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ON THE COX RING OF $\mathbf{P}^2$ BLOWN UP IN POINTS ON A LINE

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Abstract

We show that the blow-up of $\mathbf{P}^2$ in n points on a line has finitely generated Cox ring. We give explicit generators for the ring and calculate its defining ideal of relations.

1. Introduction

In 2000, Hu and Keel [7] introduced the *Cox ring* of an algebraic variety, aiming to generalize the Cox construction for toric varieties. If X is a normal variety with freely finitely generated Picard group $\mathrm{Pic}(X)$, this ring is essentially defined by

$$\mathrm{Cox}(X) = \bigoplus_{D\in \mathrm{Pic}(X)} H^0(X, \mathscr{O}_X(D)),$$

where the ring product is given by multiplication of sections as rational functions. Varieties whose Cox ring is finitely generated are called *Mori dream spaces*, and have interesting properties from the viewpoint of birational geometry.

Even though the definition of the Cox ring is quite explicit, calculating its presentation for a concrete variety can be a hard problem. For example, the Cox rings of del Pezzo surfaces have been the subject of much recent literature in algebraic geometry (e.g., [2], [9] and [11]) and show that the behaviour of the Cox ring under blow-up is highly non-trivial. For general blow-ups of $\mathbf{P}^2$, $\mathrm{Cox}(X)$ may even fail to be finitely generated, since the surfaces may have infinitely many curves of negative self-intersection.

In this paper, we consider blow-ups of $\mathbf{P}^2$ in points $p_1, \ldots, p_n$ lying on a fixed line $Y \subset \mathbf{P}^2$. The blow-up $\pi : X \to \mathbf{P}^2$ in these points is a smooth projective surface with Picard group generated by the classes of the exceptional divisors $E_1, \ldots, E_n$ and L, the pullback of a general line in $\mathbf{P}^2$. Our first result is that $\mathrm{Cox}(X)$ is finitely generated for any number of points on a line. This result was first shown in [5] and correlates with recent results of Hausen and Süss in [8], since the surface X has a complexity one torus action.

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Furthermore, we also find explicit generators for $\operatorname{Cox}(X)$, i.e., generating sections $x_1, \dots, x_r$ from respective vector spaces $H^0(X, \mathscr{O}_X(D_1)), \dots, H^0(X, \mathscr{O}_X(D_r))$, so that $\operatorname{Cox}(X)$ may be regarded as a quotient

$$\operatorname{Cox}(X) \simeq k[x_1, \dots, x_r]/I.$$

Here we consider a $\operatorname{Pic}(X)$-grading on $k[x_1, \dots, x_r]$ and I given by $\deg(x_i) = D_i$. In Section 4 we find explicit generators and a Gröbner basis for the ideal I. Our main result is the following theorem:

THEOREM. *Let X be the blow-up of $\mathbf{P}^2$ in $n \geq 3$ distinct points lying on a line. Then* $\operatorname{Cox}(X)$ *is a complete intersection ring and its defining ideal is generated by quadric trinomials.*

In particular, this means that the Cox ring is Gorenstein and a Koszul algebra.

NOTATION. We make the following standard shorthand notation for sheaf cohomology:

$$H^i(D) := H^i(X, \mathscr{O}_X(D)), \quad h^i(D) := \dim_k H^i(X, \mathscr{O}_X(D)), \quad i = 0, 1, 2.$$

2. Nef divisors and vanishing on X

The surface X has good vanishing properties for nef divisors. For example, $H^2(D) = H^0(K - D) = 0$ is immediate by Serre duality, since K cannot be effective on X. It turns out that also $H^1(D) = 0$ for D nef, so all cohomology can be calculated from the Riemann-Roch theorem. To prove this, we first need some preparatory lemmas.

LEMMA 2.1. *The monoid of effective divisor classes of X is finitely generated by the classes $L - E_1 - \cdots - E_n, E_1, E_2, \dots, E_n$.*

PROOF. It is clear that the classes above are all effective, so their semigroup span is in the effective monoid. Conversely, note that these divisor classes actually form a $\mathbf{Z}$-basis for $\operatorname{Pic} X$. So given an *irreducible* effective divisor D, we let

$$m(L - E_1 - \cdots - E_n) + a_1 E_1 + \cdots + a_n E_n$$

represent the corresponding divisor class. If D is not one of the generators above we have $D.E_i = m - a_i \geq 0$ and $D.(L - E_1 - \cdots - E_n) = m - \sum_{i=1}^n (m - a_i) \geq 0$. Together these inequalities imply that $m, a_i \geq 0$, and we are done.

