 1, \quad t_i^* t_i = \sum_{j=1}^N A(i, j) t_j t_j^* \quad \text{for } i = 1, \ldots, N.$$

Consider the correspondence $\varphi(\alpha_i) = t_i^*$, $\varphi(\beta_i) = t_i$, $i = 1, \ldots, N$. Then a word $w$ of $\{\alpha_1, \ldots, \alpha_N, \beta_1, \ldots, \beta_N\}$ is defined to be admissible for the subshift
precisely if the corresponding element to $w$ through $\varphi$ in $O_A$ is not zero. If $A$ satisfies condition (I) in the sense of [5], the subshifts $D_A$ are not sofic (Proposition 2.1). If all entries of $A$ are 1’s, $D_A$ is reduced to $D_N$. We consider the Cantor horizon $\lambda$-graph system $Q_{Ch(D_A)}$ for the topological Markov Dyck shift $D_A$. The $\lambda$-graph system will be proved to be $\lambda$-irreducible with $\lambda$-condition (I) in the sense of [23] if the matrix $A$ is irreducible with condition (I) (Proposition 2.5). Hence the associated $C^*$-algebra $O_{Q_{Ch(D_A)}}$ is simple and purely infinite. We will show:

**Theorem 1.1.** Let $A$ be an $N \times N$ matrix with entries in $\{0, 1\}$. Suppose that $A$ is irreducible with condition (I). The $C^*$-algebra $O_{Q_{Ch(D_A)}}$ associated with the $\lambda$-graph system $Q_{Ch(D_A)}$ is unital, separable, nuclear, simple and purely infinite. It is the unique $C^*$-algebra generated by $2N$ partial isometries $S_i, T_i, i = 1, \ldots, N$ subject to the following relations:

\[(1.3) \quad \sum_{j=1}^{N} (S_j S_j^* + T_j T_j^*) = 1,\]

\[(1.4) \quad \sum_{j=1}^{N} S_j S_j^* = 1,\]

\[(1.5) \quad T_i^* T_i = \sum_{j=1}^{N} A(i, j) S_j^* S_j, \quad i = 1, 2, \ldots, N,\]

\[(1.6) \quad E_{\mu_1, \ldots, \mu_k} = \sum_{j=1}^{N} A(j, \mu_1) S_j S_j^* E_{\mu_1, \ldots, \mu_k} S_j S_j^* + T_{\mu_1} E_{\mu_2, \ldots, \mu_k} T_{\mu_1}, \quad k > 1\]

where $E_{\mu_1, \ldots, \mu_k} = S_{\mu_1}^* \ldots S_{\mu_k}^* S_{\mu_k} \ldots S_{\mu_1}$ for $\mu_1, \ldots, \mu_k \in \{1, \ldots, N\}$.

Let $X_A$ be the right one-sided topological Markov shift

\[X_A = \{(x_i)_{i \in \mathbb{N}} \in \{1, \ldots, N\}^\mathbb{N} \mid A(x_i, x_{i+1}) = 1, i \in \mathbb{N}\}\]

for the matrix $A$ and $\sigma_A$ the shift on $X_A$ defined by $\sigma_A((x_i)_{i \in \mathbb{N}}) = (x_{i+1})_{i \in \mathbb{N}}$ for $(x_i)_{i \in \mathbb{N}} \in X_A$. Let $\sigma_{A_A}$ and $\lambda_A$ be endomorphisms of the abelian group $C(X_A, \mathbb{Z})$ of all $\mathbb{Z}$-valued continuous functions on $X_A$ defined by

\[\sigma_{A_A}(f)(x) = f(\sigma_A(x)), \quad \lambda_A(f)(x) = \sum_{j=1}^{N} A(j, x_1) f(jx)\]

for $f \in C(X_A, \mathbb{Z})$ and $x = (x_i)_{i \in \mathbb{N}} \in X_A$, where $jx = (j, x_1, x_2, \ldots) \in X_A$ for $A(j, x_1) = 1$. Then we will show:
Theorem 1.2.
(i) \( K_0(\mathcal{O}_{\text{Ch}(DA)}) = C(X_A, \mathbb{Z})/(\text{id} - (\sigma_{A_A} + \lambda_{A_A}))C(X_A, \mathbb{Z}) \).
(ii) \( K_1(\mathcal{O}_{\text{Ch}(DA)}) = \text{Ker}(\text{id} - (\sigma_{A_A} + \lambda_{A_A})) \) in \( C(X_A, \mathbb{Z}) \).

If all entries of \( A \) are 1's, the \( \lambda \)-graph system \( \mathcal{Q}_{\text{Ch}(DA)} \) becomes \( \mathcal{Q}_{\text{Ch}(DN)} \) so that the \( C^* \)-algebra \( \mathcal{O}_{\text{Ch}(DA)} \) goes to \( \mathcal{O}_{\text{Ch}(DN)} \). If \( A \) is the Fibonacci matrix \( F = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \), the \( C^* \)-algebra \( \mathcal{O}_{\text{Ch}(DF)} \) is simple and purely infinite. Its K-groups are
\[
\begin{align*}
K_0(\mathcal{O}_{\text{Ch}(DF)}) & \cong \mathbb{Z} \oplus C(\mathbb{C}/\mathbb{Z})^\infty, \\
K_1(\mathcal{O}_{\text{Ch}(DF)}) & = 0
\end{align*}
\]
where \( C(\mathbb{C}/\mathbb{Z})^\infty \) denotes the countable infinite direct sum of the group \( C(\mathbb{C}/\mathbb{Z}) \) (cf. [25]). In general, the \( C^* \)-algebra \( \mathcal{O}_{\text{Ch}(DA)} \) associated with a \( \lambda \)-graph system \( \mathcal{Q}_{\text{Ch}(DA)} \) has an infinite family of generators. Both of the \( C^* \)-algebras \( \mathcal{O}_{\text{Ch}(DN)}, \mathcal{O}_{\text{Ch}(DF)} \) are finitely generated, and their \( K_0 \)-groups however are not finitely generated. Therefore they are not semiprojective whereas Cuntz algebras and Cuntz-Krieger algebras are semiprojective (cf. [1], [2], [21], [26]).

2. The topological Markov Dyck shifts

Throughout the paper \( N \) is a fixed positive integer larger than 1.

We consider the Dyck shift \( D_N \) with alphabet \( \Sigma = \Sigma^- \cup \Sigma^+ \) where \( \Sigma^- = \{\alpha_1, \ldots, \alpha_N\}, \Sigma^+ = \{\beta_1, \ldots, \beta_N\} \). The symbols \( \alpha_i, \beta_i \) correspond to the brackets \( (, ) \) respectively, and have the relations
\[
\alpha_i \beta_j = \begin{cases} 
1 & \text{if } i = j, \\
0 & \text{otherwise}
\end{cases}
\]
for \( i, j = 1, \ldots, N \) (cf. (1.2), [11],[12]). A word \( \gamma_1 \ldots \gamma_n \) of \( \Sigma \) is defined to be admissible for \( D_N \) precisely if \( \prod_{m=1}^n \gamma_m \neq 0 \), where \( \prod_{m=1}^n \gamma_m \) means the product \( \gamma_1 \ldots \gamma_n \) obtained by applying (2.1).

Let \( A = [A(i, j)]_{i,j=1,\ldots,N} \) be an \( N \times N \) matrix with entries in \{0, 1\} having no zero rows or columns. Consider the Cuntz-Krieger algebra \( \mathcal{O}_A \) for the matrix \( A \) that is the universal \( C^* \)-algebra generated by \( N \) partial isometries \( t_1, \ldots, t_N \) subject to the following relations ([5]):
\[
\sum_{j=1}^N t_j^* t_j = 1, \quad t_i^* t_i = \sum_{j=1}^N A(i, j) t_j t_j^* \quad \text{for } i = 1, \ldots, N.
\]
Define a correspondence \( \varphi_A : \Sigma \longrightarrow \{t_1^*, \ldots, t_N^*, t_1, \ldots, t_N\} \) by setting
\[
\varphi_A(\alpha_i) = t_i^*, \quad \varphi_A(\beta_i) = t_i \quad \text{for } i = 1, \ldots, N.
\]
We denote by \( \Sigma^* \) the set of all words \( \gamma_1 \ldots \gamma_n \) of elements of \( \Sigma \). Define the set
\[
\tilde{\Sigma}_A = \{\gamma_1 \ldots \gamma_n \in \Sigma^* \mid \varphi_A(\gamma_1) \ldots \varphi_A(\gamma_n) = 0\}.
\]
Let $D_A$ be the subshift over $\Sigma$ whose forbidden words are $\widehat{\Gamma}_A$. The subshift is called the topological Markov Dyck shift defined by $A$. These kinds of subshifts have first appeared in [7] in semigroup setting and in [15] in more general setting without using $C^*$-algebras. If all entries of $A$ are 1's, the partial isometries $\phi_A(\alpha_1), \ldots, \phi_A(\alpha_N), \phi_A(\beta_1), \ldots, \phi_A(\beta_N)$ satisfy the same relations as (2.1) so that the subshift $D_A$ becomes the Dyck shift $D_N$. We note the fact that $\alpha_i \beta_j \in \widehat{\Gamma}_A$ if $i \neq j$, and $\alpha_{i_1} \ldots \alpha_{i_n} \in \widehat{\Gamma}_A$ if and only if $\beta_{i_1} \ldots \beta_{i_n} \in \widehat{\Gamma}_A$.

Consider the following two subsystems of $D_A$

$$D^+_A = \{(\gamma_i)_{i \in \mathbb{Z}} \in D_A \mid \gamma_i \in \Sigma^+ \text{ for all } i \in \mathbb{Z}\},$$

$$D^-_A = \{(\gamma_i)_{i \in \mathbb{Z}} \in D_A \mid \gamma_i \in \Sigma^- \text{ for all } i \in \mathbb{Z}\}.$$ 

The subshift $D^+_A$ is identified with the topological Markov shift $\Lambda_A = \{(x_i)_{i \in \mathbb{Z}} \in \{1, \ldots, N\}^\mathbb{Z} \mid A(x_i, x_{i+1}) = 1, i \in \mathbb{Z}\}$ defined by $A$ through the one block code $\beta_i \mapsto i$. Similarly $D^-_A$ is identified with the topological Markov shift $\Lambda_{A'}$ defined by the transposed matrix $A'$ of $A$. Hence the subshift $D_A$ contains copies of both of the topological Markov shifts $\Lambda_A$ and $\Lambda_{A'}$. The following proposition implies that most irreducible matrices $A$ yield non Markov subshifts $D_A$.

