THE UNICITY OF BEST APPROXIMATION IN A SPACE OF COMPACT OPERATORS

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Abstract
Chebyshev subspaces of \( \mathcal{K}(c_0, c_0) \) are studied. A \( k \)-dimensional non-interpolating Chebyshev subspace is constructed. The unicity of best approximation in non-Chebyshev subspaces is considered.

1. Introduction
Let \( K \) be the field of real or complex numbers and let \( (X, \| \cdot \|) \) be a normed space over \( K \). Let \( \text{ext}_{S_X^*} \) denote the set of all extreme points of \( S_X^* \), where \( S_X^* \) is the unit sphere in \( X^* \).
For every \( x \in X \) we put
\[
E(x) = \{ f \in \text{ext}_{S_X^*} : f(x) = \|x\| \}.
\]

By the Hahn-Banach and the Krein-Milman Theorems, \( E(x) \neq \emptyset \).
Let for \( Y \subset X \),
\[
P_Y(x) = \{ y \in Y : \|x - y\| = \text{dist}(x, Y) \}.
\]

A linear subspace \( Y \subset X \) is called a Chebyshev subspace if for every \( x \in X \) the set \( P_Y(x) \) contains one and only one element.

THEOREM 1 (see [3]). Assume \( X \) is a normed space, \( Y \subset X \) is a linear subspace, and let \( x \in X \setminus Y \). Then \( y_0 \in P_Y(x) \) if and only if for every \( y \in Y \) there exists \( f \in E(x - y_0) \) with \( \Re f(y) \leq 0 \).

DEFINITION (see, e.g., [8]). An element \( y_0 \in Y \) is called a strongly unique best approximation for \( x \in X \) if there exists \( r > 0 \) such that for every \( y \in Y \),
\[
\|x - y\| \geq \|x - y_0\| + r \|y - y_0\|.
\]

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The biggest constant $r$ satisfying the above inequality is called a **strong unicity constant**. There exist two main applications of a strong unicity constant:

- the error estimate of the Remez algorithm (see e.g. [13]),
- the Lipschitz continuity of the best approximation mapping at $x_0$ (assuming that there exists a strongly unique best approximation to $x_0$) (see e.g. [5], [9], [11]).

**Theorem 2** (see [17]). Let $x \in X \setminus Y$ and let $Y$ be a linear subspace of $X$. Then $y_0 \in Y$ is a strongly unique best approximation for $x$ with a constant $r > 0$ if and only if for every $y \in Y$ there exists $f \in E(x - y_0)$ with $\Re f(y) \leq -r \|y\|$.

Recall that a $k$-dimensional subspace $\mathcal{V}$ of a normed space $X$ is called an **interpolating subspace** if for any linearly independent $f_1, f_2, \ldots, f_k \in \text{ext} S_X$ and for every $v \in \mathcal{V}$ the following holds:

$$
\text{if } f_i(v) = 0, \ i = 1, 2, \ldots, k \ \text{ then } \ v = 0.
$$

Every interpolating subspace is a finite dimensional Chebyshev subspace. If $\mathcal{V} \subset X$ is an interpolating subspace then every $x \in X$ has a strongly unique best approximation in $\mathcal{V}$ (see [2]).

In this paper we consider $X = \mathcal{K}(c_0, c_0)$ (the space of all compact operators from $c_0$ to $c_0$ equipped with the operator norm). Here $c_0$ denotes the space of all real sequences convergent to zero. For any $x = (x_k) \in c_0$ we put

$$
\|x\|_{\infty} = \sup_k |x_k|.
$$

In [8, Theorem 3.1] it has been proved that if $\mathcal{V} \subset \mathcal{K}(c_0, c_0)$ is a finite-dimensional Chebyshev subspace then every $A \in \mathcal{K}(c_0, c_0)$ has a strongly unique best approximation in $\mathcal{V}$. However, in [8] no example of a non-interpolating Chebyshev subspace has been proposed. If it were true that any finite-dimensional Chebyshev subspace of $\mathcal{K}(c_0, c_0)$ is an interpolating subspace we would have obtained the proof of Theorem 3.1, [8] immediately (see [2] for more details).

The aim of this paper is to show that for every $k < \infty$ there exists a $k$-dimensional non-interpolating Chebyshev subspace of $\mathcal{K}(c_0, c_0)$. This result is quite different from the result obtained in [7]. In the space $\mathcal{L}(l_1^n, c_0)$ any finite-dimensional Chebyshev subspace is an interpolating subspace.