LEMMA 2.2. *The nef monoid is generated by the divisor classes $L, L - E_1, L - E_2, \dots, L - E_n$.*

PROOF. Let $D = dL - \sum a_i E_i$ be a nef divisor class. Intersecting D with the classes in Lemma 2.1 gives the following set of inequalities:

$$d \geq a_1 + a_2 + \cdots + a_n, \qquad a_i \geq 0, \quad \forall i = 1, \ldots, n$$

Now it is easy to see that we can decompose each D as a sum of the $L - E_i$'s by using a_i of $L - E_i$ and finally add $d - a_1 - a_2 - \cdots - a_n \geq 0$ times L.

Note that since the classes above are effective, so is every nef divisor on X.

LEMMA 2.3. *Let $D = dL - a_1 E_1 - \cdots - a_n E_n$ be a divisor class on X, with $d + 1 \geq \sum_{i=1}^n a_i$ and $a_i \geq 0$. Then $h^1(D) = 0$.*

PROOF. If $D = dL$, we have $h^1(D) = h^1(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(d)) = 0$ for $d \geq -1$. If say $a_1 > 0$, consider the divisor class $D' = D - (L - E_1)$. D' satisfies the conditions of the lemma, so by induction on d we have $h^1(D') = 0$. Let C be a smooth rational curve with class $L - E_1$, then $h^1(C, D|_C) = h^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(D.C)) = 0$, since $D.C \geq -1$. Now taking cohomology of the exact sequence

$$0 \to \mathcal{O}_X(D') \to \mathcal{O}_X(D) \to \mathcal{O}_C(D|_C) \to 0$$

gives $h^1(D) = 0$.

This gives us the multigraded Hilbert function of $\operatorname{Cox}(X)$ in nef degrees:

COROLLARY 2.4. *For a nef divisor class D, we have $h^1(D) = 0$ and*

$$\dim_k \operatorname{Cox}(X)_D = \chi(\mathcal{O}_X(D)).$$

3. Generators for Cox(X)

We are now in position to find explicit generators for $\operatorname{Cox}(X)$ as a $k-$algebra. When $n = 1$, the surface $\operatorname{Bl}_p \mathbf{P}^2$ is a toric variety, and by [3], its Cox ring coincides with the usual homogenous coordinate ring

$$\operatorname{Cox}(\operatorname{Bl}_p \mathbf{P}^2) = k[x, s_1, s_2, e]$$

where $\deg x = L$, $\deg s_i = L - E$ and $\deg e = E$. Therefore, we will in the following suppose that $n \geq 2$.

We first choose generators $e_1, \ldots, e_n$ for the 1-dimensional vector spaces $H^0(E_i)$ for $i = 1, \ldots, n$ and a generator l of $H^0(L - E_1 - \cdots - E_n)$. For the classes $L - E_i$, for which $H^0(L - E_i)$ is 2-dimensional, we need in addition to the section $le_1 \ldots e_{i-1}e_{i+1} \ldots e_n$, a new section s_i to form a basis. To specify these explicitly, we fix a point $q \in \mathbf{P}^2$ and for each i take a section

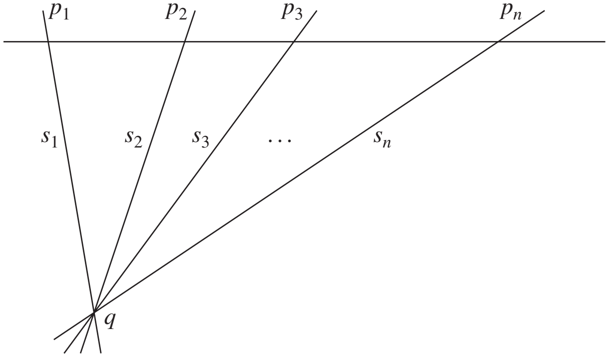

FIGURE 1. The choice of the sections $s_1, s_2, \dots, s_n$

corresponding to the strict transform of the line going through q and p_i. The projections of these sections to P^2 are shown in Figure 1.