**Proposition 2.1.** If $A$ satisfies condition (I) in the sense of [5], the subshift $D_A$ is not sofic.

**Proof.** Recall that $X_A$ is the right one-sided topological Markov shift $\{(x_i)_{i \in \mathbb{N}} \mid (x_i)_{i \in \mathbb{Z}} \in \Lambda_A\}$ for $A$. Put $X_{D^+_A} = \{(y_i)_{i \in \mathbb{N}} \mid (y_i)_{i \in \mathbb{Z}} \in D^+_A\}$. Since $A$ satisfies condition (I), we can find elements $(n(i))_{i \in \mathbb{N}} \in X_A$ such that $n(i)_{i \in \mathbb{N}} \neq k(i)_{i \in \mathbb{N}}$ for $n \neq k$. Put $x(n) = (\beta_{n(i)})_{i \in \mathbb{N}} \in X_{D^+_A}$ for $n \in \mathbb{N}$. Let $\Gamma^-(x(n)) = \{(\ldots, y_{-2}, y_{-1}, y_0) \mid (\ldots, y_{-2}, y_{-1}, y_0, \beta_{n(1)}, \beta_{n(2)}, \ldots) \in D_A\}$. The left one-sided sequence $(\ldots, \alpha_{k(2)}, \alpha_{k(1)})$ belongs to $\Gamma^-(x(n))$ if and only if $n = k$. Thus the predecessor sets $\Gamma^-(x(n))$, $n = 1, 2, \ldots$ are mutually distinct, so that $D_A$ is not sofic (cf. [18, Theorem 3.2.10]).

A $\lambda$-graph system $\mathcal{G}$ is said to present a subshift $\Lambda$ if the set of all admissible words of $\Lambda$ coincides with the set of all finite labeled sequences appearing in concatenating edges of $\mathcal{G}$. There are many $\lambda$-graph systems that present a given subshift. Among them the canonical $\lambda$-graph system is a generalization of the left-Krieger cover graph for a sofic shift([19]). The canonical $\lambda$-graph system $\mathcal{G}^{C(D_N)}$ for the Dyck shift $D_N$ together with its K-groups has been studied in [22]. One however sees that the $\lambda$-graph system $\mathcal{G}^{C(D_N)}$ is not irreducible,
We define an edge labeled \( \alpha_j \) for \( \Lambda_1 \) such that the resulting \( C^* \)-algebra \( \mathcal{O}_{\mathcal{C}(D_N)} \) is not simple. The Cantor horizon \( \lambda \)-graph system \( \mathcal{C}(D_N) \) for \( D_N \) is an irreducible component of \( \mathcal{L}(\Lambda_1) \) so that the associated \( C^* \)-algebra \( \mathcal{O}_{\mathcal{C}(D_N)} \) is simple and purely infinite whose K-groups have been computed as (1.1) [24].

In the paper we will study the Cantor horizon \( \lambda \)-graph systems \( \mathcal{C}(D_N) \) for the topological Markov Dyck shifts \( D_N \) and its associated \( C^* \)-algebras \( \mathcal{O}_{\mathcal{C}(D_N)} \). In what follows we fix an \( N \times N \) matrix \( A \) with entries in \( \{0, 1\} \) having no zero rows or columns. We denote by \( B_1(D_N) \) and \( B_1(\Lambda_1) \) the set of admissible words of length \( l \) of \( D_N \) and that of \( \Lambda_1 \) respectively. Let \( m(l) \) be the cardinal number of \( B_1(\Lambda_1) \). We use lexicographic order from the left on the words of \( B_1(\Lambda_1) \), so that we assign a word \( \mu_1 \ldots \mu_l \in B_1(\Lambda_1) \) the number \( N(\mu_1 \ldots \mu_l) \) from 1 to \( m(l) \). For example, if \( A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \), then

\[
B_1(\Lambda_1) = \{1, 2\}, \quad N(1) = 1, \quad N(2) = 2, \\
B_2(\Lambda_1) = \{11, 12, 21\}, \quad N(11) = 1, \quad N(12) = 2, \quad N(21) = 3,
\]

and so on. Hence the set \( B_1(\Lambda_1) \) bijectively corresponds to the set of natural numbers less than or equal to \( m(l) \). Let us now describe the Cantor horizon \( \lambda \)-graph system \( \mathcal{C}(D_N) \) of \( D_N \). The vertices \( V_l \) at level \( l \) for \( l \in \mathbb{Z}_+ \) are given by the admissible words of length \( l \) consisting of the symbols of \( \Sigma^+ \). We regard \( V_0 \) as a one point set of the empty word \( \{\emptyset\} \). Since \( V_l \) is identified with \( B_l(\Lambda_1) \), we may write \( V_l \) as

\[
V_l = \{v_{N(\mu_1 \ldots \mu_l)}^l \mid \mu_1 \ldots \mu_l \in B_l(\Lambda_1)\}.
\]

The mapping \( \iota (\equiv \iota_{l,l+1}) : V_{l+1} \to V_l \) is defined by deleting the rightmost symbol of a corresponding word such as

\[
(\iota(v_{N(\mu_1 \ldots \mu_l)}^{l+1}))_{N(\mu_1 \ldots \mu_{l+1})} = v_{N(\mu_1 \ldots \mu_{l+1})}^l \quad \text{for} \quad v_{N(\mu_1 \ldots \mu_{l+1})}^{l+1} \in V_{l+1}.
\]

We define an edge labeled \( \alpha_j \) from \( v_{N(\mu_1 \ldots \mu_j)}^l \in V_l \) to \( v_{N(\mu_0 \mu_1 \ldots \mu_j)}^{l+1} \in V_{l+1} \) precisely if \( \mu_0 = j \), and an edge labeled \( \beta_j \) from \( v_{N(j \mu_1 \ldots \mu_{l+1})}^l \in V_l \) to \( v_{N(\mu_1 \ldots \mu_{l+1})}^{l+1} \in V_{l+1} \). For \( l = 0 \), we define an edge labeled \( \alpha_j \) from \( v_0^l \) to \( v_0^{N(j)} \), and an edge labeled \( \beta_j \) from \( v_0^l \) to \( v_1^{N(\mu_j)} \) if \( A(j, i) = 1 \). We denote by \( E_{l,l+1} \) the set of edges from \( V_l \) to \( V_{l+1} \). Set \( E = \bigcup_{l=0}^{\infty} E_{l,l+1} \). It is easy to see that the resulting labeled Bratteli diagram with \( \iota \)-map becomes a \( \lambda \)-graph system over \( \Sigma \), that is denoted by \( \mathcal{C}(D_N) \).

In the \( \lambda \)-graph system \( \mathcal{C}(D_N) \), we consider two \( \lambda \)-graph subsystems \( \mathcal{C}(D_N^+ \lambda) \) and \( \mathcal{C}(D_N^\lambda) \). Both of the \( \lambda \)-graph subsystems have the same vertex sets as \( \mathcal{C}(D_N) \) together with the same \( \iota \)-maps as \( \mathcal{C}(D_N) \). The edge set of \( \mathcal{C}(D_N^+ \lambda) \) consists of edges labeled \( \Sigma^+ \) in the edges of \( \mathcal{C}(D_N) \), whereas that of \( \mathcal{C}(D_N^\lambda) \)
consists of edges labeled \( \Sigma^- \). Hence \( \mathcal{Ch}(D^+_\lambda) \) and \( \mathcal{Ch}(D^-_\lambda) \) are \( \lambda \)-graph systems over \( \Sigma^+ \) and over \( \Sigma^- \) respectively. The latter \( \lambda \)-graph system \( \mathcal{Ch}(D^-_\lambda) \) is called the word \( \lambda \)-graph system in [15]. Since the union of the edge sets of \( \mathcal{Ch}(D^+_\lambda) \) and \( \mathcal{Ch}(D^-_\lambda) \) coincides with the edge set of \( \mathcal{Ch}(D_A) \), we may write \( \mathcal{Ch}(D_A) \) as

\[
\mathcal{Ch}(D_A) = \mathcal{Ch}(D^+_\lambda) \cup \mathcal{Ch}(D^-_\lambda).
\]

We will prove that \( \mathcal{Ch}(D_A) \) presents the subshift \( D_A \).

**Lemma 2.2.** For \( \gamma_1 \ldots \gamma_k \in B_k(D_A) \) and \( \mu_1 \ldots \mu_l \in B_l(\Lambda_A) \), if the word \( \gamma_1 \ldots \gamma_k \beta_{\mu_2} \ldots \beta_{\mu_l} \) is admissible in \( D_A \), so is the word \( \gamma_1 \ldots \gamma_k \alpha_{\mu_1} \beta_{\mu_1} \beta_{\mu_2} \ldots \beta_{\mu_l} \).

**Proof.** As the word \( \gamma_1 \ldots \gamma_k \beta_{\mu_2} \ldots \beta_{\mu_l} \) is admissible in \( D_A \), one has

\[
\varphi_A(\gamma_1) \ldots \varphi_A(\gamma_k)t_{\mu_2} \ldots t_{\mu_1}t_{\mu_2}^* \ldots t_{\mu_2}^* \neq 0.
\]

By the condition \( \mu_1 \ldots \mu_l \in B_l(\Lambda_A) \) with the relations (2.2), one sees

\[
t_{\mu_1}^* t_{\mu_2} \ldots t_{\mu_1} t_{\mu_1} \ldots t_{\mu_1}^* t_{\mu_2} \ldots t_{\mu_1}^* \neq 0
\]

so that

\[
\varphi_A(\gamma_1) \ldots \varphi_A(\gamma_k)t_{\mu_1}^* t_{\mu_2} \ldots t_{\mu_1} t_{\mu_1} \ldots t_{\mu_1}^* t_{\mu_2} \ldots t_{\mu_1}^* \neq 0
\]

and hence the word \( \gamma_1 \ldots \gamma_k \alpha_{\mu_1} \beta_{\mu_1} \beta_{\mu_2} \ldots \beta_{\mu_l} \) is admissible in \( D_A \).