Additionally, we discuss the strong unicity of best approximation in some (not necessarily Chebyshev) subspaces of $\mathcal{K}(c_0, c_0)$. 
2. \(k\)-dimensional Chebyshev subspaces of \(\mathcal{H}(c_0, c_0)\)

Let \(A \in \mathcal{H}(c_0, c_0)\) be represented by a matrix \([a_{ij}]_{i,j \in \mathbb{N}}\). Note that

\[
(a_{ij})_{i=1}^{\infty} \in c_0 \quad \text{for every} \quad j \in \mathbb{N}.
\]

Since each row of a matrix \([a_{ij}]_{i,j \in \mathbb{N}}\) corresponds to a linear functional on \(c_0\),

\[
(a_{ij})_{j=1}^{\infty} \in l^1 \quad \text{for every} \quad i \in \mathbb{N}.
\]

Moreover, by the Schur Theorem (see [6])

\[
\lim_{i \to \infty} \left( \sum_{j=1}^{\infty} |a_{ij}| \right) = 0.
\]

Recall (see [4]) that \(\text{ext}_{\mathcal{H}^*(c_0,c_0)}\) consists of functionals of the form \(e_i \otimes x\), where \(x \in \text{ext}_{S_{l^\infty}}\) and

\[
(2) \quad (e_i \otimes x)(A) = \sum_{j=1}^{\infty} x_j a_{ij}.
\]

It is easy to see that

\[
\|A\| = \sup_{i \geq 1} \sum_{j=1}^{\infty} |a_{ij}|.
\]

**Remark 1.** Let \(X\) be a Banach space and let \(\mathcal{V}\) be a finite-dimensional subspace with \(V_1, V_2, \ldots, V_k\) as a basis. Then \(\mathcal{V}\) is an interpolating subspace if and only if for any linearly independent \(f_1, f_2, \ldots, f_k \in \text{ext}_{S_X^*}\) the determinant of \([f_i(V_j)]_{i,j=1,2,\ldots,k}\) is not equal to zero.

**Proof.** We apply the definition of a \(k\)-dimensional interpolating subspace and the theory of linear equations. This completes the proof.

In the sequel, we denote by \(\text{lin}\{V_1, V_2, \ldots, V_k\}\) the \(k\)-dimensional subspace of \(\mathcal{H}(c_0, c_0)\) with \(V_1, V_2, \ldots, V_k\) as a basis.

**Example 1.** Let \(V = \{v_{ij}\}_{i,j \in \mathbb{N}}, \) where \(v_{i1} = \frac{1}{2^i}, v_{ij} = 0, i, j \in \mathbb{N}, j \geq 2\). It is obvious that \(\mathcal{V} = \text{lin}\{V\}\) is a one-dimensional interpolating subspace of \(\mathcal{H}(c_0, c_0)\).
THEOREM 3. Let $\mathcal{V} = \text{lin}(V_1, V_2, \ldots, V_n)$. Let $V_m = [(v_{m})_{ij}]_{i,j \in \mathbb{N}}, m = 1, 2, \ldots, n$. If $\mathcal{V}$ is a Chebyshev subspace then

$$\begin{vmatrix} f_1(V_1) & \cdots & f_1(V_n) \\ \vdots & \ddots & \vdots \\ f_n(V_1) & \cdots & f_n(V_n) \end{vmatrix} \neq 0$$

(3)

for any $f_1, \ldots, f_n \in \text{ext } S_{\mathcal{H}^\ast(c_0, c_0)}$ such that $f_m = e_{i_m} \otimes x^{i_m}, m = 1, 2, \ldots, n$, where $i_m \neq i_k$ for $m \neq k$.

PROOF. Assume (3) does not hold. Therefore there exist $f_1, \ldots, f_n \in \text{ext } S_{\mathcal{H}^\ast(c_0, c_0)}$, $f_m = e_{i_m} \otimes x^{i_m}, m = 1, 2, \ldots, n$, where $i_m \neq i_k$ for $m \neq k$ such that det $D = 0$, where

$$D = \begin{bmatrix} f_1(V_1) & \cdots & f_1(V_n) \\ \vdots & \ddots & \vdots \\ f_n(V_1) & \cdots & f_n(V_n) \end{bmatrix}$$

Since det $D = \text{det } D^T$, there exists $y = (y_1, y_2, \ldots, y_n) \neq 0$ such that $D^T y = 0$. Consequently,

$$\sum_{m=1}^{n} y_m f_m |_{\mathcal{V}} = 0.$$  

(4)

Since $y \neq 0$, replacing $f_m$ by $-f_m$ if necessary, we may assume $y_m \geq 0$ for $m = 1, 2, \ldots, n$ and

$$\sum_{m=1}^{n} y_m = 1.$$  

Set $\mathcal{C} = \{l \in \{1, 2, \ldots, n\} : y_l > 0\}$.

