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C^*-ALGEBRA RELATIONS

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Abstract

We investigate relations on elements in C^*-algebras, including $*$-polynomial relations, order relations and all relations that correspond to universal C^*-algebras. We call these C^*-relations and define them axiomatically. Within these are the compact C^*-relations, which are those that determine universal C^*-algebras, and we introduce the more flexible concept of a closed C^*-relation.

In the case of a finite set of generators, we show that closed C^*-relations correspond to the zero-sets of elements in a free σ-C^*-algebra. This provides a solid link between two of the previous theories on relations in C^*-algebras.

Applications to lifting problems are briefly considered in the last section.

1. Introduction

In the contexts of operator inequalities, lifting problems, K-theory and universal C^*-algebras, the need arises for relations on an element x in a C^*-algebra A that that are best described in terms of $\mathbf{M}_2(A)$. An example is the relation

$$0 \leq \begin{bmatrix} |x| & x^* \\ x & |x| \end{bmatrix} \leq 1$$

on x. We also need relations such as

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}^2 = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$$

that do not determine universal C^*-algebras.

The variety of relations that arise in operator theory is impressive. In [13] we study questions about operators that can be reduced to questions about matrices. The relations that arise include

$$\alpha \leq e^{x+x^*} \leq \beta$$

and

$$\left\| y\sqrt{|x|} - \sqrt{|x|}y \right\| \leq \delta.$$

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This example-rich environment will support a general theory, a theory of C^*-relations.

Two existing theories are compelling: that of Phillips in [15, §1.3] and that of Hadwin, Kaonga and Mathes in [8, §6]. The allowed class of relations is, for our purposes, too large in first instance, too small in the second. Our compromise is an axiomatic approach that is slightly more restrictive than allowed by Phillips. In the case of finitely many generators, Theorem 6.11 shows that each of these relations is equivalent to a relation in the same basic form as considered by Hadwin et al.

The lack of free C^*-algebras forces us to consider pro-C^*-algebras. For background on this class of $*$-algebras, see [6] or [15].

Another name for a pro-C^*-algebra is locally-C^*-algebra. A pro-C^*-algebra is a topological $*$-algebra whose topology arises from, and is complete with respect to, a set of C^*-seminorms. Those seminorms are not part of the object in this category. The morphisms are all continuous $*$-homomorphisms.

This terminology is in conflict with Grothendieck's notion of a pro-category ([1, p. 4]). The conflict is slight, as continuous $*$-homomorphisms give rise to families of $*$-homomorphisms between C^*-algebras, as in Lemma 3.3.

When a pro-C^*-algebra has a topology described by a sequence of C^*-seminorms, it is metrizable and called a σ-C^*-algebra.

The free pro-C^*-algebras $\mathsf{F}\langle x_1, \ldots, x_n\rangle$ are σ-C^*-algebras. They contain in a nice way the $*$-polynomials in finitely many noncommuting variables. The elements of $\mathsf{F}\langle x_1, \ldots, x_n\rangle$ are the noncommutative functions of Hadwin, Kaonga and Mathes, and their zero sets provide a rich class of C^*-algebra relations.

There is a lot of confusion in the definition of a relation for C^*-algebras, mostly arising from the fact that free C^*-algebras do not exist (except on zero generators). We cannot simply define the relations as being elements of the free object that have been set to zero. The free object we can access is in the wrong category, and is not easily understood as it arises from completion with respect to a uncomputable sequence of seminorms.

We can define a relation as a "statement about elements in a C^*-algebra," but must take care. It is easy to have hidden ideas of what statements are allowed. We only need to know the class of functions $f : \mathscr{X} \to A$ that are to be representations of a relation, so we work directly with categories whose objects are functions from sets to C^*-algebras.

