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CONVERGENCE IN CAPACITY AND APPLICATIONS

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Abstract

In this article we prove that if $u_j, v_j, w \in \mathscr{E}(\Omega)$ such that $u_j, v_j \geq w, \forall\, j \geq 1$, and $|u_j - v_j| \to 0$ in C_n-capacity, then $\lim_{j\to\infty} h(\varphi_1, \dots, \varphi_m)[(dd^c u_j)^n - (dd^c v_j)^n] = 0$ in the weak-topology of measures for all $\varphi_1, \dots, \varphi_m \in \mathrm{PSH} \cap L^\infty_{\mathrm{loc}}(\Omega)$, $h \in C(\mathbf{R}^m)$. We shall then use this result to give some applications.

1. Introduction

Let Ω be a bounded hyperconvex domain in $\mathbf{C}^n$, by $\mathrm{PSH}(\Omega)$ we denote the set of plurisubharmonic (psh) functions on Ω, and by $\mathrm{PSH}^-(\Omega)$ the subclass of negative functions. The complex Monge-Ampère operator $(dd^c)^n$ is well defined over the class of locally bounded psh functions, according to the fundamental work of Bedford and Taylor in [3], [4]. In [9], Demailly generalized the work of Bedford and Taylor for the class of locally psh functions with bounded values near the boundary. In [7], Cegrell then introduced a general class $\mathscr{E}(\Omega)$ of psh functions on which the complex Monge-Ampère operator can be defined. The aim of the present paper is to study the convergence within the class $\mathscr{E}(\Omega)$.

In Section 2, we introduce some definitions, and known results that are needed for our paper. Our main result is the following theorem.

THEOREM 3.1. *Let $u_j, v_j, w \in \mathscr{E}(\Omega)$ be such that $u_j, v_j \geq w$, $\forall\, j \geq 1$. Assume that $|u_j - v_j| \to 0$ in C_n-capacity. Then $\lim_{j\to\infty} \varphi[(dd^c u_j)^n - (dd^c v_j)^n] = 0$ in the weak-topology of measures for all $\varphi \in \mathrm{PSH} \cap L^\infty_{loc}(\Omega)$.*

Theorem 3.1 is a generalization of Theorem 1.1 in [8]. As an application we obtain in Theorem 3.3 that if $u_j, u, v \in \mathscr{E}(\Omega)$ such that $u_j \geq v$, $\forall\, j \geq 1$, $\overline{\lim}_{j\to\infty} u_j \leq u$ and $(dd^c u_j)^n \to \mu$ in the weak-topology of measures then

$$\mu \geq 1_{\{u=-\infty\}}(dd^c u)^n,$$

in the weak sense of measures.

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2. Preliminaries

First we recall some elements of pluripotential theory that will be used throughout the paper. All this can be found in [1], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [16], [17], [18], [19], and [20].

NOTATION 2.1. Unless otherwise specified, Ω will be a bounded hyperconvex domain in $\mathbb{C}^n$ meaning that there exists a negative exhaustive psh function for Ω.

DEFINITION 2.2. The C_n-capacity in the sense of Bedford and Taylor on Ω is the set function given by

$$C_n(E) = C_n(E, \Omega) = \sup\left\{\int_E (dd^c\varphi)^n : \varphi \in \mathrm{PSH}(\Omega), -1 \le \varphi \le 0\right\}$$

for every Borel set E in Ω. It is known [4] that

$$C_n(E) = \int_\Omega (dd^c h^*_{E,\Omega})^n.$$

where $h^*_{E,\Omega}$ is the relative extremal psh function for E (relative to Ω) defined as the smallest upper semicontinuous majorant of $h_{E,\Omega}$

$$h_{E,\Omega}(z) = \sup\{\varphi(z) : \varphi \in \mathrm{PSH}^-(\Omega),\ \varphi \le -1 \text{ on } E\}.$$

The following definition was introduced in [19]: A sequence $u_j \in \mathrm{PSH}^-(\Omega)$ converges to u in C_n−capacity if

$$C_n(K \cap \{|u_j - u| > \delta\}) \to 0 \text{ as } j \to +\infty,\ \forall\, K \subset\subset \Omega,\ \forall\, \delta > 0.$$

A family of positive measures $\{\mu_\alpha\}$ on Ω is said to be uniformly absolutely continuous with respect to C_n-capacity in a set $E \subset \Omega$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that for each Borel subset $F \subset E$ with $C_n(F) < \delta$ the inequality $\mu_\alpha(F) < \epsilon$ holds for all α. We write $\mu_\alpha \ll C_n$ in E uniformly for α.