Fix $(d_j)_{j \in \mathbb{N}}$ with the following properties:

$$d_j > 0, \quad j \in \mathbb{N} \quad \text{and} \quad \sum_{j=1}^{\infty} d_j = 1.$$  

Define $A = [a_{ip,j}]_{p,i,j \in \mathcal{H}(c_0, c_0)}$ by

$$a_{ip,j} = 0 \quad \text{for} \quad p \notin \mathcal{C}, j \in \mathbb{N},$$

$$a_{ip,j} = d_j \cdot \text{sgn } x^{i_p}(j) \quad \text{for} \quad p \in \mathcal{C}, j \in \mathbb{N}.$$
Note that \( \|A\| = 1 \) and
\[
E(A) = \{ f_p : p \in C \}.
\]

By (4) and Theorem 1, \( 0 \in \mathcal{P}_\mathcal{V}(A) \).

Since \( \det D = 0 \), there exists \( x = (x_1, x_2, \ldots, x_n) \neq 0 \) such that \( Dx = 0 \).

Put
\[
V = \sum_{m=1}^{n} x_m V_m.
\]

Note that \( V \neq 0 \) and \( f_m(V) = 0, m = 1, 2, \ldots, n \). By Theorem 2, 0 is not a strongly unique best approximation for \( A \) in \( \mathcal{V} \). By [8, Theorem 3.1], \( \mathcal{V} \) is not a Chebyshev subspace and the proof is complete.

**Theorem 4.** Let \( \mathcal{V} = \text{lin}\{V\}, V \in \mathcal{K}(c_0, c_0), V \neq 0 \). Then \( \mathcal{V} \) is a Chebyshev subspace if and only if \( \mathcal{V} \) is an interpolating subspace.

**Proof.** The classical work here is [12]. In \( l^1 \), the one-dimensional subspace \( \text{lin}\{v\} \) is Chebyshev iff for every \( x \in \text{ext } S_\infty \) the following holds
\[
\sum_{j=1}^{\infty} x(j)v(j) \neq 0.
\]

Note that for any \( x \in c_0 \) we obtain \( V(x) = [f_1(x), f_2(x), \ldots] \), where the functionals \( f_i \) correspond to elements of \( l^1 \).

It is obvious that if for any \( j \), \( \text{lin}\{f_j\} \) is not a Chebyshev subspace of \( l^1 \), then \( \text{lin}\{V\} \) is not a Chebyshev subspace of \( \mathcal{K}(c_0, c_0) \). This proves the theorem.

Note that by a result of Malbrock (see [10], Theorem 3.3) each one-dimensional subspace \( \mathcal{V} = \text{lin}\{V\} \subset \mathcal{K}(c_0, c_0) \) is a Chebyshev subspace iff there exists \( \delta > 0 \) such that
\[
\left| \sum_{j=1}^{\infty} x(j)v_{ij} \right| \geq \delta,
\]
where \( |x(j)| = 1, j \in \mathbb{N} \).

**Corollary.** Let \( \mathcal{V} \subset \mathcal{K}(c_0, c_0) \) be a one-dimensional Chebyshev subspace. Every operator \( A \in \mathcal{K}(c_0, c_0) \) has a strongly unique best approximation in \( \mathcal{V} \).

**Proof.** Obvious. For more details we refer the reader to [2].

It is clear that (3) is satisfied for any \( n \)-dimensional interpolating subspace. However, (3) is not sufficient for an \( n \)-dimensional \( (n \geq 2) \) subspace to be Chebyshev.
Example 2. Let $Y = \text{lin}\{V_1, V_2\}$, where

$$V_1 = \begin{bmatrix} 1 & 0 & \ldots & \frac{1}{4} & 0 & \ldots \\ \frac{1}{4} & 0 & \ldots & \frac{1}{4} & 0 & \ldots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ \frac{1}{8} & 0 & \ldots & \frac{1}{8} & 0 & \ldots \end{bmatrix}, \quad V_2 = \begin{bmatrix} 1 & 0 & \ldots & \frac{1}{9} & 0 & \ldots \\ \frac{1}{9} & 0 & \ldots & \frac{1}{9} & 0 & \ldots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ \frac{1}{16} & 0 & \ldots & \frac{1}{16} & 0 & \ldots \end{bmatrix}. $$

Note that $Y$ satisfies (3). We claim that $Y$ is a non-Chebyshev subspace. Indeed, define $A = [a_{ij}]_{i,j \in \mathbb{N}}$ by

$$a_{12} = 100, \quad a_{ij} = 0 \quad \text{for each} \quad (i, j) \neq (1, 2), \quad i, j \in \mathbb{N}. $$