The statement

$$0 \leq a_1 \leq a_2 \leq 1$$

is to be thought of as shorthand for the category whose objects are functions

$$f : \{x_1, x_2\} \to A$$

for which

$$0 \leq f(x_1) \leq f(x_2) \leq 1$$

and whose morphisms are intertwining $*$-homomorphisms. The desired universal representation

$$\iota : \{x_1, x_2\} \to C^* \langle x_1, x_2 \mid 0 \leq x_1 \leq x_2 \leq 1 \rangle$$

is the initial object in that category.

2. C^*-Algebra Relations

We identify within a general class of relations those that correspond to universal C^*-algebras.

DEFINITION 2.1. Given a set $\mathscr{X}$, the *null C^*-relation* on $\mathscr{X}$ is the category $\mathscr{F}_{\mathscr{X}}$ with objects of the form (j, A), where A is a C^*-algebra and $j : \mathscr{X} \to A$ is a function. The morphisms from (j, A) to (k, B) all $*$-homomorphisms $\varphi : A \to B$ for which $\varphi \circ j = k$.

Given any nonempty set Λ and C^*-algebras A_λ for $\lambda \in \Lambda$, we use one of

$$\prod_{\lambda \in \Lambda} A_\lambda \qquad \text{or} \qquad \prod_{\lambda \in \Lambda}^{C^*} A_\lambda$$

to denote the C^*-algebra of families $\langle a_\lambda \rangle_{\lambda \in \Lambda}$ that are bounded in norm and have $a_\lambda \in A_\lambda$.

DEFINITION 2.2. Given a set $\mathscr{X}$, a *C^*-relation on $\mathscr{X}$* is a full subcategory $\mathscr{R}$ of $\mathscr{F}_{\mathscr{X}}$ of such that:

C1: the unique map $\mathscr{X} \to \{0\}$ is an object;

C2: if $\varphi : A \hookrightarrow B$ is an injective $*$-homomorphism and $f : \mathscr{X} \to A$ is a function, then

$$f \text{ is an object} \Leftarrow \varphi \circ f \text{ is an object;}$$

C3: if $\varphi : A \to B$ is a $*$-homomorphism and $f : \mathscr{X} \to A$ is a function, then

$$f \text{ is an object} \Rightarrow \varphi \circ f \text{ is an object;}$$

C4f: if $f_j : \mathscr{X} \to A_j$ is an object for $1 \leq j \leq n$ then

$$\prod f_j : \mathscr{X} \to \prod_{j=1}^{n} A_j$$

is an object.

The admissible relations defined in [15] are only required to satisfy **C3** in the case where φ is a surjection. Such a relation can be extended to a C^*-relation by adding in every push-forward by an inclusion.

The *intersection* of two or more C^*-relations on the same set $\mathscr{X}$ will be the full subcategory whose objects are the $f : \mathscr{X} \to A$ that are representations of all the given relations. This intersection is again a C^*-relation.

We will generally not mention the morphisms as they are determined by the objects.

DEFINITION 2.3. The C^*-relation $\mathscr{R}$ on $\mathscr{X}$ is called *compact* if

C4: for any nonempty set Λ, if $f_\lambda : \mathscr{X} \to A_\lambda$ is an object for all $\lambda \in \Lambda$ then

$$\prod f_\lambda : \mathscr{X} \to \prod_{\lambda \in \Lambda} A_\lambda$$

is an object.

EXAMPLE 2.4. Let $\mathscr{R}$ be the subcategory of $\mathscr{F}_\emptyset$ whose only object is the unique function from $\emptyset$ to the zero C^*-algebra. This is satisfies **C1**, **C2** and **C4** but only the weaker form of **C3** where φ is only allowed to be a surjection.

Usually we will have a statement that determines the objects in a C^*-relation. We will call this statement a C^*-relation and the objects in the associated category *representations of the relation*. If we start with $\mathscr{R}$ we can use

$$f : \mathscr{X} \to A \text{ is an object in } \mathscr{R}$$

as a relation whose representations are the objects in $\mathscr{R}$. For this reason, we generally call an object a representation.