DEFINITION 2.3. The following classes of psh functions were introduced by Cegrell in [5], [6] and [7]:

$$\mathcal{E}_0 = \mathcal{E}_0(\Omega) = \left\{\varphi \in \mathrm{PSH}^-(\Omega) \cap L^\infty(\Omega) : \lim_{z\to\partial\Omega} \varphi(z) = 0, \int_\Omega (dd^c\varphi)^n < +\infty\right\},$$

$$\mathcal{F} = \mathcal{F}(\Omega) = \left\{\varphi \in \mathrm{PSH}^-(\Omega) : \exists\, \mathcal{E}_0(\Omega) \ni \varphi_j \searrow \varphi, \sup_{j\geq 1} \int_\Omega (dd^c\varphi_j)^n < +\infty\right\},$$

$$\mathcal{E} = \mathcal{E}(\Omega) = \{\varphi \in \mathrm{PSH}^-(\Omega) : \exists\, \varphi_K \in \mathcal{F}(\Omega) \text{ such that } \varphi_K = \varphi \text{ on } K,\ \forall K \subset\subset \Omega\},$$

$$\delta\,\mathrm{PSH} \cap L^\infty(\Omega) = \{\varphi - \psi : \varphi, \psi \in \mathrm{PSH} \cap L^\infty(\Omega)\},$$

$$\delta\mathcal{E}_0(\Omega) = \{\varphi - \psi : \varphi, \psi \in \mathcal{E}_0(\Omega)\}.$$

Next we introduce some results needed for our work.

PROPOSITION 2.4. i) *Let* $\varphi, \psi \in \delta\,\mathrm{PSH} \cap L^\infty(\Omega)$. *Then* $\varphi\psi \in \delta\,\mathrm{PSH} \cap L^\infty(\Omega)$.

ii) *Let* $\varphi, \psi \in \delta\mathcal{E}_0(\Omega)$. *Then* $\varphi\psi \in \delta\mathcal{E}_0(\Omega)$.

PROOF. i) Without loss of generality we can assume that $\varphi, \psi \in \mathrm{PSH} \cap L^\infty(\Omega)$. Set $c = \sup\{|\varphi(z)| + |\psi(z)| : z \in \Omega\}$. We have

$$\varphi\psi = \frac{1}{2}[(\varphi + \psi + c)^2 - (\varphi + c)^2 - (\psi + c)^2 + c^2].$$

Hence $\varphi\psi \in \delta\,\mathrm{PSH} \cap L^\infty(\Omega)$.

ii) Without loss of generality we can assume that $\varphi, \psi \in \delta\mathcal{E}_0(\Omega)$. Set $c = \sup\{|\varphi(z)| + |\psi(z)| : z \in \Omega\}$. We prove that $(\varphi + c)^2 - c^2 \in \mathcal{E}_0(\Omega)$. Indeed, we have $dd^c[(\varphi + c)^2 - c^2] = 2[(\varphi + c)dd^c\varphi + d\varphi \wedge d^c\varphi] \geq 0$. Thus

$$\begin{aligned}
&\int_\Omega (dd^c[(\varphi + c)^2 - c^2])^n \\
&\quad= \int_\Omega 2^n[(\varphi + c)^n(dd^c\varphi)^n + n(\varphi + c)^{n-1}d\varphi \wedge d^c\varphi \wedge (dd^c\varphi)^{n-1}] \\
&\quad\leq \int_\Omega 2^n c^n (dd^c\varphi)^n + 2^n n c^{n-1} \int_\Omega d\varphi \wedge d^c\varphi \wedge (dd^c\varphi)^{n-1} \\
&\quad= \int_\Omega 2^n c^n (dd^c\varphi)^n + 2^n n c^{n-1} \int_\Omega -\varphi(dd^c\varphi)^n \\
&\quad\leq \int_\Omega 2^n c^n (n+1)(dd^c\varphi)^n < +\infty.
\end{aligned}$$

Moreover, since $\lim_{z\to\partial\Omega}[(\varphi + c)^2 - c^2] = 0$ we get $(\varphi + c)^2 - c^2 \in \mathcal{E}_0(\Omega)$. On the other hand, we have

$$\varphi\psi = \frac{(\varphi + \psi + c)^2 - c^2}{2} - \frac{(\varphi + c)^2 - c^2}{2} - \frac{(\psi + c)^2 - c^2}{2},$$

Hence $\varphi\psi \in \delta\mathcal{E}_0(\Omega)$.