It follows that

$$A - (\alpha_1 V_1 + \alpha_2 V_2) = \begin{bmatrix} -\alpha_1 - \alpha_2 & 100 & 0 & \ldots \\ -\frac{1}{2}\alpha_1 - \frac{1}{3}\alpha_2 & 0 & \ldots \\ -\frac{1}{4}\alpha_1 - \frac{1}{9}\alpha_2 & 0 & \ldots \\ \vdots & \vdots & \ddots & \vdots \end{bmatrix}, \quad \alpha_1, \alpha_2 \in \mathbb{R}. $$

Hence

$$\|A\| = \|A - (600 V_1 - 600 V_2)\| = 100 = \inf_{\alpha_1, \alpha_2 \in \mathbb{R}} \|A - (\alpha_1 V_1 + \alpha_2 V_2)\|. $$

Theorem 5. Let $V_1, V_2, \ldots, V_n$ be given by

$$V_j = \begin{bmatrix} 0 & 0 & \ldots & v_{1j} & 0 & \ldots \\ 0 & 0 & \ldots & v_{2j} & 0 & \ldots \\ 0 & 0 & \ldots & v_{3j} & 0 & \ldots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \end{bmatrix}, $$

where $v_{ij} \neq 0$ for each $i \in \mathbb{N}, \ j \in \{1, 2, \ldots, n\}$ and

$$\lim_{i \to \infty} v_{ij} = 0 \quad \text{for each} \quad j \in \{1, 2, \ldots, n\}. $$

The following statements are equivalent:

(i) For every choice of distinct $j_1, \ldots, j_k$ from $\{1, 2, \ldots, n\}$, $Y(j_1, \ldots, j_k) := \text{lin}\{V_{j_1}, \ldots, V_{j_k}\}$ is a Chebyshev subspace of $\mathcal{H}(c_0, c_0)$.
Proof. First, we assume that (ii) holds.
If \( k = 1 \) then \( V(j_1) \) is an interpolating subspace for every \( j_1 \in \{1, 2, \ldots, n\} \).
Let \( 1 < k < n \) and assume that for any \( j_1, \ldots, j_k \in \{1, 2, \ldots, n\} \), \( j_p \neq j_q \), \( p \neq q \), \( V_k := V(j_1, \ldots, j_k) \) is a Chebyshev subspace.
Suppose that there exist \( 1 \leq j_1 < j_2 < \cdots < j_k < j_{k+1} \leq n \) such that
\[
V_{k+1} := V(j_1, \ldots, j_k, j_{k+1})
\]
is a non-Chebyshev subspace. Without loss of generality we can assume that for any \( k+1 \in \{1, 2, \ldots, n\} \), \( j_m = m, m = 1, 2, \ldots, k+1 \). This means precisely that \( V_{j_m} = [(V_{j_m})_{ij}]_{i,j \in \mathbb{N}}, \) where
\[
(V_{j_m})_{ij} = \begin{cases} v_{ijm}, & j = m \\ 0, & j \neq m \end{cases}.
\]
for \( i \in \mathbb{N}, m \in \{1, 2, \ldots, k, k+1\} \).

Since \( V_{k+1} \) is a non-Chebyshev subspace, there exists \( A = [a_{ij}]_{i,j \in \mathbb{N}} \in \mathcal{K}(c_0, c_0) \) such that \( \#P_{V_{k+1}}(A) > 1 \). We can assume that \( 0, W \in P_{V_{k+1}}(A), \) where \( W \neq 0 \). Let \( \mathcal{U} = \{i : \|e_i \circ A\| = \|A\|\} \). Since \( A \in \mathcal{K}(c_0, c_0), \#\mathcal{U} < \infty \). For every \( i \in \mathcal{U} \) we put
\[
E_i = \{x \in \text{ext } S_{l^\infty} : (e_i \otimes x)(A) = \|A\|\}.
\]
Since \( 0, W \in P_{V_{k+1}}(A), \) we conclude that for all \( i \in \mathcal{U} \) and \( x \in E_i \)
\[
(e_i \otimes x)(W) \geq 0. \tag{5}
\]
Let
\[
\mathcal{U}_1 = \{i \in \mathcal{U} : \exists x \in E_i : (e_i \otimes x)(W) = 0\}.
\]
Since \( 0 \in P_{V_{k+1}}(A), \mathcal{U}_1 \neq \emptyset \).
We will prove that for any \( i \in \mathcal{U}_1 \) and \( x, y \in E_i \) such that
\[
(e_i \otimes x)(W) = (e_i \otimes y)(W) = 0, \tag{6}
\]
\( x(l) = y(l), \quad l = 1, 2, \ldots, k+1 \).
On the contrary, suppose that (6) does not hold. Let \( x, y \in E_i \) be such that
\[
(e_i \otimes x)(W) = 0, \quad (e_i \otimes y)(W) = 0,
\]
and 
\[ x(l) \neq y(l) \quad \text{for some} \quad l \in \{1, 2, \ldots, k + 1\}. \]