We will show that $\mathrm{Cox}(X)$ is generated by the sections l, e_i, s_i for $i = 1, \dots, n$. The following lemma is a variant of Castelnuovo's base point free pencil trick ([4, Ex. 17.18]) and will be our main tool for proving this.

LEMMA 3.1. *Let X be an algebraic variety over a field k, let $\mathscr{F}$ be a locally free sheaf of $\mathcal{O}_X$-modules on X, let $\mathscr{L}$ be an invertible sheaf on X and V a two-dimensional base-point free subspace of $H^0(X, \mathscr{L})$. If $H^1(\mathscr{L}^{-1} \otimes \mathscr{F}) = 0$, then the multiplication map*

$$V \otimes H^0(X, \mathscr{F}) \to H^0(X, \mathscr{L} \otimes \mathscr{F})$$

is surjective.

PROPOSITION 3.2. *Let X be the blow-up of $\mathbf{P}^2$ in $n \geq 2$ distinct points on a line. Then there is a multigraded surjection*

$$k[l, e_1, \dots, e_n, s_1, \dots, s_n] \to \mathrm{Cox}(X). \tag{1}$$

where $\deg(l) = L - E_1 - \cdots - E_n$*,* $\deg e_i = E_i$ *and* $\deg s_i = L - E_i$*.*

PROOF. Let D be an effective divisor on X. We need to show that $H^0(D)$ has a basis of sections which are polynomials in l, e_i, s_i.

We first show that we may take D to be nef. Indeed, suppose E is a curve such that $D \cdot E < 0$. Without loss of generality, we may suppose that E is one of the divisor classes generating the effective monoid. Let $x_E \in \{l, e_1, \dots, e_n\}$ be the corresponding section in $H^0(E)$. Then since E is a base component of the linear system $|D|$, multiplication by x_E induces an isomorphism $H^0(D-E) \to H^0(X, D)$. By induction on the number of fixed components, $H^0(D - E)$ is generated by monomials in the l, e_i, s_i and hence the same applies to $H^0(D)$.

Now suppose that D is a nef divisor class and write D (uniquely) in terms of the nef classes of Lemma 2.2:

$$D = aL + a_1(L - E_1) + a_2(L - E_2) + \cdots + a_n(L - E_n)$$

where $a, a_i \geq 0$. If say, $a_1 \geq 2$, then $H^1(D - 2(L - E_1)) = 0$, since $D - 2(L - E_1)$ is nef, and so applying Lemma 3.1 with $V = H^0(L - E_1)$ and $\mathscr{F} = \mathcal{O}_X(D - (L - E_1))$, we get a surjection

$$H^0(D - (L - E_1)) \otimes H^0(L - E_1) \to H^0(D).$$

By induction on the number $D.L \geq 0$, $H^0(D - (L - E_1))$ is generated by monomials in l, e_i, s_i, and therefore so is $H^0(D)$.

If $a_i \leq 1$ for all i, and say, $a_1 = 1$, then $D - 2(L - E_1) = N + E_1$ for some divisor N satisfying the assumptions of Lemma 2.3 and $N.E_1 = 0$. In particular, $h^1(N) = 0$. Now also $h^1(N + E_1) = 0$, by the exact sequence

$$0 \to \mathcal{O}_X(N) \to \mathcal{O}_X(N + E_1) \to \mathcal{O}_{E_1}(-1) \to 0$$

and we proceed as above.

If $a_i = 0$ for all i, then $D = aL$ for some $a \geq 1$ and $H^0(X, \mathcal{O}_X(D)) \simeq H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(a))$. This implies that

$$H^0(X, (a-1)L) \otimes H^0(X, L) \to H^0(X, aL).$$

is surjective. By induction on a, $H^0((a-1)L)$ is generated by monomials in l, e_i, s_i, and therefore so is $H^0(D)$.