For \( \mu_1 \ldots \mu_l \in B_l(\Lambda_A) \) and \( k \leq l \) we set

\[
\Gamma^k_{D_A}(\beta_{\mu_1} \ldots \beta_{\mu_l}) = \{ \gamma_1 \ldots \gamma_k \in B_k(D_A) \mid \gamma_1 \ldots \gamma_k \beta_{\mu_1} \ldots \beta_{\mu_l} \in B_{k+l}(D_A) \}
\]

the \( k \)-predecessor set of the word \( \beta_{\mu_1} \ldots \beta_{\mu_l} \) in \( D_A \) and

\[
\Gamma^k_{\mathcal{Ch}(D_A)}(v^j_{N(\mu_1 \ldots \mu_l)})
\]

\[
= \{ \gamma_1 \ldots \gamma_k \in B_k(D_A) \mid \text{there exist } e_i \in E, i = 1, \ldots, k \text{ such that } \gamma_i = \lambda(e_i) \text{ for } i = 1, \ldots, k \text{ and } t(e_k) = v^j_{N(\mu_1 \ldots \mu_l)} \}
\]

the \( k \)-predecessor set of the vertex \( v^j_{N(\mu_1 \ldots \mu_l)} \) in \( \mathcal{Ch}(D_A) \).

**Lemma 2.3.** \( \Gamma^k_{\mathcal{Ch}(D_A)}(\beta_{\mu_1} \ldots \beta_{\mu_l}) = \Gamma^k_{\mathcal{Ch}(D_A)}(v^j_{N(\mu_1 \ldots \mu_l)}). \)

**Proof.** We will prove the desired equality by induction on the length \( k \).

(1) Assume that \( k = 1 \).

For \( \mu_0 \in \{1, \ldots, N\} \), one sees \( \alpha_{\mu_0} \in \Gamma^1_{D_A}(\beta_{\mu_1} \ldots \beta_{\mu_l}) \) if and only if \( \mu_0 = \mu_1 \), which is equivalent to \( \alpha_{\mu_0} \in \Gamma^1_{\mathcal{Ch}(D_A)}(v^j_{N(\mu_1 \ldots \mu_l)}). \) Similarly \( \beta_{\mu_0} \in \)
\[ \Gamma_{D_A}^1(\beta_{\mu_1} \ldots \beta_{\mu_l}) \] if and only if \( A(\mu_0, \mu_1) = 1 \), which is equivalent to \( \beta_{\mu_0} \in \Gamma_{\mathcal{Ch}(D_A)}^1(v_{N(\mu_1 \ldots \mu_l)}^l) \).

(2) Assume next that the desired equality holds for a fixed \( k \) with \( k + 1 \leq l \).

For a word \( \gamma_1 \ldots \gamma_{k+1} \in B_{k+1}(D_A) \), we have two cases.

**Case 1:** \( \gamma_{k+1} = \alpha_{\mu_0} \) for some \( \mu_0 \in \{1, \ldots, N\} \).

Assume \( \gamma_1 \ldots \gamma_{k+1} \in \Gamma_{D_A}^{k+1}(\beta_{\mu_1} \ldots \beta_{\mu_l}) \) and hence \( \gamma_1 \ldots \gamma_k \alpha_{\mu_0} \beta_{\mu_1} \ldots \beta_{\mu_l} \) is admissible in \( D_A \) so that \( \mu_0 = \mu_1 \). Since \( t_{\mu_0}^*t_{\mu_0} \) is a projection in the algebra \( \mathcal{O}_A \), the word \( \gamma_1 \ldots \gamma_k \beta_{\mu_2} \ldots \beta_{\mu_l} \) is admissible in \( D_A \). Hence

\[ \gamma_1 \ldots \gamma_k \in \Gamma_{D_A}^k(\beta_{\mu_2} \ldots \beta_{\mu_l}). \]

By the hypothesis of induction, one has

\[ \gamma_1 \ldots \gamma_k \in \Gamma_{\mathcal{Ch}(D_A)}^k(v_{N(\mu_2 \ldots \mu_l)}^{l-1}). \]

Since \( \mu_1 \mu_2 \ldots \mu_l \) is admissible in \( \Lambda_A \), there exists an edge \( e \in E_{l-1,l} \) in \( \mathcal{Q}_{\mathcal{Ch}(D_A)}(v_{N(\mu_2 \ldots \mu_l)}^{l-1}) \) such that \( \lambda(e) = \alpha_{\mu_1} \) and \( s(e) = v_{N(\mu_2 \ldots \mu_l)}^{l-1} \), \( t(e) = v_{N(\mu_1 \ldots \mu_l)}^l \). Hence we know that

\[ \gamma_1 \ldots \gamma_{k+1} \in \Gamma_{\mathcal{Ch}(D_A)}^{k+1}(v_{N(\mu_1 \ldots \mu_l)}^l). \]

Conversely assume \( \gamma_1 \ldots \gamma_{k+1} \in \Gamma_{\mathcal{Ch}(D_A)}^{k+1}(v_{N(\mu_1 \ldots \mu_l)}^l) \) so that \( \mu_0 = \mu_1 \). Hence

\[ \gamma_1 \ldots \gamma_k \in \Gamma_{\mathcal{Ch}(D_A)}^k(v_{N(\mu_2 \ldots \mu_l)}^{l-1}). \]

By the hypothesis of induction, the word \( \gamma_1 \ldots \gamma_k \beta_{\mu_2} \ldots \beta_{\mu_l} \) is admissible in \( D_A \). By the preceding lemma, \( \gamma_1 \ldots \gamma_k \alpha_{\mu_1} \beta_{\mu_1} \beta_{\mu_2} \ldots \beta_{\mu_l} \) is admissible in \( D_A \) so that

\[ \gamma_1 \ldots \gamma_{k+1} \in \Gamma_{D_A}^{k+1}(\beta_{\mu_1} \ldots \beta_{\mu_l}). \]

**Case 2:** \( \gamma_{k+1} = \beta_{\mu_0} \) for some \( \mu_0 \in \{1, \ldots, N\} \).

Assume \( \gamma_1 \ldots \gamma_{k+1} \in \Gamma_{D_A}^{k+1}(\beta_{\mu_1} \ldots \beta_{\mu_l}) \). Then

\[ \gamma_1 \ldots \gamma_k \in \Gamma_{D_A}^k(\beta_{\mu_0} \beta_{\mu_1} \ldots \beta_{\mu_{l-2}}). \]

By the hypothesis of induction, we have

\[ \gamma_1 \ldots \gamma_k \in \Gamma_{\mathcal{Ch}(D_A)}^k(v_{N(\mu_0 \ldots \mu_{l-2})}^{l-1}). \]

Since \( \mu_0 \mu_1 \ldots \mu_l \) is admissible in \( \Lambda_A \), there exists an edge \( e \in E_{l-1,l} \) in \( \mathcal{Q}_{\mathcal{Ch}(D_A)}(v_{N(\mu_0 \ldots \mu_{l-2})}^{l-1}) \) such that \( \lambda(e) = \beta_{\mu_0} \) and \( s(e) = v_{N(\mu_0 \ldots \mu_{l-2})}^{l-1} \), \( t(e) = v_{N(\mu_1 \ldots \mu_l)}^l \). Hence we know

\[ \gamma_1 \ldots \gamma_{k+1} \in \Gamma_{\mathcal{Ch}(D_A)}^{k+1}(v_{N(\mu_1 \ldots \mu_l)}^l). \]
Conversely assume $\gamma_1 \ldots \gamma_k+1 \in \Gamma_{\text{Ch}(\Lambda)^k}(v^{l}_{N(\mu_0 \ldots \mu_l)})$. Hence

$$\gamma_1 \ldots \gamma_k \in \Gamma_{\text{Ch}(\Lambda)^k}(v^{l-1}_{N(\mu_0 \ldots \mu_{l-2})}).$$

As $\Gamma_{\text{Ch}(\Lambda)^k}(v^{l-1}_{N(\mu_0 \ldots \mu_{l-2})}) = \Gamma_{\text{Ch}(\Lambda)^k}(v^{l+1}_{N(\mu_0 \ldots \mu_{l+1})})$, by the hypothesis of induction

$$\gamma_1 \ldots \gamma_k \in \Gamma_{\Lambda}(\beta_{\mu_0} \ldots \beta_{\mu_l}).$$

Hence we have

$$\gamma_1 \ldots \gamma_{k+1} \in \Gamma_{\Lambda}(\beta_{\mu_1} \ldots \beta_{\mu_l}).$$

Therefore the desired equality holds for all $k$ with $k \leq l$.

**Proposition 2.4.** The $\lambda$-graph system $\text{Ch}(\Lambda)^k$ presents the subshift $\Lambda_{\text{Ch}(\Lambda)}$.

**Proof.** Put $X_{\Lambda} = \{((\mu_i)_{i \in \mathbb{N}} \mid (\mu_i)_{i \in \mathbb{Z}} \in \Lambda_{\Lambda}) \text{ and } X_{\text{Ch}(\Lambda)} = \{((\gamma_i)_{i \in \mathbb{N}} \mid (\gamma_i)_{i \in \mathbb{Z}} \in \Lambda_{\Lambda})\}$. Let $\hat{S}$ be the Hilbert space $\hat{S}$ whose complete orthonormal basis are given by the vectors

$$e_{\mu_1} \otimes e_{\mu_2} \otimes \cdots \quad \text{for} \quad (\mu_1, \mu_2, \ldots) \in X_{\Lambda}.$$

We faithfully represent $\Lambda_{\text{Ch}(\Lambda)}$ on $\hat{S}$ by using the creation operators $t_i$, $i = 1, \ldots, N$ defined by

$$t_i(e_{\mu_1} \otimes e_{\mu_2} \otimes \cdots) = \begin{cases} e_i \otimes e_{\mu_1} \otimes e_{\mu_2} \otimes \cdots & \text{if } A(i, \mu_1) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We may identify $\varphi_A(\alpha_i)$ and $\varphi_A(\beta_i)$ with the operators $t_i^*$ and $t_i$ on $\hat{S}$ respectively. For a word $\gamma_1 \ldots \gamma_k \in \Sigma^*$, it follows that $\gamma_1 \ldots \gamma_k$ is admissible in $\Lambda_{\text{Ch}(\Lambda)}$ if and only if there exists a sequence $(\mu_1, \mu_2, \ldots) \in X_{\Lambda}$ such that $\varphi_A(\gamma_1) \ldots \varphi_A(\gamma_k)e_{\mu_1} \otimes e_{\mu_2} \otimes \cdots$ is a nonzero vector. The latter condition is equivalent to the condition $(\gamma_1, \ldots, \gamma_k, \mu_1, \mu_2, \ldots) \in X_{\Lambda}$. This is equivalent to the condition $\gamma_1 \ldots \gamma_k \in \Gamma_{\text{Ch}(\Lambda)}(\beta_{\mu_1} \beta_{\mu_2} \ldots \beta_{\mu_l})$ for all $l \geq k$. Therefore by the preceding lemma, the subshift $\Lambda_{\text{Ch}(\Lambda)^k}$ presented by the $\lambda$-graph system $\text{Ch}(\Lambda)^k$ is $\Lambda_{\text{Ch}(\Lambda)}$.