Without loss of generality we can assume 
\[ x(j) = y(j) \quad \text{for} \quad j = 1, 2, \ldots, p, \quad p < k + 1 \]
and 
\[ x(j) = -y(j) \quad \text{for} \quad j = p + 1, p + 2, \ldots, k + 1. \]

Hence

\[ \sum_{j=1}^{p} x(j)w_{ij} = 0, \quad \sum_{j=p+1}^{k+1} x(j)w_{ij} = 0. \]

As 
\[ x(j) = -y(j) \quad \text{for} \quad j = p + 1, p + 2, \ldots, k + 1 \]
we obtain

\[ a_{ij} = 0 \quad \text{for} \quad j = p + 1, p + 2, \ldots, k + 1. \]

By (5),
\[ \sum_{j=p+1}^{k} x(j)w_{ij} - x(k + 1)w_{i,k+1} \geq 0 \]
\[ \sum_{j=p+1}^{k} -x(j)w_{ij} + x(k + 1)w_{i,k+1} \geq 0. \]

Therefore
\[ \sum_{j=p+1}^{k} x(j)w_{ij} = x(k + 1)w_{i,k+1}. \]

By (7), \( x(k + 1)w_{i,k+1} = 0 \). Consequently, \( w_{i,k+1} = 0 \). Hence \( W \in \mathcal{V}_k \). Since \( 0 \in \mathcal{V}'_k \) and \( \mathcal{V}'_k \) is a Chebyshev subspace, (6) is proved.

We will show that there exists \( \alpha_0 > 0 \) such that for every \( \alpha \in (0, \alpha_0] \),

\[ E(A - \alpha W) = \{ e_i \otimes x : i \in \mathcal{U}_1, (e_i \otimes x)(W) = 0, (e_i \otimes x)(A) = \|A\| \}. \]

We first prove that

\[ \sup\{ f(A) : f = e_i \otimes x, i \in \mathcal{U} : f(W) < 0 \} \leq \|A\| - 2 \min\{|a_{ij}| : i \in \mathcal{U}, j \in \{1, 2, \ldots, n\}, a_{ij} \neq 0\}, \]
where $A = [a_{ij}]_{i,j \in \mathbb{N}}$.

Let $i \in \mathcal{U}$, $f = e_i \otimes x$, $f(W) < 0$. Hence there exists $j_0 \in \{1, 2, \ldots, n\}$ satisfying
\[
x(j_0) \neq \text{sgn}(a_{ij_0}) \quad \text{for} \quad a_{ij_0} \neq 0.
\]

Now, we will show
\[
f(A) = \sum_{j=1}^{\infty} x(j)a_{ij} \leq \|A\| - 2|a_{ij_0}|
\]
\[
\leq \|A\| - 2 \min\{|a_{ij}| : i \in \mathcal{U}, j = 1, 2, \ldots, n, \ |a_{ij}| \neq 0\},
\]
and (9) is proved.

We conclude from (9) that there exist $\alpha_0 > 0, b > 0$ such that for every $\alpha \in (0, \alpha_0]$,
\[
f(A - \alpha W) < b < \|A\|,
\]
where $f \in \text{ext } S_{\mathcal{H}^* (c_0, c_0)}$, $f(W) < 0$.

Assume $\alpha_0$ is so small that
\[
\sup_{i \in \mathbb{N} \setminus \mathcal{U}} \|e_i \circ (A - \alpha_0 W)\| < \|A\|.
\]

Consequently, if $f \in E(A - \alpha_0 W)$ then $f = e_i \otimes x$, where $i \in \mathcal{U}_1$ and $f(W) = 0$. Since
\[
\|A - \alpha_0 W\| = \|A\| = \text{dist}(A, \mathcal{Y}_{k+1}),
\]
(8) is proved.