It remains to show that $H^0(L)$ has a basis of monomials in l, e_i, s_i. But $H^0(L) = \pi^* H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1))$, so it suffices to find three monomials of degree L that project to linearly independent sections in $\mathcal{O}_{\mathbf{P}^2}(1)$. By construction, this works for the three sections

$$\sigma_1 = s_1 e_1, \quad \sigma_2 = s_2 e_2, \quad \sigma_3 = l e_1 e_2 e_3 \dots e_n.$$

4. Relations

We now turn to the defining ideal I of relations of $\mathrm{Cox}(X)$, i.e., the kernel of the map (1). Consider again the divisor class L: We have $h^0(L) = 3$, while there are $n + 1$ monomials of degree L in $k[l, e_i, s_i]$:

$$s_1 e_1, \quad s_2 e_2, \quad \dots, \quad s_n e_n, \quad l e_1 e_2 e_3 \dots e_n$$

This means that there are $n - 2$ linear dependence relations between them. To see what they look like, consider again the projection of these sections in Figure 1. Of course any three of these lines through q satisfy a linear dependence relation, and these pull back via π to relations in $\mathrm{Cox}(X)$ of the following form:

$$
\begin{aligned}
g_1 &= \underline{s_1 e_1} + a_1 s_{n-1} e_{n-1} + b_1 s_n e_n = 0 \\
g_2 &= \underline{s_2 e_2} + a_2 s_{n-1} e_{n-1} + b_2 s_n e_n = 0 \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
g_{n-2} &= \underline{s_{n-2} e_{n-2}} + a_{n-2} s_{n-1} e_{n-1} + b_{n-2} s_n e_n = 0
\end{aligned}
\tag{2}
$$

where each of the coefficients a_i, b_i are non-zero. We denote the ideal generated by these relations by J. The leftmost terms above are underlined since as the next lemma shows, they form an initial ideal for J.

LEMMA 4.1. *The set* $\{g_1, \dots, g_{n-2}\}$ *is a Gröbner basis for J with respect to the graded lexicographical order, and* $(s_1 e_1, \dots, s_{n-2} e_{n-2})$ *is an initial ideal of J.*

PROOF. It is well-known (e.g., see [1]) that a collection of polynomials with relatively prime leading terms is a Gröbner basis for the ideal they generate.

We will show that that the expressions (2) in fact generate all the relations, i.e., that $I = J$. For this, we will make use of the $\mathrm{Pic}(X)-$grading on $R = k[l, e_i, s_i]$ and I. The next lemma shows that it is sufficient to consider generators for the ideal of degrees corresponding to nef divisor classes.

LEMMA 4.2. *The ideal I is generated by elements of degree D, where D is a nef divisor class.*

PROOF. Suppose D is an effective divisor class and that there is a negative curve E such that $D.E < 0$. Then this implies that E is a fixed component of $|D|$ and as above every monomial in $k[l, s_i, e_i]_D$ is divisible by x_E, the variable corresponding to E. This shows that any element of I_D can be written as a product of x_E and a relation in I_{D-E}. Now the claim follows by induction on the number of fixed components of D.

We will now prove our main theorem.

THEOREM 4.3. *Let X be the blow-up of $n \geq 2$ points on a line. Then* $\mathrm{Cox}(X)$ *is a complete intersection with $n-2$ quadratic defining relations given in* (2).

PROOF. By Lemma 4.2, it is sufficient to show that $I_D = J_D$ for all nef classes $D = dL - a_1 E_1 - a_2 E_2 - \cdots - a_n E_n$, (here $d \geq a_1 + \cdots + a_n$). Note that since $J \subseteq I$, we have in any case a surjective homomorphism

$$R/J \to \mathrm{Cox}(X).$$

To show that this is an isomorphism in degree D, we calculate the multigraded Hilbert function of both sides. From Corollary 2.4 and Riemann-Roch, we have

$$\dim_k \operatorname{Cox}(X)_D = h^0(D) = \binom{d+2}{2} - \binom{a_1+1}{2} - \cdots - \binom{a_n+1}{2}. \tag{3}$$

To calculate $\dim_k(R/J)_D$, we use the Gröbner basis for J. Since the Hilbert function is preserved when going to initial ideals, we have

$$\dim_k(R/J)_D = \dim_k R/(s_1e_1, \dots, s_{n-2}e_{n-2})_D$$

We use a counting argument to evaluate this number. Note that any monomial in $R/(s_1e_1, \dots, s_{n-2}e_{n-2})_D$ corresponds to a way of writing D as a non-negative sum of divisor classes from

$$L - E_1 - \cdots - E_n, \quad L - E_1, \quad \dots, \quad L - E_n, \quad E_1, \quad \dots, \quad E_n$$