PROPOSITION 2.5. *Let* $u_1, \dots, u_n \in \mathcal{F}(\Omega)$ *be such that* $u_1, \dots, u_k \geq -1$. *Then*

$$\int_B dd^c u_1 \wedge \cdots \wedge dd^c u_n \leq \left[\int_\Omega (dd^c u_{k+1})^n\right]^{\frac{1}{n}} \dots \left[\int_\Omega (dd^c u_n)^n\right]^{\frac{1}{n}} C_n(B)^{\frac{k}{n}},$$

for all Borel sets $B \subset \Omega$.

PROOF. For each $u \in \mathcal{E}(\Omega)$ we set

$$h_{B,u}(z) = \sup\{\varphi(z) : \varphi \in \mathrm{PSH}^-(\Omega), \varphi \leq u \text{ on } B\}.$$

For each open set $B \subset\subset \Omega$, by Corollary 5.6 in [7] we get

$$\begin{aligned}
&\int_B dd^c u_1 \wedge \cdots \wedge dd^c u_n \\
&\quad = \int_B dd^c h_{B,u_1} \wedge \cdots \wedge dd^c h_{B,u_k} \wedge dd^c u_{k+1} \wedge \cdots dd^c u_n \\
&\quad \leq \int_\Omega dd^c h_{B,u_1} \wedge \cdots \wedge dd^c h_{B,u_k} \wedge dd^c u_{k+1} \wedge \cdots \wedge dd^c u_n \\
&\quad \leq \int_\Omega (dd^c h_B)^k \wedge dd^c u_{k+1} \wedge \cdots \wedge dd^c u_n \\
&\quad \leq \left[\int_\Omega (dd^c h_B)^n\right]^{\frac{k}{n}} \left[\int_\Omega (dd^c u_{k+1})^n\right]^{\frac{1}{n}} \dots \left[\int_\Omega (dd^c u_n)^n\right]^{\frac{1}{n}} \\
&\quad \leq \left[\int_\Omega (dd^c u_{k+1})^n\right]^{\frac{1}{n}} \dots \left[\int_\Omega (dd^c u_n)^n\right]^{\frac{1}{n}} [C_n(B)]^{\frac{k}{n}}.
\end{aligned}$$

Hence

$$\int_B dd^c u_1 \wedge \cdots \wedge dd^c u_n \leq \left[\int_\Omega (dd^c u_{k+1})^n\right]^{\frac{1}{n}} \dots \left[\int_\Omega (dd^c u_n)^n\right]^{\frac{1}{n}} [C_n(B)]^{\frac{k}{n}}.$$

for all Borel sets $B \subset \Omega$.

PROPOSITION 2.6. *Let* $u, v, w_1, \dots, w_{n-1} \in \mathcal{F}(\Omega)$, $s > 0$ *be such that* $u \geq v$. *Then*

$$\int_\Omega [\max(v, -s) - v] dd^c u \wedge T \leq \int_\Omega [\max(2v, -s) - 2v] dd^c v \wedge T,$$

where $T = dd^c w_1 \wedge \cdots \wedge dd^c w_{n-1}$.

Proof. By Stokes formula and Theorem 4.1 in [15] we get

$$\begin{aligned}\int_\Omega [\max(v,-s)-v]dd^c u\wedge T &= \int_\Omega -udd^c[v-\max(v,-s)]\wedge T\\ &= \int_{\{v\le -s\}} -udd^c[v-\max(v,-s)]\wedge T\\ &\le \int_{\{v\le -s\}} -udd^c v\wedge T\\ &\le \int_{\{v\le -s\}} -vdd^c v\wedge T\\ &\le \int_{\{v\le -s\}} [\max(2v,-s)-2v]dd^c v\wedge T\\ &\le \int_\Omega [\max(2v,-s)-2v]dd^c v\wedge T.\end{aligned}$$

We generalize Theorem 3.1 in [13]:

Proposition 2.7. *Let* $\mathcal{E}(\Omega)\ni u_j\to u\in \mathrm{PSH}^-(\Omega)$ *in* C_n*-capacity. Assume that* $(dd^c u_j)^n\to\mu$ *in the weak-topology of measures. Then*

$$\mu\ge 1_{\{u>v\}}(dd^c\max(u,v))^n,$$

in the weak sense of measures for all $v\in \mathrm{PSH}^-\cap L^\infty(\Omega)$*, where* 1_E *is the characteristic function for the set* E.