We automatically know that the $\lambda$-graph systems $\text{Ch}(\Lambda_{\text{Ch}(\Lambda)^k})$ and $\text{Ch}(\Lambda_{\text{Ch}(\Lambda)})$ present the subshifts $\Lambda_{\text{Ch}(\Lambda)^k}$ and $\Lambda_{\text{Ch}(\Lambda)}$ respectively. A $\lambda$-graph system $\mathcal{L}$ satisfies $\lambda$-condition (I) if for every vertex $v \in V_{\mathcal{L}}$ of $\mathcal{L}$ there exist at least two paths with distinct label sequences starting with the vertex $v$ and terminating with a same vertex. $\mathcal{L}$ is said to be $\lambda$-irreducible if for an ordered pair of vertices $u, v \in V_{\mathcal{L}}$, there exists a number $L_{\mathcal{L}}(u, v) \in \mathbb{N}$ such that for a vertex $w \in V_{\mathcal{L}+L_{\mathcal{L}}(u, v)}$ with $L_{\mathcal{L}}(u, v)(w) = u$, there exists a path $\xi$ in $\mathcal{L}$ such that $s(\xi) = v, t(\xi) = w$, where $L_{\mathcal{L}}(u, v)$ means the $L_{\mathcal{L}}(u, v)$-times compositions of $\mathcal{L}$, and $s(\xi), t(\xi)$ denote the source vertex, the terminal vertex of $\xi$ respectively ([23]).
Proposition 2.5. Let \( A \) be an \( N \times N \) matrix with entries in \( \{0, 1\} \).

(i) If \( A \) satisfies condition (I) in the sense of [5], the \( \lambda \)-graph system \( \mathcal{Q}^{\text{Ch}(D^+_1)} \) satisfies \( \lambda \)-condition (I).

(ii) If \( A \) is irreducible, the \( \lambda \)-graph system \( \mathcal{Q}^{\text{Ch}(D^+_1)} \) is \( \lambda \)-irreducible.

Hence if \( A \) is an irreducible matrix with condition (I), then both the \( \lambda \)-graph systems \( \mathcal{Q}^{\text{Ch}(D^+_1)} \) and \( \mathcal{Q}^{\text{Ch}(D^+_1)} \) are \( \lambda \)-irreducible with \( \lambda \)-condition (I).

Proof. (i) Suppose that \( A \) satisfies condition (I). In the \( \lambda \)-graph system \( \mathcal{Q}^{\text{Ch}(D^+_1)} \), let \( v^j_i \) be a vertex in \( V_j \). We write \( i = N(i_1 \ldots i_l) \) for \( i_1 \ldots i_l \in B_l(\Lambda_A) \).

By condition (I) for \( A \), there exist \( \mu = \mu_1 \ldots \mu_r, v = v_1 \ldots v_r \in B_r(\Lambda_A) \) such that \( \mu \neq v, \mu_1 = v_1 = i \) and \( \mu_r = v_r \).

Take \( \eta_{r+1} \ldots \eta_{2l+2r-1} \in B_{2l+r-1}(\Lambda_A) \) such that \( \mu_r \eta_{r+1} \ldots \eta_{2l+2r-1} \in B_{2l+r-1}(\Lambda_A) \).

We put \( \mu_n = v_n = \eta_n \) for \( n = r+1, \ldots, 2l+2r-1 \) and \( L' = 2l+2r-1 \).

Let \( v^L_j \in V_{L'} \) be the vertex in \( \mathcal{Q}^{\text{Ch}(D^+_1)} \) such that \( j = N(\mu_r \mu_{r+1} \ldots \mu_{2l+2r-2})(= N(v_r v_{r+1} \ldots v_{2l+2r-2})) \).

Then there exist two paths labeled \( \beta_{i_1} \ldots \beta_{i_l} \beta_{\mu_1} \ldots \beta_{\mu_r-1} \) and \( \beta_{i_1} \ldots \beta_{i_l} \beta_{v_1} \ldots \beta_{v_{r-1}} \) whose sources are both \( v^j_i \) and terminals are both \( v^L_j \).

Hence \( \mathcal{Q}^{\text{Ch}(D^+_1)} \) satisfies \( \lambda \)-condition (I).

(ii) In the \( \lambda \)-graph system \( \mathcal{Q}^{\text{Ch}(D^+_1)} \), let \( v^j_i, v^j_j \) be vertices in \( V_j \).

We write \( i = N(i_1 \ldots i_l), j = N(j_1 \ldots j_l) \) for \( i_1 \ldots i_l, j_1 \ldots j_l \in B_l(\Lambda_A) \) respectively. As \( \Lambda_A \) is irreducible, there exists a word \( \eta_1 \ldots \eta_L \in B_L(\Lambda_A) \) such that \( j_1 \ldots j_l \eta_1 \ldots \eta_l i_1 \ldots i_l \in B_{2l+L}(\Lambda_A) \).

We may assume \( L \geq l \).

For \( v^j_h \in V_{2l+L} \) with \( t^{L+L}(v^j_h) = v^j_i, h = 1, \ldots, m(2l + L) \) we have \( h = N(i_1 \ldots i_l \mu_{l+1} \ldots \mu_{2l+L}) \) for some \( \mu_{l+1} \ldots \mu_{2l+L} \in B_{l+L}(\Lambda_A) \).

Then there exists a path labeled \( \beta_{j_1} \ldots \beta_{j_l} \beta_{\eta_1} \ldots \beta_{\eta_L} \) whose source is \( v^j_i \) and whose terminal is \( v^j_h \).

This means that \( \mathcal{Q}^{\text{Ch}(D^+_1)} \) is \( \lambda \)-irreducible.

Therefore we have by [23]

Theorem 2.6. Let \( A \) be an \( N \times N \) matrix with entries in \( \{0, 1\} \). If \( A \) is an irreducible matrix with condition (I), then the \( C^* \)-algebra \( \mathcal{O}^{\text{Ch}(D^+_1)} \) is simple and purely infinite.

We note that the \( \lambda \)-graph systems \( \mathcal{Q}^{\text{Ch}(D^+_1)} \) are examples of \( \lambda \)-synchronizing \( \lambda \)-graph systems for \( D_A \) introduced in [17].

3. The \( C^* \)-algebra \( \mathcal{O}^{\text{Ch}(D^+_1)} \)

This section is devoted to studying operator relations among generators of the algebra \( \mathcal{O}^{\text{Ch}(D^+_1)} \) to prove Theorem 1.1. A general structure for the \( C^* \)-algebra \( \mathcal{O} \) associated with a \( \lambda \)-graph system \( \Sigma \) has been studied in [20]. For a \( \lambda \)-graph system \( \Sigma = (V, E, \lambda, \iota) \) over \( \Sigma \), let \( \{v^j_1, \ldots, v^j_{m(l)}\} \) be the vertex set \( V_j \). We
set for \( i = 1, 2, \ldots, m(l), j = 1, 2, \ldots, m(l + 1), \gamma \in \Sigma \),

\[
A_{l,l+1}(i, \gamma, j) = \begin{cases} 
1 & \text{if } s(e) = v^l_j, \lambda(e) = \gamma, t(e) = v^{l+1}_j \\
0 & \text{for some } e \in E_{l,l+1},
\end{cases}
\]

\[
I_{l,l+1}(i, j) = \begin{cases} 
1 & \text{if } t_{l,l+1}(v^{l+1}_j) = v^l_j, \\
0 & \text{otherwise},
\end{cases}
\]

**Lemma 3.1** ([20, Theorem A and B], cf. [23]). Suppose that a \( \lambda \)-graph system \( \mathcal{L} \) satisfies \( \lambda \)-condition (I). Then the \( C^* \)-algebra \( \mathcal{O}_{\mathcal{L}} \) is the unique \( C^* \)-algebra generated by nonzero partial isometries \( s_\gamma \), \( \gamma \in \Sigma \) and nonzero projections \( e_i \), \( i = 1, 2, \ldots, m(l) \), \( l \in \mathbb{Z}_+ \) satisfying the following operator relations:

\[
\sum_{\gamma \in \Sigma} s_\gamma s^*_\gamma = 1, \quad \text{(3.1)}
\]

\[
\sum_{j=1}^{m(l)} e^l_j = 1, \quad e^l_i = \sum_{j=1}^{m(l+1)} I_{l,l+1}(i, j) e^{l+1}_j, \quad \text{(3.2)}
\]

\[
s_\gamma s^*_\gamma e^l_i = e^l_i s_\gamma s^*_\gamma, \quad \text{(3.3)}
\]

\[
s^*_\gamma e^l_i s_\gamma = \sum_{j=1}^{m(l+1)} A_{l,l+1}(i, \gamma, j) e^{l+1}_j, \quad \text{(3.4)}
\]

for \( i = 1, 2, \ldots, m(l) \), \( l \in \mathbb{Z}_+ \), \( \gamma \in \Sigma \). If in particular \( \mathcal{L} \) is \( \lambda \)-irreducible, the \( C^* \)-algebra \( \mathcal{O}_{\mathcal{L}} \) is simple and purely infinite.

We first consider the \( C^* \)-algebra \( \mathcal{O}_{\mathcal{L}_{\text{Ch}}(D^+_A)} \) for the \( \lambda \)-graph system \( \mathcal{L}_{\text{Ch}}(D^+_A) \).

**Proposition 3.2.** Suppose that \( A \) satisfies condition (I). The \( C^* \)-algebra \( \mathcal{O}_{\mathcal{L}_{\text{Ch}}(D^+_A)} \) is canonically isomorphic to the Cuntz-Krieger algebra \( \mathcal{O}_A \).