Since $\alpha_0 W \in \mathcal{P}_{\mathcal{Y}_{k+1}}(A)$, we conclude (see [16]) that
\[
\exists 1 \leq q \leq k + 2, \quad \exists \lambda_1, \ldots, \lambda_q > 0, \quad \sum_{m=1}^{q} \lambda_m = 1
\]
such that
\[
\sum_{m=1}^{q} \lambda_m (e_{i_m} \otimes x^{i_m})|_{\mathcal{Y}_{k+1}} = 0,
\]
where $(e_{i_m} \otimes x^{i_m})(A - \alpha_0 W) = \|A - \alpha_0 W\|$. Let $q$ be the smallest number having property (10). By (6), $i_j \neq i_l$ for $j \neq l, j, l \in \{1, 2, \ldots, q\}$. If $q = k + 2$ then (see [18]) $\alpha_0 W$ is the strongly unique best approximation for $A$ in $\mathcal{Y}_{k+1}$, a contradiction. Suppose that $1 \leq q \leq k + 1$. This contradicts (ii).
Let us assume that $\mathcal{H}_k$ is a Chebyshev subspace of $\mathcal{H}(c_0, c_0)$ for every $1 \leq k \leq n$. Suppose that (ii) is false. Consequently, there exist

$$1 \leq k \leq n, \quad 1 \leq j_1 < j_2 < \cdots < j_k \leq n,$$

$$1 \leq i_1 < i_2 < \cdots < i_k,$$

$$x_{ml} \in \mathbb{R} : |x_{ml}| = 1, \quad m, l = 1, 2, \ldots, k$$

satisfying

$$\det[x_{ml}v_{mji}]_{m=1,2,\ldots,k,\quad i=1,2,\ldots,k} = 0.$$

It follows that there exist

$$\lambda_1, \ldots, \lambda_k \in \mathbb{R}, \quad \sum_{m=1}^k |\lambda_m| > 0$$

such that

$$(11) \quad \sum_{m=1}^k \lambda_m (e_{im} \otimes x^{im})|_{\mathcal{H}_k} = 0,$$

where $x^{im} = (x^{im}(1), x^{im}(2), \ldots), x^{im}(l) = x_{ml}$.

Without loss of generality we can assume

$$\lambda_m > 0, \quad m = 1, 2, \ldots, k, \quad \sum_{m=1}^k \lambda_m = 1.$$

We define an operator $B = [b_{ij}]_{i,j \in \mathbb{N}}$ by

$$b_{ij} = \frac{\text{sgn} x^i(j)}{2^j}, \quad i \in \{i_1, i_2, \ldots, i_k\},$$

$$b_{ij} = 0, \quad i \notin \{i_1, i_2, \ldots, i_k\}, \quad j \in \mathbb{N}.$$

Hence $(e_{im} \otimes x^{im})(B) = \|B\|, m = 1, 2, \ldots, k$. By (11), $0 \in \mathcal{P}_{\mathcal{H}_k}(B)$ and

$$\dim \text{span}\{e_{im} \otimes x^{im} | \mathcal{H}_k\} < k,$$

where $\dim \mathcal{H}_k = k$. Therefore there exists $V \in \mathcal{H}_k \setminus \{0\}$ such that

$$(e_{im} \otimes x^{im})(V) = 0, \quad m = 1, 2, \ldots, k.$$

Note that (see the proof of the formula (9))

$$\sup\{f(B) : f = e_{im} \otimes x, \quad m = 1, 2, \ldots, k, \quad f(V) < 0\}$$

$$< \|B\| - \min\{|b_{ij}| : i = i_1, i_2, \ldots, i_k, \quad j = 1, 2, \ldots, n\}.$$
Hence there exist $\alpha_0 > 0$, $b > 0$ such that
\[
f(B - \alpha_0 V) \leq b < \|B\|, \quad f \in \text{ext} \mathcal{K}^*(c_0, c_0), \quad f(V) \leq 0.
\]
Consequently, $\|B - \alpha_0 V\| = \|B\|$, a contradiction. The proof is complete.

**Example 3.** We will construct an $n$-dimensional Chebyshev subspace $V \subset \mathcal{K}(c_0, c_0)$. Let $0 < t_1 < t_2 < \cdots < t_{n-1}$ be such that
\[
\lim_{i \to \infty} \frac{1}{2^i} t_m^i = 0, \quad m = 1, 2, \ldots, n - 1.
\]
Define $V_m = [(v_m)_{ij}]_{i,j \in \mathbb{N}}$ by
\[
(v_m)_{im} = \frac{1}{2^i} t_m^i, \quad (v_m)_{ij} = 0, \quad i \in \mathbb{N}, \quad j \neq m.
\]
Hence $V_m \in \mathcal{K}(c_0, c_0)$ for every $m = 1, 2, \ldots, n - 1$.

Let $\mathcal{V}_{n-1} := \text{lin}\{V_1, V_2, \ldots, V_{n-1}\}$ satisfy the formula (ii) for every $1 \leq k \leq n - 1$.