Proof. By Theorem 4.1 in [15] we get

$$\begin{aligned}(dd^c u_j)^n &\ge 1_{\{u_j>v-1\}}(dd^c u_j)^n\\ &= 1_{\{u_j>v-1\}}(dd^c\max(u_j,v-1))^n\\ &= (dd^c\max(u_j,v-1))^n - 1_{\{u_j\le v-1\}}(dd^c\max(u_j,v-1))^n\\ &\ge (dd^c\max(u_j,v-1))^n - 1_{\{u_j\le v-1\}\cap\{|u_j-u|\le 1\}}(dd^c\max(u_j,v-1))^n\\ &\qquad - 1_{\{|u_j-u|>1\}}(dd^c\max(u_j,v-1))^n\\ &\ge (dd^c\max(u_j,v-1))^n - 1_{\{u\le v\}}(dd^c\max(u_j,v-1))^n\\ &\qquad - 1_{\{|u_j-u|>1\}}(dd^c\max(u_j,v-1))^n.\end{aligned}$$

From the quasicontinuity of u and v (Theorem 3.5 in [4]) and from $(dd^c\max(u_j,v-1))^n\ll C_n$ in Ω uniformly for j we get

$$\varlimsup_{j\to\infty} 1_{\{u\le v\}}(dd^c\max(u_j,v-1))^n\le 1_{\{u\le v\}}(dd^c\max(u,v-1))^n.$$

On the other hand, since $u_j \to u$ in C_n-capacity and $(dd^c \max(u_j, v-1))^n \ll C_n$ in Ω uniformly for j we get

$$\lim_{j\to\infty} 1_{\{|u_j-u|>1\}}(dd^c \max(u_j, v-1))^n = 0.$$

Moreover, by Theorem 1.1 in [8] and Theorem 4.1 in [15] we obtain

$$\begin{aligned}\mu &\geq (dd^c \max(u, v-1))^n - 1_{\{u\leq v\}}(dd^c \max(u, v-1))^n \\ &= 1_{\{u>v\}}(dd^c \max(u, v-1))^n \\ &= 1_{\{u>v\}}(dd^c \max(u, v))^n.\end{aligned}$$

Let X be a compact Kähler manifold with a fundamental form $\omega = \omega_X$ such that $\int_X \omega^n = 1$. An upper semicontinuous function $\varphi : X \to [-\infty, +\infty)$ is called ω-plurisubharmonic (ω-psh) if $\varphi \in L^1(X)$ and $\omega + dd^c\varphi \geq 0$. In [18] Kołodziej introduced the capacity $C_{X,\omega}$ on X by

$$C_X(E) = C_{X,\omega}(E) = \sup\left\{\int_E \omega_\varphi^n : \varphi \in \mathrm{PSH}(X, \omega), -1 \leq \varphi \leq 0\right\},$$

where $\omega_\varphi^n = (\omega + dd^c\varphi)^n$ and $n = \dim X$. In [10] Guedj and Zeriahi proved that C_X is a Choquet capacity on X and

$$C_X(E) = \int_X (-h^*_{E,\omega})\omega^n_{h^*_{E,\omega}}.$$

where $h^*_{E,\omega}$ denotes the upper semicontinuous regularization of $h_{E,\omega}$ given by

$$h_{E,\omega}(z) = \sup\{\varphi(z) : \varphi \in \mathrm{PSH}^-(X, \omega), \varphi|_E \leq -1\}.$$

In [11] they introduced the new class of ω-psh functions

$$\mathscr{E}(X, \omega) = \left\{\varphi \in \mathrm{PSH}(X, \omega) : \lim_{j\to\infty} \int_{\{\varphi>-j\}} \omega^n_{\max(\varphi,-j)} = 1\right\}.$$

They proved that the complex Monge-Ampère is well defined on $\mathscr{E}(X, \omega)$ by

$$\omega_\varphi^n = \lim_{j\to\infty} 1_{\{\varphi>-j\}}\omega^n_{\max(\varphi,-j)}.$$

From the proof of Proposition 2.7 we give a simple proof of Theorem 1.9 in [11] (see [20]):