**Proof.** Both the algebras \( \mathcal{O}_{\mathcal{L}_{\text{Ch}}(D^+_A)} \) and \( \mathcal{O}_A \) are uniquely determined by certain operator relations of their canonical generators. We write the canonical generating partial isometries and the projections in \( \mathcal{O}_{\mathcal{L}_{\text{Ch}}(D^+_A)} \) as \( s_\beta \), \( i = 1, \ldots, N \) and \( e^l_{N(i_1 \ldots i_l)}, i_1 \ldots i_l \in B_l(\Lambda_A), l \in \mathbb{Z}_+ \) respectively. By (3.1), (3.3) and (3.4), one has

\[
e^l_{N(i_1 \ldots i_l)} = \sum_{i_{l+1}, i_{l+2}=1}^N s_{\beta_{i_1}} e^{l+1}_{N(i_2 \ldots i_{l+1} i_{l+2})} s^*_{\beta_{i_{l+1}}}.\]
For \( l = 1 \), one sees that by (3.2)
\[
e_1^{N(i_1)} = \sum_{i_2, i_3=1}^{N} s_{\beta_{i_1}} e_{N(i_2 i_3)}^{2} s_{\beta_{i_1}^*} = s_{\beta_{i_1}} s_{\beta_{i_1}^*}.
\]

As \( v^{l+1}(N(i_2...i_{l+1})_{i_1}) = v^{l-1}_{N(i_2...i_l)} \), (3.2) implies the equality
\[
\sum_{i_1+1, i_3=1}^{N} e_{N(i_2...i_{l+1})_{i_1}}^{l+1} = e_{N(i_2...i_l)}^{l-1}
\]
so that by induction one obtains
\[
e_{N(i_1...i_l)}^{l} = s_{\beta_{i_1}} ... s_{\beta_{i_l}} s_{\beta_{i_1}^*} ... s_{\beta_{i_l}^*}.
\]

One also sees that (3.4) implies the equality
\[
s_{\beta_{i}}^* s_{\beta_{i}} = \sum_{j=1}^{N} A(i, j) s_{\beta_{j}} s_{\beta_{j}^*}.
\]

As the equality \( \sum_{i=1}^{N} s_{\beta_{i}} s_{\beta_{i}^*} = 1 \) holds, the \( C^* \)-algebra generated by partial isometries \( s_{\beta_{i}}, i = 1, \ldots, N \) is canonically isomorphic to the Cuntz-Krieger algebra \( O_A \).

In what follows, an \( N \times N \) matrix \( A \) is assumed to be irreducible with entries in \( \{0, 1\} \), and satisfy condition (I). We will describe concrete operator relations among the canonical generators of the algebra \( O_{\mathcal{Ch}(D_{A})} \). Let \( A_{l, l+1}, I_{l, l+1} \) be the matrices as in Lemma 3.1 for the \( \lambda \)-graph system \( \mathcal{Ch}(D_{A}) \). We denote by \( m(l) \) the number of the vertex set \( V_{l} = \{v_{l}^1, \ldots, v_{m(l)}^l\} \) of \( \mathcal{Ch}(D_{A}) \). Let \( s_{\gamma}, \gamma \in \Sigma \) and \( e_{i}^{l}, i = 1, \ldots, m(l), l \in \mathbb{Z}_+ \) be the canonical generating partial isometries and projections of \( O_{\mathcal{Ch}(D_{A})} \). They satisfy the relations (3.1), (3.2), (3.3) and (3.4) for \( \mathcal{Ch}(D_{A}) \). Define the operators \( S_{i}, T_{i}, i = 1, \ldots, N \) by setting
\[
S_{i} := s_{\alpha_{i}}, \quad T_{i} := s_{\beta_{i}} \quad \text{for} \quad i = 1, \ldots, N.
\]

**Proposition 3.3.** The operators \( S_{i}, T_{i}, i = 1, \ldots, N \) satisfy the relations (1.3), (1.4), (1.5) and (1.6), and generate the \( C^* \)-algebra \( O_{\mathcal{Ch}(D_{A})} \).

**Proof.** The equality (1.3) is nothing but (3.1). To prove (1.4), by the equality (3.4) and the first equality of (3.2), one has for a fixed \( l \in \mathbb{Z}_+ \),
\[
\sum_{j=1}^{N} S_j^* S_j = \sum_{j=1}^{N} \sum_{i=1}^{m(l)} \sum_{k=1}^{m(l+1)} A_{l, l+1}(i, \alpha_{j}, k) e_{k}^{l+1}.
\]
For $k = 1, \ldots, m(l+1)$, there exists a unique edge in $\mathcal{C}^{\text{Ch}(D_A)}$ labeled a symbol in $\Sigma^-$ whose terminal is $v_k^{l+1}$. Hence we have $\sum_{j=1}^{N} \sum_{i=1}^{m(l)} A_{l,l+1}(i, \alpha_j, k) = 1$ so that

$$\sum_{j=1}^{N} S_j^* S_j = \sum_{k=1}^{m(l+1)} e_k^{l+1} = 1.$$ 

For (1.5), one similarly has

$$T_i^* T_i = \sum_{k=1}^{m(l)} s_{\beta_i}^* e_k^l s_{\beta_i} = \sum_{h=1}^{m(l+1)} \sum_{k=1}^{m(l)} A_{l,l+1}(k, \beta_i, h) e_h^{l+1}.$$ 

On the other hand,

$$\sum_{j=1}^{N} A(i, j) S_j^* S_j = \sum_{h=1}^{m(l+1)} \left( \sum_{k=1}^{m(l)} \sum_{j=1}^{N} A(i, j) A_{l,l+1}(k, \alpha_j, h) \right) e_h^{l+1}.$$ 

Let $h$ be written as $N(h_1 \ldots h_{l+1})$. Then the condition $A_{l,l+1}(k, \beta_i, h) = 1$ is equivalent to the condition that $i h_1 \in B_2(A_A)$ and $k = N(i h_1 \ldots h_{l-1})$. On the other hand, the condition $\sum_{j=1}^{N} A(i, j) A_{l,l+1}(k, \alpha_j, h) = 1$ is equivalent to the condition that $j = h_1, A(i, j) = 1$ for some $j$ and $k = N(h_2 \ldots h_{l+1})$. Hence one has

$$\sum_{k=1}^{m(l)} A_{l,l+1}(k, \beta_i, h) = \sum_{j=1}^{m(l)} \sum_{k=1}^{N} A(i, j) A_{l,l+1}(k, \alpha_j, h).$$

This implies the equality (1.5). For (1.6), we put

$$E_{\mu_1 \ldots \mu_k} = S_{\mu_1}^* \ldots S_{\mu_k}^* S_{\mu_k} \ldots S_{\mu_1}.$$ 

By using the first equality of (3.2), (3.3) and (3.4) recursively, $E_{\mu_1 \ldots \mu_k}$ commutes with $S_j S_j^*$ and $T_j T_j^*$ for $j = 1, \ldots, N$, so that by (1.3)

$$E_{\mu_1 \ldots \mu_k} = \sum_{j=1}^{N} S_j S_j^* E_{\mu_1 \ldots \mu_k} S_j S_j^* + \sum_{j=1}^{N} T_j T_j^* E_{\mu_1 \ldots \mu_k} T_j T_j^*.$$ 

As $S_j^* E_{\mu_1 \ldots \mu_k} S_j = A(j, \mu_1) S_j^* E_{\mu_1 \ldots \mu_k} S_j$ and $S_{\mu_1} T_j = s_{\alpha_{\mu_1}} s_{\beta_j} = 0$ if $\mu_1 \neq j$, one has

$$E_{\mu_1 \ldots \mu_k} = \sum_{j=1}^{N} A(j, \mu_1) S_j S_j^* E_{\mu_1 \ldots \mu_k} S_j S_j^* + T_{\mu_1} T_{\mu_1}^* E_{\mu_1 \ldots \mu_k} T_{\mu_1} T_{\mu_1}^*.$$
Since \( A_{0,1}(1, \alpha_{\mu_1}, j) = 1 \) if and only if \( j = \mu_1 \), it follows that by (3.4),

\[
E_{\mu_1} = s^*_{\alpha_{\mu_1}} s_{\alpha_{\mu_1}} = \sum_{j=1}^{m(1)} A_{0,1}(1, \alpha_{\mu_1}, j) e_j^1 = e_{\mu_1}^1.
\]

By (3.2) and (3.4), we similarly have

\[
E_{\mu_1...\mu_k} = s^*_{\alpha_{\mu_1}} ... s^*_{\alpha_{\mu_k}} s_{\alpha_{\mu_1}} ... s_{\alpha_{\mu_k}} \\
= \sum_{i_1=1}^{m(1)} ... \sum_{i_k=1}^{m(k)} A_{0,1}(1, \alpha_{\mu_k}, i_1) ... A_{k-1,k}(i_{k-1}, \alpha_{\mu_k}, i_k) e_k^1.
\]

As \( \sum_{i_1=1}^{m(1)} ... \sum_{i_k=1}^{m(k)} A_{0,1}(1, \alpha_{\mu_k}, i_1) ... A_{k-1,k}(i_{k-1}, \alpha_{\mu_k}, i_k) = 1 \) if and only if \( i_k = N(\mu_1 ... \mu_k) \), one knows \( E_{\mu_1...\mu_k} = e_{N(\mu_1...\mu_k)}^k \). Hence we have

\[
T_{\mu_1}^* E_{\mu_1...\mu_k} T_{\mu_1} = \sum_{j=1}^{m(k+1)} A_{k,k+1}(N(\mu_1 ... \mu_k), \beta_{\mu_1}, j) e_{j+1}^k.
\]

Since \( A_{k,k+1}(N(\mu_1 ... \mu_k), \beta_{\mu_1}, j) = 1 \) if and only if \( j = N(\mu_2 ... \mu_k \mu_{k+1}\mu_{k+2}) \) for some \( \mu_{k+1}, \mu_{k+2} = 1, \ldots, N \), and the equality

\[
\sum_{\mu_{k+1}, \mu_{k+2}=1,\ldots, N} E_{\mu_2...\mu_k \mu_{k+1}\mu_{k+2}} = E_{\mu_1...\mu_k}
\]

holds, we have \( T_{\mu_1}^* E_{\mu_1...\mu_k} T_{\mu_1} = E_{\mu_1...\mu_k} \). Thus we conclude that (1.6) holds. Consequently the operators \( S_i, T_i, i = 1, \ldots, N \) satisfy the relations (1.3), (1.4), (1.5) and (1.6).