EXAMPLE 2.5. If p is a noncommutative $*$-polynomial in n variables with zero constant term then

$$p(x_1, \ldots, x_n) = 0$$

is a C^*-relation.

EXAMPLE 2.6. The C^*-relation

$$x^*x - x = 0$$

is compact, since $x^*x = x$ implies $x^* = x$ and so x is a projection, and so has norm at most one.

EXAMPLE 2.7. The C^*-relation

$$x^2 - x = 0$$

is not compact, as idempotents can have any norm.

EXAMPLE 2.8. Consider the relation determined by the equation

$$xy - yx - 1 = 0,$$

where if x and y are in A then this relation holds if A is unital and $xy - yx$ equals the unit in A. If we allow the case $1 = 0$ in the zero C^*-algebra then **C3** will fail. If we exclude this case, then **C1** will fail. Either way, we do not obtain a C^*-relation.

For any C^*-algebra A, let

$$\mathrm{rep}_{\mathscr{R}}(\mathscr{X}, A) = \{f : \mathscr{X} \to A \mid f \text{ is a representation of } \mathscr{R}\}.$$

DEFINITION 2.9. If $\mathscr{X}$ is a set and $\mathscr{R}$ is a C^*-relation on $\mathscr{X}$ then a function $\iota : \mathscr{X} \to U$ from $\mathscr{X}$ to a C^*-algebra U is *universal for* $\mathscr{R}$ if:

U1: given a C^*-algebra A, if $\varphi : U \to A$ is a $*$-homomorphism then $\varphi \circ \iota : \mathscr{X} \to A$ is a representation of $\mathscr{R}$;

U2: given a C^*-algebra A, if a function $f : \mathscr{X} \to A$ is a representation of $\mathscr{R}$ then there is a unique $*$-homomorphism $\varphi : U \to A$ so that $f = \varphi \circ \iota$.

It should be clear that ι and U are unique up to isomorphism. Notice that ι must be a representation. The definition of a universal representation is summarized by the bijection

$$\hom(U, A) \to \mathrm{rep}_{\mathscr{R}}(\mathscr{X}, A)$$

defined by $\varphi \mapsto \varphi \circ \iota$.

Notice that **U1** is absent in [15, §1.3]. See [3, Definition 1.2].

Various versions of Theorem 2.10 can be found in [8, §1.4], [10, §3.1] and [15, Proposition 1.3.6]. The proof here uses the same techniques as Hadwin and Ma in [9, §2].

THEOREM 2.10. *If $\mathscr{R}$ is C^*-relation on $\mathscr{X}$ then $\mathscr{R}$ is compact if and only if there exists a universal representation for $\mathscr{R}$.*

PROOF. Assume such a universal representation $f : \mathscr{X} \to U$ exists. We need to verify **C4.**

Suppose Λ is a nonempty set and $f_\lambda : \mathscr{X} \to A_\lambda$ is a representation for each $\lambda \in \Lambda$. For each λ we know there exists a $*$-homomorphism $\varphi_\lambda : U \to A_\lambda$ with $f_\lambda = \varphi_\lambda \circ \iota$. Since

$$\prod f_\lambda = \left(\prod \varphi_\lambda\right) \circ \iota$$

we have proven **C4**.

As to the converse, assume $\mathscr{R}$ is a compact C^*-relation on $\mathscr{X}$.

Let S_1 be a set such that every C^*-algebra generated by a set no larger than $\mathscr{X}$ has cardinality at most the cardinality of S_1. Let S_2 be the set of all C^*-algebras whose underlying set is a subset of S_1. Let S_3 be the set of all functions from $\mathscr{X}$ to a C^*-algebra in S_2. Let S_4 be the set containing every function $f : \mathscr{X} \to A$ in S_3 whose image $f(\mathscr{X})$ generates A and so that f is a representation in $\mathscr{R}$. Let these representations be indexed as $f_\lambda : \mathscr{X} \to A_\lambda$ for λ in a set Λ.