We will construct an operator $V_n \in \mathcal{K}(c_0, c_0)$ such that $\mathcal{V}_n := \text{lin}\{V_1, V_2, \ldots, V_{n-1}, V_n\}$ satisfies the formula (ii) for every $1 \leq k \leq n$. Our goal is to find $x \in \mathbb{R}$ such that
\[
\lim_{i \to \infty} \frac{1}{2^i} x_i^i = 0
\]
and
\[
W(x, y^1, \ldots, y^k, i_1, \ldots, i_k, m_1, \ldots m_{k-1})
\]
\[
\begin{vmatrix}
  y_1^{1/2^1} t_{m_1}^{i_1} & \cdots & y_1^{k-1/2^1} t_{m_{k-1}}^{i_k} & y_1^{1/2^1} x_1^{i_1} \\
  \vdots & \ddots & \vdots & \vdots \\
  y_k^{1/2^k} t_{m_1}^{i_k} & \cdots & y_k^{k-1/2^k} t_{m_{k-1}}^{i_k} & y_k^{1/2^k} x_1^{i_k}
\end{vmatrix} \neq 0,
\]
where $k \in \{1, 2, \ldots, n\}$, $i_1, i_2, \ldots, i_k \in \mathbb{N}$, $y^1, \ldots, y^k \in \{-1, 1\}^k$, $m_1, m_2, \ldots, m_{k-1} \in \{1, 2, \ldots, n - 1\}$. Since $W(x, y^1, \ldots, y^k, i_1, \ldots, i_k, m_1, \ldots, m_{k-1})$ is not totally equal to zero, we conclude that the set of roots of $W(x, y^1, \ldots, y^k, i_1, \ldots, i_k, m_1, \ldots, m_{k-1})$ is finite for arbitrary but fixed $y^1, \ldots, y^k, i_1, \ldots, i_k, m_1, \ldots, m_{k-1}$. Therefore for all $y^1, \ldots, y^k, i_1, \ldots, i_k, m_1, \ldots, m_{k-1}$ as above, the set of roots of $W(x, y^1, \ldots, y^k, i_1, \ldots, i_k, m_1, \ldots, m_{k-1})$ is countable. Since $\mathbb{R}$ is not countable we see that there exists $x \in \mathbb{R}$ satisfying (12) and (13).
Remark 2. An $n$-dimensional Chebyshev subspace proposed in Example 3 is a non-interpolating subspace of $\mathcal{K}(c_0, c_0)$.

Proof. Let us assume that $\mathcal{V}_n = \text{lin}\{V_1, V_2, \ldots, V_n\}$ is an $n$-dimensional Chebyshev subspace, where $V_m, m = 1, 2, \ldots, n$ are defined in Example 3.

Put $V = \frac{1}{t_1}V_1 - \frac{1}{t_2}V_2$. Note that $V \neq 0$ and $v_{ij} = 0$, $j \geq 3, i \in \mathbb{N}$, where $V = [v_{ij}]_{i,j \in \mathbb{N}}$. It is obvious that there exist $x^1, x^2, \ldots, x^n \in \text{ext } S_{\mathcal{V}}$ such that $x^m(1) = x^m(2) = 1, m = 1, 2, \ldots, n$ and $f_m := e_1 \otimes x^m, m = 1, 2, \ldots, n$ are linearly independent. Note that

$$f_m(V) = 0, \quad m = 1, 2, \ldots, n.$$ 

This completes the proof.

Lemma. Let $X$ be a normed space and let $\mathcal{V}$ be a finite-dimensional subspace of $X$. Let $T \in X$. If $0 \in \mathcal{P}_T \mathcal{V}$ and $0$ is not a strongly unique best approximation for $T$ in $\mathcal{V}$ then

$$\exists V \in \mathcal{V}, \quad V \neq 0: \forall f \in E(T) \quad f(V) \geq 0.$$ 

Proof. Let us assume that

$$\forall V \in \mathcal{V}, \quad V \neq 0, \quad \exists f \in E(T): \quad f(V) < 0.$$ 

Set for any $V \in \mathcal{V}$, $\|V\| = 1$,

$$-r_V = \inf \{f(V) : f \in E(T)\},$$

$$-r = \sup \{-r_V : V \in \mathcal{V}, \|V\| = 1\}.$$ 

We show that $r > 0$. If not, there exists $(V_n) \subset S_{\mathcal{V}}$ such that $-r_{V_n} \geq -\frac{1}{n}$. Since $\mathcal{V}$ is a finite-dimensional subspace, we may assume that $V_n \rightarrow V \in S_{\mathcal{V}}$. Take $f \in E(T), f(V) < 0$. Hence for $n \geq n_0$ there exists $d > 0$ such that

$$-\frac{1}{n} \leq -r_{V_n} \leq f(V_n) < f(V) + d < 0,$$

a contradiction. Therefore

$$\forall V \in \mathcal{V}, \quad V \neq 0, \quad \exists f \in E(T) : \quad f \left( \frac{V}{\|V\|} \right) < -r.$$ 

By the above,

$$\forall V \in \mathcal{V}, \quad V \neq 0, \quad \exists f \in E(T) : \quad f(V) \leq -r\|V\|.$$
Hence 0 is a strongly unique best approximation for $T$, a contradiction. This proves the lemma.