In the above discussions, we have proved the equality

\[
e_{N(\mu_1...\mu_k)}^k = E_{\mu_1...\mu_k} (= s^*_{\mu_1} ... s^*_{\mu_k} s_{\mu_k} ... s_{\mu_1})
\]

for \( \mu_1 ... \mu_k \in B_k(\Lambda_A) \). Hence \( \mathcal{O}_{\text{ch}(\Lambda_1)} \) is generated by \( S_1, \ldots, S_N, T_1, \ldots, T_N \).

We next show that the relations (1.3), (1.4), (1.5) and (1.6) imply the relations (3.1), (3.2), (3.3) and (3.4). Let \( S_i, T_i, i = 1, \ldots, N \) be partial isometries satisfying the relations (1.3), (1.4), (1.5) and (1.6). In the relation (1.6) for \( k = 2 \), by summing up \( \mu_2 \) over \( \{1, \ldots, N\} \) and using (1.4), we have

\[
(3.5) \quad S_i^* S_i = \sum_{j=1}^{N} A(j, i) S_j S_j^* S_i S_j S_j^* + T_i T_i^*, \quad i = 1, \ldots, N.
\]
Lemma 3.4.

(i) \( T_i^* S_j^* S_j T_i = \begin{cases} T_i^* T_i & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \)

(ii) \( T_i^* E_{\mu_1 \ldots \mu_l} T_i = \begin{cases} A(i, \mu_2) E_{\mu_2 \ldots \mu_l} & \text{if } i = \mu_1, \\ 0 & \text{otherwise} \end{cases} \)

for \( l > 1 \), where \( E_{\mu_1 \ldots \mu_l} = S_{\mu_1}^* \ldots S_{\mu_l}^* S_{\mu_l} \ldots S_{\mu_1} \) for \( \mu_1 \ldots \mu_l \in B_l(\Lambda_A) \).

Proof. (i) By (3.5), we have
\[
T_i^* S_j^* S_j T_i = \sum_{j=1}^{N} A(j, i) T_j^* S_j^* S_j S_j^* T_i + T_i^* T_i T_i^* T_i.
\]

The equality (1.3) implies \( T_i^* S_j = 0 \) for \( i, j = 1, \ldots, N \) and hence we have \( T_i^* S_j^* S_j T_i = T_i^* T_i \). By (1.4), one has
\[
\sum_{j=1}^{N} T_i^* S_j^* S_j T_i = T_i^* T_i
\]
so that \( T_i^* S_j^* S_j T_i = 0 \) for \( i \neq j \).

(ii) By (1.6), we have
\[
T_i^* E_{\mu_1 \ldots \mu_l} T_i = \sum_{j=1}^{N} A(j, \mu_1) T_j^* S_j^* E_{\mu_1 \ldots \mu_l} S_j T_i + T_i^* T_{\mu_1} E_{\mu_2 \ldots \mu_l} T_{\mu_1}^* T_i
\]
for \( l > 1 \). Since \( T_i^* S_j = 0 \) for \( i, j = 1, \ldots, N \) and \( T_i^* T_{\mu_1} = 0 \) for \( i \neq \mu_1 \), we have
\[
T_i^* E_{\mu_1 \ldots \mu_l} T_i = T_i^* T_{\mu_1} E_{\mu_2 \ldots \mu_l} T_{\mu_1}^* T_i = \begin{cases} T_i^* T_i E_{\mu_2 \ldots \mu_l} T_{\mu_1}^* T_i & \text{if } i = \mu_1, \\ 0 & \text{otherwise.} \end{cases}
\]

By (1.5) one has
\[
T_i^* T_i E_{\mu_2 \ldots \mu_l} T_{\mu_1}^* T_i = \sum_{j=1}^{N} \sum_{k=1}^{N} A(i, j) A(i, k) S_j^* S_j S_{\mu_2}^* \ldots S_{\mu_l}^* S_{\mu_i} \ldots S_{\mu_2} S_k.
\]

By (1.4), one sees that \( S_j^* S_j S_{\mu_2}^* = 0 \) for \( j \neq \mu_2 \), and \( S_{\mu_2}^* S_{\mu_2} S_k = 0 \) for \( k \neq \mu_2 \). It then follows that
\[
T_i^* E_{\mu_1 \ldots \mu_l} T_i = A(i, \mu_2) E_{\mu_2 \ldots \mu_l}.
\]

Lemma 3.5. Keep the above notations. The projection \( E_{\mu_1 \ldots \mu_l} \) commutes with both \( S_j^* S_j \) and \( T_j T_j^* \).
Proof. By (1.6), we have for \( l > 1 \)

\[
S_i S^*_i E_{\mu_1 \ldots \mu_l} = \sum_{j=1}^{N} A(j, \mu_1) S_j S^*_j E_{\mu_1 \ldots \mu_l} S^*_j + S_i S^*_i T_{\mu_1} E_{\mu_2 \ldots \mu_l} T^*_{\mu_1}.
\]

By (1.3), one has \( S^*_i T_{\mu_1} = 0 \) for all \( i, \mu_1 \), and \( S^*_i S_j = 0 \) for \( i \neq j \). Hence \( S_i S^*_i E_{\mu_1 \ldots \mu_l} = A(i, \mu_1) S_i S^*_i E_{\mu_1 \ldots \mu_l} S^*_i \) and similarly \( E_{\mu_1 \ldots \mu_l} S_i S^*_i = A(i, \mu_1) S_i S^*_i E_{\mu_1 \ldots \mu_l} S^*_i \) so that \( S_i S^*_i \) commutes with \( E_{\mu_1 \ldots \mu_l} \). By (1.6) and (1.3), we have

\[
T_i T^*_i E_{\mu_1 \ldots \mu_l} = \sum_{j=1}^{N} A(j, \mu_1) T_i T^*_i S_j S^*_j E_{\mu_1 \ldots \mu_l} S^*_j + T_i T^*_i T_{\mu_1} E_{\mu_2 \ldots \mu_l} T^*_{\mu_1} = \begin{cases} T_{\mu_1} E_{\mu_2 \ldots \mu_l} T^*_{\mu_1} & \text{if } i = \mu_1, \\ 0 & \text{otherwise.} \end{cases}
\]

We similarly have the same equality for \( E_{\mu_1 \ldots \mu_l} T_i T^*_i \) as above so that \( T_i T^*_i \) commutes with \( E_{\mu_1 \ldots \mu_l} \). For \( l = 1 \), the equality \( E_{\mu_1} = \sum_{\mu_2=1}^{N} E_{\mu_1 \mu_2} \) from (1.4) implies that \( E_{\mu_1} \) commutes with both \( S_j S^*_j \) and \( T_j T^*_j \) by the above discussions.

Lemma 3.6. Keep the above notations. For \( \mu_1, \ldots, \mu_l \in \{1, \ldots, N\} \) we have \( E_{\mu_1 \ldots \mu_l} = 0 \) if \( \mu_1 \ldots \mu_l \not\in B_l(\Lambda_A) \).

Proof. As we are assuming that the matrix \( A \) has no zero rows or columns, one sees \( B_1(\Lambda_A) = \{1, \ldots, N\} \). By (3.5) one has for \( \mu_1 = 1, \ldots, N \)

\[
S^*_\mu_1 S_{\mu_1} = \sum_{j=1}^{N} A(j, \mu_1) S_j S^*_j S^*_\mu_1 S_{\mu_1} S_j S^*_j + T_{\mu_1} T^*_{\mu_1}
\]

so that for \( \mu_0 = 1, \ldots, N \)

\[
S^*_\mu_0 S^*_\mu_1 S_{\mu_0} = A(\mu_0, \mu_1) S^*_\mu_0 S^*_\mu_1 S_{\mu_0}
\]

because \( S^*_\mu_0 S_j = 0 \) if \( \mu_0 \neq j \), and \( S^*_\mu_0 T_{\mu_1} = 0 \) by (1.3). This means that \( E_{\mu_0 \mu_1} = 0 \) if \( \mu_0 \mu_1 \not\in B_2(\Lambda_A) \).

Suppose next that the assertion holds for \( l = k > 1 \). By (1.6) one has for \( \mu_1 \ldots \mu_k \in B_k(\Lambda_A) \) and \( \mu_0 = 1, \ldots, N \)

\[
S^*_{\mu_0} E_{\mu_1 \ldots \mu_k} S_{\mu_0} = \sum_{j=1}^{N} A(j, \mu_1) S^*_{\mu_0} S_j S^*_j E_{\mu_1 \ldots \mu_k} S_j S^*_j S_{\mu_0} + S^*_{\mu_0} T_{\mu_1} E_{\mu_2 \ldots \mu_k} T^*_{\mu_1} S_{\mu_0}
\]

so that we have

\[
E_{\mu_0 \mu_1 \ldots \mu_k} = A(\mu_0, \mu_1) E_{\mu_0 \mu_1 \ldots \mu_k}.
\]
For $\mu_1 \ldots \mu_k \in B_k(\Lambda_A)$, we have $\mu_0\mu_1 \ldots \mu_k \not\in B_{k+1}(\Lambda_A)$ if and only if $A(\mu_0, \mu_1) = 0$. Hence the assertion holds for $l = k + 1$, so that it holds for all $l \in \mathbb{Z}_+$. 

**Proposition 3.7.** Keep the above notations. Put

$$s_{\alpha_i} := S_i, \quad s_{\beta_i} := T_i \quad \text{for} \quad i = 1, \ldots, N, \quad \text{and}$$

$$e_1^0 := 1,$$

$$e_{N(\mu_1 \ldots \mu_l)}^l := E_{\mu_1 \ldots \mu_l} = (S_{\mu_1}^* \ldots S_{\mu_l}^* S_{\mu_l} \ldots S_{\mu_1}) \quad \text{for} \quad \mu_1 \ldots \mu_l \in B_l(\Lambda_A).$$

Then the family of operators $s_{\gamma}, \gamma \in \Sigma$, $e_{N(\mu_1 \ldots \mu_l)}^l$, $\mu_1 \ldots \mu_l \in B_l(\Lambda_A)$ satisfies the relations (3.1), (3.2), (3.3) and (3.4) for the $\lambda$-graph system $Q^{Ch(D_A)}$.