such that not both $L - E_i$ and E_i occur in the sum for $i = 1, \dots, n-2$. Abusing notation, we let the numbers s_i, e_i, l represents respectively the non-negative coefficients of $L - E_i, E_i, L - E_1 - \cdots - E_n$ in this sum. Working in $\operatorname{Pic}(X) \simeq \mathsf{Z}^{n+1}$, this translates the problem of finding $\dim_k(R/J)_D$ into following counting problem: finding the number of non-negative solutions of

$$\begin{aligned} s_1 + s_2 + \cdots + s_n &= d - l \\ s_1 - e_1 &= a_1 - l \\ \cdots\cdots&\cdots\cdots \\ s_n - e_n &= a_n - l \end{aligned} \tag{4}$$

such that $s_i \cdot e_i = 0$ for $i = 1, \dots, n-2$.

LEMMA 4.4. *For each $l \leq d$ we have $d - l \geq \sum_{k=1}^{n} \max(a_i - l, 0)$.*

PROOF. For $l = 0$, this inequality reduces to the nef condition on D. Now, increasing l by one decreases the left hand side by one, and if there is some $a_i - l > 0$, then $\max(a_i - l, 0)$ is decreased by one, as well, if not, the right hand side is zero, so in any case the inequality is preserved.

For each fixed l, we count the number of non-negative solutions $S(l)$ to the system (4). Note that s_i is completely determined as $s_i = \max(a_i - l, 0)$ for $1 \leq i \leq n-2$ by the condition $s_i.e_i = 0$. Hence by the first equation in (4), we are looking for non-negative solutions to

$$s_n + s_{n-1} = d - l - \sum_{k=1}^{n-2} \max(a_k - l, 0) \geq 0$$

such that $s_n \geq \max(a_n - l, 0)$ and $s_{n-1} \geq \max(a_{n-1} - l, 0)$, of which there are in total

$$d - l - \sum_{k=1}^{n-2} \max(a_{n-2} - l, 0) + 1 - \sum_{k=n-1,n} \max(a_k - l, 0)$$

Hence the total number of solutions to (4) is

$$\begin{aligned}\sum_{l=0}^{d} S(l) &= \sum_{l=0}^{d}\left(d + 1 - l - \sum_{k=1}^{n} \max(a_k - l, 0)\right)\\ &= \binom{d+2}{2} - \sum_{i=0}^{a_1}(a_1 - i) - \sum_{i=0}^{a_1}(a_1 - i) - \cdots - \sum_{i=0}^{a_n}(a_n - i)\\ &= \binom{d+2}{2} - \binom{a_1+1}{2} - \binom{a_2+1}{2} - \cdots - \binom{a_n+1}{2} = h^0(D).\end{aligned}$$

This finishes the proof that $I = J$. Now, from [2, Remark 1.4] we have $\dim \operatorname{Cox}(X) = n + 3$, furthermore by Proposition 3.2 we have that codim $\operatorname{Cox}(X) = (2n + 1) - (n + 3) = n - 2$, which is exactly the number of relations in I.

REMARK. The above result can also be proved in another way, using the following lemma, proved by Stillman in [10]:

LEMMA 4.5. *Let $J \subset k[x_1, x_2, \dots, x_n]$ be an ideal containing a polynomial $f = gx_1 + h$, with g, h not involving x_1 and g a non-zero divisor modulo J. Then, J is prime if and only if the elimination ideal $J \cap k[x_2, \dots, x_n]$ is prime.*

The above lemma can be used to prove that $J = (g_1, \dots, g_{n-2})$ is prime, using induction on n. For $n = 3$, this is clear. Next, note that the elimination ideal $J \cap k[s_2, \dots, s_n, e_2 \dots, e_n, l]$ is just $(g_2, \dots, g_{n-2})$ since $\{g_1, \dots, g_{n-2}\}$ is a Gröbner basis. By induction on n, $(g_2, \dots, g_{n-2})$ is the defining ideal of $\operatorname{Cox}(X)$ for $\mathbf{P}^2$ blown up in the points $p_2, \dots, p_n$ and hence is prime. Taking now $x_1 = e_1$, $g = s_1$, $h = a_1 s_{n-1} e_{n-1} + b_1 s_n e_n$ shows that J is prime. Then, since $J \subseteq I$ are two prime ideals with the same Krull dimension, they must be equal.

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