PROPOSITION 2.8. *Let $\mathscr{E}(X, \omega) \ni u_j \to u \in \mathscr{E}(X, \omega)$ in C_X-capacity. Then $\omega^n_{u_j} \to \omega^n_u$ in the weak-topology of measures.*

Proof. Without loss of generality we can assume that $\omega_{u_j}^n \to \mu$ in the weak-topology of measures. We have

$$\begin{aligned}
\omega_{u_j}^n &\geq 1_{\{u_j>-k-1\}}\omega_{\max(u_j,-k-1)}^n \\
&= \omega_{\max(u_j,-k-1)}^n - 1_{\{u_j\leq -k-1\}}\omega_{\max(u_j,-k-1)}^n \\
&\geq \omega_{\max(u_j,-k-1)}^n - 1_{\{u_j\leq -k-1\}\cap\{|u_j-u|\leq 1\}}\omega_{\max(u_j,-k-1)}^n \\
&\qquad - 1_{\{|u_j-u|>1\}}\omega_{\max(u_j,-k-1)}^n \\
&\geq \omega_{\max(u_j,-k-1)}^n - 1_{\{u\leq -k\}}\omega_{\max(u_j,-k-1)}^n - 1_{\{|u_j-u|>1\}}\omega_{\max(u_j,-k-1)}^n.
\end{aligned}$$

From the quasicontinuity of u (Corollary 3.8 in [10]) and from $\omega_{\max(u_j,-k-1)}^n \ll C_n$ in Ω uniformly for j we get

$$\overline{\lim_{j\to\infty}}\, 1_{\{u\leq -k\}}\omega_{\max(u_j,-k-1)}^n \leq 1_{\{u\leq -k\}}\omega_{\max(u,-k-1)}^n.$$

On the other hand, since $u_j \to u$ in C_n-capacity and $\omega_{\max(u_j,-k-1)}^n \ll C_n$ in Ω uniformly for j we get

$$\lim_{j\to\infty} 1_{\{|u_j-u|>1\}}\omega_{\max(u_j,-k-1)}^n = 0.$$

Moreover, by Theorem 1.1 in [8] we obtain

$$\begin{aligned}
\mu \geq \omega_{\max(u,-k-1)}^n - 1_{\{u\leq -k\}}\omega_{\max(u,-k-1)}^n &= 1_{\{u>-k\}}\omega_{\max(u,-k-1)}^n \\
&= 1_{\{u>-k\}}\omega_{\max(u,-k)}^n,
\end{aligned}$$

Letting $k \to \infty$ we obtain $\mu \geq \omega_u^n$. Moreover, we have $\mu(X) = \omega_u^n(X) = 1$. Hence $\mu = \omega_u^n$.

3. Convergence in capacity

We start with the first result which is a generalization of Theorem 1.1 in [8].

Theorem 3.1. *Let $u_j, v_j, w \in \mathscr{E}(\Omega)$ be such that $u_j, v_j \geq w$, $\forall\, j \geq 1$. Assume that $|u_j - v_j| \to 0$ in C_n-capacity. Then $\lim_{j\to\infty} \varphi[(dd^c u_j)^n - (dd^c v_j)^n] = 0$ in the weak-topology of measures for all $\varphi \in \mathrm{PSH} \cap L^\infty_{\mathrm{loc}}(\Omega)$.*

Proof. As in the proof of Proposition 2.5 in [13] we can assume that $w \in \mathscr{F}(\Omega)$ and $\varphi \in \mathscr{E}_0(\Omega)$. We prove that

$$\lim_{j\to\infty} \int_\Omega \varphi[(dd^c u_j)^n - (dd^c v_j)^n] = 0,$$

for all $\varphi \in \mathscr{E}_0(\Omega)$. We can assume that $\varphi \geq -1$. For each $s > 0$ we have

$$\int_\Omega \varphi[(dd^c u_j)^n - (dd^c v_j)^n] = A_{js} + B_{js} + C_{js},$$

where

$$\begin{aligned}
A_{js} &= \int_\Omega \varphi[(dd^c u_j)^n - (dd^c \max(u_j, -s))^n],\\
B_{js} &= \int_\Omega \varphi[(dd^c \max(v_j, -s))^n - (dd^c v_j)^n],\\
C_{js} &= \int_\Omega \varphi[(dd^c \max(u_j, -s))^n - (dd^c \max(v_j, -s))^n].
\end{aligned}$$