**Proof.** The relation (3.1) is nothing but the equality (1.3). The equality (1.4) implies $\sum_{\mu_1 \in B_1(\Lambda_A)} e_{N(\mu_1)}^1 = 1$. Suppose that $\sum_{\mu_1 \ldots \mu_l \in B_l(\Lambda_A)} e_{N(\mu_1 \ldots \mu_l)}^l = 1$ holds for $l = k$. As

$$S_{\mu_1}^* \ldots S_{\mu_k}^* S_{\mu_k} \ldots S_{\mu_1} = \sum_{h=1}^N S_{\mu_1}^* \ldots S_{\mu_k}^* S_h S_{\mu_k} \ldots S_{\mu_1},$$

the equality $\sum_{\mu_1 \ldots \mu_l \in B_l(\Lambda_A)} e_{N(\mu_1 \ldots \mu_l)}^l = 1$ holds for $l = k + 1$ by Lemma 3.6 and hence for all $l$. The above equality with the equality

$$I_{l,l+1}(N(\mu_1 \ldots \mu_l), N(\nu_1 \ldots \nu_{l+1})) = \begin{cases} 1 & \text{if } \nu_1 \ldots \nu_l = \mu_1 \ldots \mu_l, \\ 0 & \text{otherwise} \end{cases}$$

for $\nu_1 \ldots \nu_{l+1} \in B_{l+1}(\Lambda_A)$ implies the second relation of (3.2) by using Lemma 3.6. The equality (3.3) comes from Lemma 3.5.

We will finally show the equality (3.4). For $l = 0$, one has $e_1^0 = 1$ by definition. If $\gamma = \alpha_k$ for some $k = 1, \ldots, N$, one has $A_{0,1}(1, \alpha_k, j) = 1$ if and only if $j = k$. Hence

$$s_{\alpha_k}^* e_1^0 s_{\alpha_k} = s_{\alpha_k}^* s_{\alpha_k} = e_k^1 = \sum_{j=1}^m A_{0,1}(1, \alpha_k, j) e_j^1.$$

If $\gamma = \beta_k$ for some $k = 1, \ldots, N$, one has $A_{0,1}(1, \beta_k, j) = A(k, j)$. Hence by the relation (1.5) one has

$$s_{\beta_k}^* e_1^0 s_{\beta_k} = T_k^* T_k = e_k^1 = \sum_{j=1}^N A(k, j) S_j^* S_j = \sum_{j=1}^N A_{0,1}(1, \beta_k, j) e_j^1.$$
For $l = 1$, one sees that $e_i^1 = e_i^{N(i)}$. If $\gamma = \alpha_k$ for some $k = 1, \ldots, N$, as $A_{1,2}(i, \alpha_k, j) = 1$ if and only if $j = N(ki)$, one has

$$s_{\alpha_k} e_i^1 s_{\alpha_k} = S_k^* S_i^* S_j S_k = e_i^{N(ki)} = \sum_{j=1}^{m(2)} A_{1,2}(i, \alpha_k, j) e_j^2.$$ 

If $\gamma = \beta_k$ for some $k = 1, \ldots, N$, one has by Lemma 3.4(i) and (1.5)

$$s_{\beta_k} e_i^1 s_{\beta_k} = T_k^* S_i^* S_j T_k = \begin{cases} \sum_{j=1}^{N} A(i, j) S_j & \text{if } k = i, \\ 0 & \text{if } k \neq i. \end{cases}$$

By Lemma 3.6 and (1.4), one has

$$\sum_{j=1}^{N} A(i, j) S_j S_i = \sum_{\mu_1 \mu_2 \in B_2(\Lambda_A)} A(i, \mu_1) A(\mu_1, \mu_2) S_{\mu_1}^* S_{\mu_2}^* S_{\mu_1} S_{\mu_2}. $$

Since $A_{1,2}(i, \beta_k, N(\mu_1 \mu_2)) = 1$ if and only if $k = i$, $A(i, \mu_1) = A(\mu_1, \mu_2) = 1$, it follows that by $S_{\mu_1}^* S_{\mu_2}^* S_{\mu_1} S_{\mu_2} = e_i^{N(\mu_1 \mu_2)}$,

$$s_{\beta_k} e_i^1 s_{\beta_k} = \sum_{j=1}^{m(2)} A_{1,2}(i, \beta_k, j) e_j^2.$$ 

For $\mu_1 \ldots \mu_l \in B_l(\Lambda_A)$ with $l > 1$ and $\alpha_k \in \Sigma^-$, the relation (1.6) implies

$$s_{\alpha_k} e_i^l s_{\alpha_k} = A(k, \mu_1) S_{\mu_1}^* S_{\mu_2}^* \ldots S_{\mu_1}^* S_{\mu_1} \ldots S_{\mu_1} S_{\mu_2} = A_{l, l+1} (N(\mu_1 \ldots \mu_l), \alpha_k, N(k \mu_1 \ldots \mu_l)) e_i^{l+1}_{N(k \mu_1 \ldots \mu_l)}.$$ 

Since $A_{l, l+1} (N(\mu_1 \ldots \mu_l), \alpha_k, i) = 0$ if $i \neq N(k \mu_1 \ldots \mu_l)$, one has

$$s_{\alpha_k} e_i^l s_{\alpha_k} = \sum_{\nu_1 \ldots \nu_{l+1} \in B_{l+1}(\Lambda_A)} A_{l, l+1} (N(\mu_1 \ldots \mu_l), \alpha_k, N(v_1 \ldots v_{l+1})) e_i^{l+1}_{N(v_1 \ldots v_{l+1})}.$$ 

We also have by Lemma 3.4 for $j = \mu_1$

$$s_{\beta_j} e_i^l s_{\beta_j} = T_j^* E_{\mu_1 \ldots \mu_l} T_j = A(j, \mu_2) E_{\mu_2 \ldots \mu_l} = \sum_{\mu_{l+1} \mu_{l+2} \in B_2(\Lambda_A)} A(j, \mu_2) S_{\mu_2}^* \ldots S_{\mu_1}^* S_{\mu_{l+1}}^* S_{\mu_{l+2}}^* S_{\mu_{l+1}} S_{\mu_{l+2}} \ldots S_{\mu_2}.$$
and for \( j \neq \mu_1 \)
\[
s_{\beta_j}^* e_{N(\mu_1 \ldots \mu_l)}^j s_{\beta_j} = T_j^* E_{\mu_1 \ldots \mu_l} T_j = 0.
\]

Since one has
\[
A_{l,l+1}(N(\mu_1 \ldots \mu_l), \beta_j, N(\nu_1 \ldots \nu_{l+1})) = \begin{cases} 1 & \text{if } j = \mu_1, A(j, \mu_2) = 1 \text{ and } \nu_i = \mu_{i+1} \text{ for } i = 1, \ldots, l-1, \\ 0 & \text{otherwise}, \end{cases}
\]
we have
\[
s_{\beta_j}^* e_{N(\mu_1 \ldots \mu_l)}^j s_{\beta_j} = \sum_{\nu_1 \ldots \nu_{l+1} \in B_{l+1}(\Lambda_A)} A_{l,l+1}(N(\mu_1 \ldots \mu_l), \beta_j, N(\nu_1 \ldots \nu_{l+1})) e_{N(\nu_1 \ldots \nu_{l+1})}^{l+1}.
\]
Therefore (3.4) holds.

**Proof of Theorem 1.1.** By a general theory of the \( C^* \)-algebras associated with \( \lambda \)-graph systems [20], the algebras \( \mathcal{O}_{\mathcal{C}h(D_A)} \) are nuclear. By Proposition 3.3 and Proposition 3.7, the family of the operator relations (1.3), (1.4), (1.5) and (1.6) is equivalent to the family of the operator relations (3.1), (3.2), (3.3) and (3.4). Thus by Lemma 3.1 and Theorem 2.6, we conclude Theorem 1.1.

### 4. K-Theory

In this section, we will present K-theory formulae of the \( C^* \)-algebra \( \mathcal{O}_{\mathcal{C}h(D_A)} \) in terms of the topological Markov shift defined by the matrix \( A \). We will prove Theorem 1.2. Recall that the right one-sided topological Markov shift \( X_A \) for the matrix \( A \) is naturally identified with \( X_{D_A^+} \) as in the proof of Proposition 2.1. Let \( S_i, T_i, i = 1, \ldots, N \) be the generating partial isometries of the \( C^* \)-algebra \( \mathcal{O}_{\mathcal{C}h(D_A)} \) as in Theorem 1.1. Let \( \mathcal{A}_{\mathcal{C}h(D_A)} \) be the \( C^* \)-subalgebra of \( \mathcal{O}_{\mathcal{C}h(D_A)} \) generated by the projections \( E_{\mu_1 \ldots \mu_l} = S_{\mu_1}^* \ldots S_{\mu_l}^* S_{\mu_l} \ldots S_{\mu_1} \), \( \mu_1 \ldots \mu_l \in B_l(\Lambda_A) \), \( l \in \mathbb{Z}_+ \). Define two endomorphisms \( \lambda_{-\Sigma} \) and \( \lambda_{+\Sigma} \) on it by
\[
\lambda_{-\Sigma}(a) = \sum_{j=1}^N S_j^* a S_j, \quad \lambda_{+\Sigma}(a) = \sum_{j=1}^N T_j^* a T_j \quad \text{for } a \in \mathcal{A}_{\mathcal{C}h(D_A)}
\]
Let \( C(X_A, \mathbb{C}) \) be the abelian \( C^* \)-algebra of all \( \mathbb{C} \)-valued continuous functions on \( X_A \). We note that its \( K_0 \)-group \( K_0(C(X_A, \mathbb{C})) \) is naturally identified with \( C(X_A, \mathbb{Z}) \).

**Lemma 4.1.** Let \( \Phi : \mathcal{A}_{\mathcal{C}h(D_A)} \rightarrow C(X_A, \mathbb{C}) \) be a map defined by
\[
\Phi(E_{\mu_1 \ldots \mu_l}) = \chi_{\mu_1 \ldots \mu_l} \quad \text{for } \mu_1 \ldots \mu_l \in B_l(\Lambda_A), l \in \mathbb{Z}_+
\]
where $\chi_{\mu_1 \ldots \mu_l}$ is the characteristic function for the word $\mu_1 \ldots \mu_l$ on $X_A$ defined by
\[
\chi_{\mu_1 \ldots \mu_l}((x_i)_{i \in \mathbb{N}}) = \begin{cases} 
1 & \text{if } (x_1, \ldots, x_l) = (\mu_1, \ldots, \mu_l), \\
0 & \text{otherwise}.
\end{cases}
\]

Then we have

(i) $\Phi$ gives rise to an isomorphism from $\mathcal{A}_{q \text{Ch}(d_A)}$ onto $C(X_A, C)$.