Given any representation $g : \mathscr{X} \to B$, by **C2** we know that by corestricting we can factor g as $g = \alpha \circ g_0$ where $g_0 : \mathscr{X} \to B_0$ has image that generates B_0 and $\alpha : B_0 \to B$ is an inclusion. There will be an isomorphism $\beta : B_0 \to B_1$ for some B_1 in S_2. Let $g_1 : \mathscr{X} \to B_1$ be defined as $g_1 = \beta \circ g_0$. This will be a representation by **C3**, with generating image, and so $g_1 = f_\lambda$ and $B_1 = A_\lambda$ for some λ in Λ. Thus g factors as $g = \gamma \circ f_\lambda$ where $g : A_\lambda \to B$ is the injective $*$-homomorphism $g = \alpha \circ \beta^{-1}$.

To summarize the last paragraph: every representation g in $\mathscr{R}$ can be factored as $g = \varphi \circ f_\lambda$ where $\varphi : A_\lambda \to B$ is an injective $*$-homomorphism.

By **C1** there is a representation, so we know $\Lambda \neq \emptyset$.

Let

$$f = \prod_{\lambda \in \Lambda} f_\lambda : \mathscr{X} \to \prod_{\lambda \in \Lambda} A_\lambda.$$

This is well defined and a representation by **C4**. Let U denote the C^*-algebra generated by the image of f and let $\iota : \mathscr{X} \to U$ be the corestriction of f. The inclusion of U in the product we call η, so $f = \eta \circ \iota$.

Suppose $\varphi : U \to A$ is a $*$-homomorphism. Since ι is a representation, **C3** tells us that $\varphi \circ \iota$ is also a representation. We have meet the first requirement on U.

Suppose B is a C^*-algebra and that a function $g : \mathscr{X} \to B$ is a representation in $\mathscr{R}$. We can factor g as $g = \varphi \circ f_{\lambda_0}$ where $\varphi : A_{\lambda_0} \to B$ is an injective $*$-homomorphism. Let p_{λ_0} denote the coordinate projection

$$p_{\lambda_0} : \prod_{\lambda \in \Lambda} A_\lambda \to A_{\lambda_0}.$$

Define $\psi : U \to B$ as the $*$-homomorphism $\psi = \varphi \circ p_{\lambda_0} \circ \eta$. Then

$$\psi \circ \iota = \varphi \circ p_{\lambda_0} \circ f = \varphi \circ f_{\lambda_0} = g.$$

Since ι has range that generates U, the $*$-homomorphism ψ is the unique one satisfying $\psi \circ \iota = g$.

If $\mathscr{R}$ is a C^*-relation with universal representation $\iota : \mathscr{X} \to U$ then we call U the *universal C^*-algebra for $\mathscr{R}$* and use for notation $U = C^* \langle \mathscr{X} \mid \mathscr{R} \rangle$. Sometimes we will use $\iota_{\mathscr{R}}$ in place of the generic ι.

EXAMPLE 2.11. There is one free C^*-algebra, namely $C^* \langle \emptyset \mid \mathscr{F}_{\emptyset} \rangle$, which is just $\{0\}$.

EXAMPLE 2.12. For any C^*-algebra A,

$$C^* \langle A \mid A \to B \text{ is a } *\text{-homomorphism} \rangle \cong A.$$

That is, if we let $\mathscr{R}_A$ be the full subcategory of $\mathscr{F}_A$ with objects $f : A \to B$ that are $*$-homomorphisms, then A is isomorphic to $C^* \langle A \mid \mathscr{R}_A \rangle$.

Neither the zero sets of noncommutative polynomials, not even $a^2 = 0$, nor basic order relations like $a \leq b$ are compact. The story must continue, and that means leaving our familiar category.