By Stokes formula and Proposition 2.6 we get

$$\begin{aligned}
|A_{js}| &= \Bigg| \int_\Omega \varphi \Big[dd^c u_j - dd^c \max(u_j, -s) \Big] \\
&\qquad \wedge \Big[\sum_{k=0}^{n-1} (dd^c u_j)^k \wedge (dd^c \max(u_j, -s))^{n-1-k} \Big] \Bigg| \\
&= \int_\Omega [\max(u_j, -s) - u_j] dd^c \varphi \\
&\qquad \wedge \Big[\sum_{k=0}^{n-1} (dd^c u_j)^k \wedge (dd^c \max(u_j, -s))^{n-1-k} \Big] \\
&\leq \sum_{k=0}^{n-1} \int_\Omega [\max(w, -s) - w] dd^c \varphi \wedge (dd^c u_j)^k \wedge (dd^c \max(u_j, -s))^{n-1-k} \\
&\leq \sum_{k=0}^{n-1} \int_\Omega [\max(2w, -s) - 2w] dd^c \varphi \wedge dd^c w \\
&\qquad \wedge (dd^c u_j)^{k-1} \wedge (dd^c \max(u_j, -s))^{n-1-k} \\
&\leq \cdots \\
&\leq n \int_\Omega [\max(2^{n-1} w, -s) - 2^{n-1} w] dd^c \varphi \wedge (dd^c w)^{n-1} \\
&= n \int_\Omega \varphi dd^c [\max(2^{n-1} w, -s) - 2^{n-1} w] \wedge (dd^c w)^{n-1}.
\end{aligned}$$

Similarly we have

$$|B_{js}| \leq n \int_\Omega \varphi dd^c[\max(2^{n-1}w, -s) - 2^{n-1}w] \wedge (dd^c w)^{n-1}.$$

To simplify the notation we set

$$\begin{aligned}\varphi_\epsilon &= \max(\varphi, -\epsilon),\\ T_{js} &= \sum_{k=0}^{n-1}(dd^c \max(u_j, -s))^k \wedge (dd^c \max(v_j, -s))^{n-1-k}.\end{aligned}$$

Next by Stokes formula and Proposition 2.5 we get

$$\begin{aligned}
&|C_{js}|\\
&\quad= \left|\int_\Omega [\max(u_j, -s) - \max(v_j, -s)] \wedge dd^c\varphi \wedge T_{js}\right|\\
&\quad\leq \int_\Omega |\max(u_j, -s) - \max(v_j, -s)| dd^c\varphi \wedge T_{js}\\
&\quad\leq \int_\Omega \epsilon dd^c\varphi \wedge T_{js} + \int_{\{|u_j - v_j| > \epsilon\}} |\max(u_j, -s) - \max(v_j, -s)| dd^c\varphi \wedge T_{js}\\
&\quad\leq n\epsilon \int_\Omega dd^c\varphi \wedge (dd^c w)^{n-1}\\
&\qquad+ \int_{\{|u_j - v_j| > \epsilon\}} |\max(u_j, -s) - \max(v_j, -s)| [dd^c\varphi - dd^c\varphi_\epsilon] \wedge T_{js}\\
&\qquad+ \int_{\{|u_j - v_j| > \epsilon\}} |\max(u_j, -s) - \max(v_j, -s)| dd^c\varphi_\epsilon \wedge T_{js}\\
&\quad\leq n\epsilon \int_\Omega dd^c\varphi \wedge (dd^c w)^{n-1} + \int_{\{|u_j - v_j| > \epsilon\} \cap \{\varphi \leq -\epsilon\}} 2s[dd^c\varphi + dd^c\varphi_\epsilon] \wedge T_{js}\\
&\qquad+ \int_\Omega -2w dd^c\varphi_\epsilon \wedge T_{js}\\
&\quad\leq n\epsilon \int_\Omega dd^c\varphi \wedge (dd^c w)^{n-1}\\
&\qquad+ 4sn\left[\int_\Omega (dd^c w)^n\right]^{\frac{n-1}{n}} \left[C_n(\{|u_j - v_j| > \epsilon\} \cap \{\varphi \leq -\epsilon\})\right]^{\frac{1}{n}}\\
&\qquad+ 2n \int_\Omega -w dd^c\varphi_\epsilon \wedge (dd^c w)^{n-1}
\end{aligned}$$