**Theorem 6.** Let $\mathcal{V} \subset \mathcal{K}(c_0, c_0)$ be an $n$-dimensional subspace such that
\[
\forall V \in \mathcal{V}, \forall i \in \mathbb{N} \quad \exists \{j \in \mathbb{N} : v_{ij} \neq 0\} < \infty,
\]
where $V = [v_{ij}]_{i,j \in \mathbb{N}}$ and let $T \in \mathcal{K}(c_0, c_0)$. Then $T$ has a unique best approximation in $\mathcal{V}$ if and only if $T$ has a strongly unique best approximation in $\mathcal{V}$.

**Proof.** Let us assume that 0 is the unique best approximation for $T$ in $\mathcal{V}$. Suppose that 0 is not a strongly unique best approximation. Hence (see Lemma)
\[
\exists V \in \mathcal{V}, V \neq 0 : \forall f \in E(T) \quad f(V) \geq 0,
\]
where $f = e_i \otimes x^i$ for some $x^i \in \text{ext} S_{l^\infty}$.

Put $\mathcal{N} = \{i \in \mathbb{N} : \exists x^i \in \text{ext} S_{l^\infty} : e_i \otimes x^i \in E(T)\}$.

Since $T$ is compact, we conclude that $\#\mathcal{N} < \infty$.

For every $i \in \mathcal{N}$ we set
\[
E_i = \{x^i \in \text{ext} S_{l^\infty} : (e_i \otimes x^i)(T) = \|T\|\}.
\]
Let $i \in \mathbb{N} \setminus \mathcal{N}$. Hence there exists $b > 0$ such that
\[
(e_i \otimes x)(T) < b < \|T\|, \quad x \in \text{ext} S_{l^\infty}.
\]
Consequently, there exists $\alpha_0 > 0$ such that for every $\alpha \in (0, \alpha_0]$,
\[
| (e_i \otimes x)(T - \alpha V) | < b.
\]
Therefore
\[
\sup_{i \in \mathbb{N} \setminus \mathcal{N}} | (e_i \otimes x)(T - \alpha V) | \leq b < \|T\|.
\]
Let $i \in \mathcal{N}$ and let $x^i \notin E_i$. From this we conclude that there exists $j_0 \in \mathbb{N}$ such that
\[
\text{sgn} x^i(j_0) \neq \text{sgn}(t_{ij_0}), \quad t_{ij_0} \neq 0,
\]
where $T = [t_{ij}]_{i,j \in \mathbb{N}}$.

Set $J = \{j \in \mathbb{N} : v_{ij} \neq 0\}$. If $\text{sgn} x^i(j) = \text{sgn}(t_{ij})$ for any $j \in J$, then there exists $y^i \in E_i$ such that
\[
(e_i \otimes y^i)(T) = \|T\|, \quad (e_i \otimes y^i)(V) = (e_i \otimes x^i)(V).
\]
By the above,
\[(e_i \otimes x^i)(T - \alpha V) \leq \|T\| - (e_i \otimes y^i)(\alpha V) \leq \|T\| .\]

Let \(\text{sgn} x^i(j_0) \neq \text{sgn}(t_{ij_0})\) for some \(j_0 \in J\), where \(t_{ij_0} \neq 0\). Since \(J\) is finite, there exists \(\alpha_0 > 0\) such that
\[
\|\alpha_0 V\| < \min\{|t_{ij}| : j \in J, t_{ij} \neq 0\}.
\]

Let \(\alpha \in (0, \alpha_0]\). Hence
\[
(e_i \otimes x^i)(T - \alpha V) = \sum_{j \in J} x^i(j)(t_{ij} - \alpha v_{ij}) + \sum_{j \notin J} x^i(j)(t_{ij} - \alpha v_{ij}) \\
\leq \sum_{j \in J} |t_{ij}| - 2|t_{ij_0}| + \sum_{j \notin J} |t_{ij}| + \alpha \|V\| \\
= \|T\| + \alpha \|V\| - 2|t_{ij_0}| < \|T\|.
\]

Finally,
\[
\|T - \alpha V\| = f(T - \alpha V),
\]
where \(f = e_i \otimes x^i, i \in \mathcal{N}, x^i \in E_i\). Hence
\[
\|T - \alpha V\| = f(T - \alpha V) \leq \|T\|.
\]

The proof is complete.

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