3. Relations in Pro-C^*-algebras

For any pro-C^*-algebra A, let $S(A)$ denote the set of all continuous C^*-seminorms on A. For p in $S(A)$ we have the C^*-algebra $A_p = A/\ker(p)$ and the surjection $\pi_p : A \to A_p$. For $q \geq p$ we have also surjections $\pi_{q,p} : A_q \to A_p$. If $S \subseteq S(A)$ is cofinal then $A = \varprojlim A_p$ where p ranges over S.

Starting from an inverse system of C^*-algebras, $\rho_{\lambda',\lambda} : A_{\lambda'} \to A_\lambda$ for $\lambda \preceq \lambda'$ in Λ, we can take the inverse limit and get a pro-C^*-algebra $A = \varprojlim A_\lambda$. However, the induced $*$-homomorphisms $\rho_\lambda : A \to A_\lambda$ may fail to be surjective. However, if $\Lambda = \mathsf{N}$ then the ρ_λ are always surjections. For proofs of these facts, see [15, §1] and [14, §5].

LEMMA 3.1. *Suppose $A = \varprojlim A_\lambda$ is a pro-C^*-algebra and $\rho_\lambda : A \to A_\lambda$ is a surjection for all λ in Λ. There is an order-preserving, cofinal map $\gamma : \Lambda \to S(A)$ and there are isomorphisms $\varphi_\lambda : A_{\gamma(\lambda)} \to A_\lambda$ so that $\rho_\lambda = \varphi_\lambda \circ \pi_{\gamma(\lambda)}$.*

PROOF. Simply define $\gamma(\lambda)(a) = \|\rho_\lambda(a)\|$. There is clearly an injective $*$-homomorphism φ_λ defined by

$$\varphi_\lambda(a + \ker\gamma(\lambda)) = \rho_\lambda(a)$$

and it is onto because ρ_λ is assumed to be onto. If $\lambda \preceq \lambda'$ then $\rho_{\lambda',\lambda}$ is norm decreasing, which is easily seen to imply $\gamma(\lambda) \leq \gamma(\lambda')$. The inverse limit topology on A is determined by the $\gamma(\lambda)$ and so $\gamma(\Lambda)$ must be cofinal.

LEMMA 3.2. *Suppose A and B are pro-C^*-algebras and that $T \subseteq S(B)$ is cofinal. If $\varphi : A \to B$ is a $*$-homomorphism that is a homeomorphism onto its image then there is a cofinal function $\theta : T \to S(A)$ and injective $*$-homomorphisms $\varphi_p : A_{\theta(p)} \hookrightarrow B_p$ so that, for all p in T, we have $\pi_p \circ \varphi = \varphi_p \circ \pi_{\theta(p)}$.*

PROOF. For any p in T we know that $p \circ \varphi$ is in $S(A)$, so we define $\theta(p) = p \circ \varphi$. Since $a \in \ker(\theta(p))$ implies

$$\|\pi_p \circ \varphi(a)\| = p(\varphi(a)) = 0$$

we find that φ induces a $*$-homomorphism φ_p from $A_{\theta(p)}$ to B_p with $\pi_p \circ \varphi = \varphi_p \circ \pi_{\theta(p)}$. It is injective since

$$\|\varphi_p(\pi_{\theta(p)}(a))\| = \|\pi_p(\varphi(a))\| = p(\varphi(a)) = \theta(p)(a) = \|\pi_{\theta(p)}(a)\|.$$

We wish to show that $\theta(T)$ is cofinal. For p in T let

$$B(p, \epsilon) = \{b \in B \mid p(b) < \epsilon\}$$

and define $B(q, \epsilon)$ similarly for q in $S(A)$. These sets form neighborhood bases at the respective origins.

Suppose q is in $S(A)$. Since φ is open, there is an $\epsilon > 0$ and a p in T so that

$$B(p, \epsilon) \subseteq \varphi(B(q, 1)).$$

For a in A,

$$\theta(p)(a) < \epsilon \Longrightarrow \exists a_1 \in A \text{ s.t. } q(a_1) < 1 \text{ and } \varphi(a_1) = \varphi(a)$$

and, since φ is one-to-one,

$$\theta(p)(a) < \epsilon \Longrightarrow q(a) < 1.$$

Standard facts about C^*-algebras show that this implies $q \leq \theta(p)$.