$$
\begin{aligned}
&= n\epsilon \int_\Omega dd^c\varphi \wedge (dd^c w)^{n-1} \\
&\quad + 4sn\left[\int_\Omega (dd^c w)^n\right]^{\frac{n-1}{n}} \left[C_n(\{|u_j - v_j| > \epsilon\} \cap \{\varphi \le -\epsilon\})\right]^{\frac{1}{n}} \\
&\quad + 2n \int_\Omega -\varphi_\epsilon (dd^c w)^n \\
&\le n\epsilon \int_\Omega dd^c\varphi \wedge (dd^c w)^{n-1} \\
&\quad + 4sn\left[\int_\Omega (dd^c w)^n\right]^{\frac{n-1}{n}} \left[C_n(\{|u_j - v_j| > \epsilon\} \cap \{\varphi \le -\epsilon\})\right]^{\frac{1}{n}} \\
&\quad + 2n\epsilon \int_\Omega (dd^c w)^n.
\end{aligned}
$$

Hence

$$
\overline{\lim_{j\to\infty}}\, C_{js} \le n\epsilon \int_\Omega dd^c\varphi \wedge (dd^c w)^{n-1} + 2n\epsilon \int_\Omega (dd^c w)^n,
$$

for all $\epsilon > 0$. Letting $\epsilon \to 0$ we have

$$
\lim_{j\to\infty} C_{js} = 0.
$$

Combining these inequalities we obtain

$$
\begin{aligned}
&\overline{\lim_{j\to\infty}} \left| \int_\Omega \varphi[(dd^c u_j)^n - (dd^c v_j)^n] \right| \\
&\qquad\qquad \le 2n \int_\Omega \varphi dd^c[\max(2^{n-1}w, -s) - 2^{n-1}w] \wedge (dd^c w)^{n-1},
\end{aligned}
$$

for all $s > 0$. Letting $s \to \infty$ by Proposition 5.1 in [7] we have

$$
\lim_{j\to\infty} \int_\Omega \varphi[(dd^c u_j)^n - (dd^c v_j)^n] = 0.
$$

Moreover, from $C_0^\infty(\Omega) \subset \delta\mathcal{E}_0(\Omega)$ and from Proposition 2.4 ii) we obtain

$$
\lim_{j\to\infty} \int_\Omega f\varphi[(dd^c u_j)^n - (dd^c v_j)^n] = 0,
$$

for all $f \in C_0^\infty(\Omega)$.

COROLLARY 3.2. *Let $u_j, v_j, w \in \mathcal{E}(\Omega)$ be such that $u_j, v_j \ge w, \forall\, j \ge 1$. Assume that $|u_j - v_j| \to 0$ in C_n-capacity. Then* $\lim_{j\to\infty} h(\varphi_1, \dots, \varphi_m)[(dd^c u_j)^n$

$-(dd^c v_j)^n] = 0$ in the weak-topology of measures for all $\varphi_1, \dots, \varphi_m \in$ PSH $\cap L^\infty_{loc}(\Omega)$, $h \in C(\mathbb{R}^m)$.

PROOF. As in the proof of Proposition 2.5 in [13] we can assume that $w \in \mathscr{F}(\Omega)$ and $\varphi_1, \dots, \varphi_m \in$ PSH $\cap L^\infty(\Omega)$. Set $A = \sup\{\max(|\varphi_1(z)|, \dots, |\varphi_m(z)|) : z \in \Omega\}$. We choose a sequence of polynomials P_j such that

$$\lim_{j\to\infty} \sup\{|P_j(x) - h(x)| : x \in [-A, A]^m\} = 0.$$

On the other hand, by Proposition 2.4 i) we have $P_j(\varphi_1, \dots, \varphi_m) \in \delta\,\mathrm{PSH} \cap L^\infty(\Omega)$. Moreover, by Theorem 3.1 we obtain $\lim_{j\to\infty} h(\varphi_1, \dots, \varphi_m)[(dd^c u_j)^n - (dd^c v_j)^n] = 0$.

By using Theorem 3.1 we also obtain the following application.