(ii) Both of the diagrams
\[
\begin{array}{ccc}
K_0(\mathcal{A}_{q \text{Ch}(d_A)}) & \xrightarrow{\Phi_*} & C(X_A, \mathbb{Z}) \\
\xrightarrow{\lambda_{\Sigma^-}} & \downarrow & \xrightarrow{\lambda_{\lambda_A}} \\
K_0(\mathcal{A}_{q \text{Ch}(d_A)}) & \xrightarrow{\Phi_*} & C(X_A, \mathbb{Z})
\end{array}
\]
are commutative, where $\Phi_*$ is the induced isomorphism from $K_0(\mathcal{A}_{q \text{Ch}(d_A)})$ to $K_0(C(X_A, C)) = C(X_A, \mathbb{Z})$, and $\lambda_{\Sigma^-}, \lambda_{\Sigma^+}$ are induced endomorphisms on $K_0(\mathcal{A}_{q \text{Ch}(d_A)})$ by $\lambda_{\Sigma^-}, \lambda_{\Sigma^+}$ respectively.

\textbf{Proof.} (i) The assertion is straightforward.

(ii) The equality
\[
\Phi(\lambda_{\Sigma^-}(E_{\mu_1 \ldots \mu_l})) = \sum_{j=1}^N \chi_{j \mu_1 \ldots \mu_l}
\]
is immediate. As
\[
\sigma_{\lambda_A}(\chi_{\mu_1 \ldots \mu_l})(x) = \begin{cases} 
1 & \text{if } (x_2, \ldots, x_{l+1}) = (\mu_1, \ldots, \mu_l), \\
0 & \text{otherwise},
\end{cases}
\]
for $x = (x_i)_{i \in \mathbb{N}} \in X_A$, the equality
\[
\sigma_{\lambda_A}(\chi_{\mu_1 \ldots \mu_l}) = \Phi(\lambda_{\Sigma^-}(E_{\mu_1 \ldots \mu_l}))
\]
is clear. Hence the first diagram is commutative.

For the second diagram, as
\[
T_j^*E_{\mu_1 \ldots \mu_l}T_j = \begin{cases} 
A(j, \mu_2)E_{\mu_2 \ldots \mu_l} & \text{if } j = \mu_1, \\
0 & \text{if } j \neq \mu_1
\end{cases}
\]
by Lemma 3.4, it follows that
\[
\Phi(T_j^*E_{\mu_1 \ldots \mu_l}T_j)(x) = \begin{cases} 
A(j, \mu_2)\chi_{\mu_2 \ldots \mu_l}(x) & \text{if } j = \mu_1, \\
0 & \text{if } j \neq \mu_1
\end{cases}
\]
\[
= \begin{cases} 
1 & \text{if } (\mu_1, \ldots, \mu_l) = (j, x_1, x_2, \ldots, x_{l-1}), \\
0 & \text{otherwise}.
\end{cases}
\]
On the other hand, one sees for $x = (x_i)_{i \in \mathbb{N}} \in X_A$

$$\lambda_{A_A}(\chi_{\mu_1 \ldots \mu_l})(x) = \sum_{j=1}^{N} \chi_{\mu_1 \ldots \mu_l}(jx) = \begin{cases} 
1 & \text{if } (\mu_1, \ldots, \mu_l) = (j, x_1, x_2, \ldots, x_{l-1}) \\
0 & \text{otherwise}
\end{cases}$$

for some $j = 1, \ldots, N$

so that one obtains

$$\lambda_{A_A}(\chi_{\mu_1 \ldots \mu_l}) = \sum_{j=1}^{N} \Phi(T_j^* E_{\mu_1 \ldots \mu_l} T_j) = \Phi(\lambda_{\Sigma^+}(E_{\mu_1 \ldots \mu_l})).$$

Hence the second diagram is commutative.

Therefore we have

**Theorem 4.2.**

(i) $K_0(\mathcal{O}_{\text{Ch}(D_N)}) = C(X_A, \mathbb{Z})/(\text{id} - (\lambda_{A_A} + \lambda_{\Sigma^+}))C(X_A, \mathbb{Z}).$

(ii) $K_1(\mathcal{O}_{\text{Ch}(D_A)}) = \text{Ker}(\text{id} - (\lambda_{A_A} + \lambda_{\Sigma^+}))$ in $C(X_A, \mathbb{Z}).$

**Proof.** By discussions in [20, Theorem 5.5], one knows

$$K_0(\mathcal{O}_{\text{Ch}(D_A)}) = K_0(\mathcal{A}_{\text{Ch}(D_A)}/(\text{id} - \lambda_{\Sigma^+})),
\quad K_1(\mathcal{O}_{\text{Ch}(D_A)}) = \text{Ker}(\text{id} - \lambda_{\Sigma^+})$$

where $\lambda_{\Sigma^+}$ is an endomorphism on $K_0(\mathcal{A}_{\text{Ch}(D_A)})$ induced by the map

$$\lambda_{\Sigma^+} : \mathcal{A}_{\text{Ch}(D_A)} \to \mathcal{A}_{\text{Ch}(D_A)}$$

defined by

$$\lambda_{\Sigma^+}(a) = \sum_{y \in \Sigma^- \cup \Sigma^+} S_y^* a S_y \quad \text{for } a \in \mathcal{A}_{\text{Ch}(D_A)}.$$

As $\lambda_{\Sigma^+}(a) = \lambda_{\Sigma^-}(a) + \lambda_{\Sigma^+}(a)$, one sees the desired formulae by the previous lemma.

**5. Examples**

**Example 1 (Dyck shifts).** For the matrix $A$ all of whose entries are 1’s, Theorem 1.1 goes to:

**Proposition 5.1 ([24]).** The $C^*$-algebra $\mathcal{O}_{\text{Ch}(D_N)}$ associated with the Cantor horizon $\lambda$-graph system $\mathcal{O}_{\text{Ch}(D_N)}$ for the Dyck shift $D_N$ is unital, separable, nuclear, simple and purely infinite. It is the unique $C^*$-algebra generated by $N$
partial isometries $S_i, i = 1, \ldots, N$ and $N$ isometries $T_i, i = 1, \ldots, N$ subject to the following operator relations:

$$\sum_{j=1}^{N} S_j^* S_j = 1, \quad E_{\mu_1 \ldots \mu_k} = \sum_{j=1}^{N} S_j S_j^* E_{\mu_1 \ldots \mu_k} S_j S_j^* + T_{\mu_1} E_{\mu_2 \ldots \mu_k} T_{\mu_1}^*$$

where $E_{\mu_1 \ldots \mu_k} = S_{\mu_1}^* \ldots S_{\mu_k}^* S_{\mu_k} \ldots S_{\mu_1}$, $\mu_1, \ldots, \mu_k \in \{1, \ldots, N\}$. The $K$-groups are

$$K_0(\mathcal{O}_{\mathcal{Q}^{Ch(D_F)}}) \cong \mathbb{Z}/N\mathbb{Z} \oplus C(\mathfrak{sl}, \mathbb{Z}), \quad K_1(\mathcal{O}_{\mathcal{Q}^{Ch(D_F)}}) \cong 0.$$ 

**Proof.** The relation (1.5) implies that $T_i, i = 1, \ldots, N$ are isometries. By summing up $\mu_2$ over $\{1, \ldots, N\}$ in the second relation above for $k = 2$, one has the equalities

$$S_i^* S_i = \sum_{j=1}^{N} S_j S_j^* S_i S_j S_j^* + T_i T_i^*, \quad i = 1, \ldots, N$$

by using the first relation above. By summing up $i = 1, 2, \ldots, N$ in the above equalities, one sees the relation (1.3).

**Example 2** (Fibonacci Dyck shift). Let $F$ be the $2 \times 2$ matrix $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. It is the smallest matrix in the irreducible square matrices with condition (I) such that the associated topological Markov shift $\Lambda_F$ is not conjugate to any full shift. The topological entropy of $\Lambda_F$ is $\log \frac{1 + \sqrt{5}}{2}$ the logarithm of the Perron eigenvalue of $F$. We call the subshift $D_F$ the Fibonacci Dyck shift. As the matrix is irreducible with condition (I), the associated $C^*$-algebra $\mathcal{O}_{\mathcal{Q}^{Ch(D_F)}}$ is simple and purely infinite.

**Proposition 5.2.** The $C^*$-algebra $\mathcal{O}_{\mathcal{Q}^{Ch(D_F)}}$ associated with the $\lambda$-graph system $\mathcal{Q}^{Ch(D_F)}$ is unital, separable, nuclear, simple and purely infinite. It is the unique $C^*$-algebra generated by one isometry $T_1$ and three partial isometries $S_1, S_2, T_2$ subject to the following operator relations:

$$\sum_{j=1}^{2} (S_j S_j^* + T_j T_j^*) = \sum_{j=1}^{2} S_j^* S_j = 1, \quad T_2^* T_2 = S_1^* S_1,$$

$$E_{\mu_1 \ldots \mu_k} = \sum_{j=1}^{2} F(j, \mu_1) S_j S_j^* E_{\mu_1 \ldots \mu_k} S_j S_j^* + T_{\mu_1} E_{\mu_2 \ldots \mu_k} T_{\mu_1}^*, \quad k > 1$$

where $E_{\mu_1 \ldots \mu_k} = S_{\mu_1}^* \ldots S_{\mu_k}^* S_{\mu_k} \ldots S_{\mu_1}$, $(\mu_1, \ldots, \mu_k) \in B_k(\Lambda_F)$ and $B_k(\Lambda_F)$ is the set of admissible words of the topological Markov shift $\Lambda_F$ defined by
the matrix $F$. The $K$-groups are

$$K_0(\mathcal{O}_{\text{Ch}(\mathcal{D})}) \cong \mathbb{Z} \oplus C(\mathcal{D}, \mathbb{Z})^\infty, \quad K_1(\mathcal{O}_{\text{Ch}(\mathcal{D})}) \cong 0.$$ 

**Proof.** The operator relations above directly come from Theorem 1.1. The $K$-group formulae above are not direct. Its computations need some technical steps as in [25]. The full proof of the above $K$-group formulae are written in [25].

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**References**