THEOREM 3.3. *Let $u_j, u, v \in \mathscr{E}(\Omega)$ be such that $u_j \geq v$, $\forall\, j \geq 1$ and $\overline{\lim}_{j\to\infty} u_j \leq u$. Assume that the sequence of measures $(dd^c u_j)^n$ has a limit point μ in the weak-topology of measures. Then $\mu \geq 1_{\{u=-\infty\}}(dd^c u)^n$ in the weak sense of measures.*

We need:

LEMMA 3.4. *Let $u \in \mathscr{E}(\Omega)$ and a compact subset K in $\{u = -\infty\}$. Given open sets $D_j \subset\subset \Omega$ such that $\bar{D}_j \searrow K$. Then there exist $\varphi_j \in \mathscr{E}_0 \cap C(\Omega)$ such that $\varphi_j \searrow u$ and $\lim_{j\to\infty} 1_{D_j}(dd^c\varphi_j)^n = 1_K(dd^c u)^n$ in the weak-topology of measures.*

PROOF. By Theorem 2.1 in [7] we can find

$$\mathscr{E}_0 \cap C(\Omega) \ni \psi_j \searrow u.$$

Since $(dd^c\psi_j)^n \to (dd^c u)^n$ as $j \to \infty$ we can find a increasing sequence $\{k_j\}$ such that

$$\int_{D_j} (dd^c\psi_{k_j})^n \geq \int_{D_j} (dd^c u)^n - \frac{1}{j}.$$

Set $\varphi_j = \psi_{k_j}$. We have

$$\lim_{j\to\infty} 1_{D_j}(dd^c\varphi_j)^n \leq \lim_{j\to\infty} 1_{\bar{D}_k}(dd^c\varphi_j)^n = 1_{\bar{D}_k}(dd^c u)^n$$

for all $k \geq 1$. Letting $k \to \infty$ we get

$$\lim_{j\to\infty} 1_{D_j}(dd^c\varphi_j)^n \leq 1_K(dd^c u)^n.$$

On the other hand, we have

$$\begin{aligned}\varliminf_{j\to\infty}\int_\Omega 1_{D_j}(dd^c\varphi_j)^n &= \varliminf_{j\to\infty}\int_{D_j}(dd^c\psi_{k_j})^n \geq \varliminf_{j\to\infty}\left[\int_{D_j}(dd^c u)^n - \frac{1}{j}\right]\\ &= \int_\Omega 1_K(dd^c u)^n.\end{aligned}$$

Therefore $\lim_{j\to\infty} 1_{D_j}(dd^c\varphi_j)^n = 1_K(dd^c u)^n$.

PROOF OF THEOREM 3.3. Let K be a compact subset in $\{u = -\infty\}$. Given $f \in C_0^\infty(\Omega)$, $f \geq 0$. We only have to prove that

$$\int_\Omega f\,d\mu \geq \int_\Omega f 1_K(dd^c u)^n.$$

We choose open sets $D_j \subset\subset \Omega$ such that $D_j \searrow K$. By Lemma 3.3 we can find $\varphi_j \in \mathscr{E}_0(\Omega)\cap C(\Omega)$ such that $\varphi_j \searrow u$ and $\lim_{j\to\infty} 1_{D_j}(dd^c\varphi_j)^n = 1_K(dd^c u)^n$. Since $\sup\{\varphi_j(z) : z \in D_j\} \searrow -\infty$ we can assume that

$$\sup\{\varphi_j(z) : z \in D_j\} < -j(j+1).$$

By [14] we can choose a increasing $\{k_j\}$ such that $u_{k_j} \leq \varphi_j + 1$ on $\bar{D}_j$. Set

$$v_j = \max\left(u_{k_j}, \left(1 - \frac{1}{j}\right)\varphi_j - j\right).$$

Since $u_{k_j} \leq \varphi_j + 1 < \left(1 - \frac{1}{j}\right)\varphi_j - j$ on D_j we get $v_j = \left(1 - \frac{1}{j}\right)\varphi_j - j$ on D_j. This implies that

$$\begin{aligned}\varliminf_{j\to\infty}\int_\Omega f(dd^c v_j)^n &\geq \varliminf_{j\to\infty}\left(1 - \frac{1}{j}\right)^n\int_\Omega f 1_{D_j}(dd^c\varphi_j)^n\\ &= \int_\Omega f 1_K(dd^c u)^n\end{aligned}\tag{1}$$

Since $\{|u_{k_j} - v_j| \neq 0\} \subset \{u_{k_j} < -j\} \subset \{v < -j\}$ we get $u_{k_j} - v_j \to 0$ in C_n-capacity. Moreover, by Theorem 3.1 we have

$$\lim_{j\to\infty}\int_\Omega f(dd^c u_{k_j})^n = \lim_{j\to\infty}\int_\Omega f(dd^c v_j)^n.\tag{2}$$

From (1) and (2), letting $j \to \infty$ we get

$$\int_\Omega f\mu \geq \int_\Omega f 1_K(dd^c u)^n.$$

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