LEMMA 3.3. *Suppose A and B are pro-C^*-algebras and that $S \subseteq S(A)$ and $T \subseteq S(B)$ are cofinal. If $\varphi : A \to B$ is a continuous $*$-homomorphism then there is a function $\theta : T \to S$ and there are $*$-homomorphisms $\varphi_p : A_{\theta(p)} \to B_p$ so that $\pi_p \circ \varphi = \varphi_p \circ \pi_{\theta(p)}$ for all p in T.*

PROOF. For any p in T, we know that $p \circ \varphi$ is in $S(A)$, so choose $\theta(p) \in S$ with $\theta(p) \geq p \circ \varphi$. Since $a \in \ker(\theta(p))$ implies

$$\|\pi_p \circ \varphi(a)\| = p(\varphi(a)) = 0$$

we find that φ induces a $*$-homomorphism φ_p from $A_{\theta(p)}$ to B_p with $\pi_p \circ \varphi = \varphi_p \circ \pi_{\theta(p)}$.

In the last two lemmas, the function θ and the maps $\varphi_p : A_{\theta(p)} \to B_p$ are a morphism in the sense of Grothendieck ([1]) between the pro-objects $(A, \{A_p\}, \{\pi_p\})$ and $(B, \{B_p\}, \{\pi_p\})$. That is, one can show that if $p \leq p'$ and $q \geq \theta(p)$ and $q \geq \theta(p')$ then

$$\varphi_p \circ \pi_{q,\theta(p)} = \pi_{p',p} \circ \varphi_{p'} \circ \pi_{q,\theta(p')}.$$

LEMMA 3.4. *Suppose* $A = \varprojlim A_\lambda$ *is a pro-*C^**-algebra and* $\rho_\lambda : A \to A_\lambda$ *is a surjection for all* λ *in* Λ. *Suppose* B *is a* C^**-algebra. If* $\varphi : A \to B$ *is a continuous* $*$*-homomorphism then there exists* λ *in* Λ *and* $\varphi' : A_\lambda \to B$ *so that* $\varphi = \varphi' \circ \rho_\lambda$.

PROOF. Lemma 3.1 reduces this to a special case of Lemma 3.3.

LEMMA 3.5. *Suppose* $\mathscr{R}$ *is a* C^**-relation on* $\mathscr{X}$. *Suppose* $f : \mathscr{X} \to A$ *is a function and* A *is a pro-*C^**-algebra. If* $\pi_p \circ f$ *is a representation of* $\mathscr{R}$ *in* A_p *for all* p *in a cofinal set* S *in* $S(A)$ *then* $\varphi \circ f$ *is a representation of* $\mathscr{R}$ *for every continuous* $*$*-homomorphisms* φ *from* A *to a* C^**-algebra.*

PROOF. Composition with a $*$-isomorphism preserves representations of $\mathscr{R}$, so it suffices to show $\pi_p \circ f$ is a representation for any p in $S(A)$. Since S is cofinal, we know $\pi_q \circ f$ is a representation for some $q \geq p$. Therefore

$$\pi_p \circ f = \pi_{q,p} \circ \pi_q \circ f$$

is a representation.

Given A_λ a pro-C^*-algebra for each λ in a set Λ, we denote the $*$-algebra of all families $\langle a_\lambda \rangle$ with $a_\lambda \in A_\lambda$ by

$$A = \prod_{\lambda \in \Lambda}^{\text{pro}\, C^*} A_\lambda,$$

with projection maps $\rho_\lambda : A \to A_\lambda$. This becomes a pro-$C^*$-algebra if we endow it with the product topology.

LEMMA 3.6. *If* A_λ *is a family of pro-*C^**-algebras and*

$$A = \prod_{\lambda \in \Lambda}^{\text{pro}\, C^*} A_\lambda,$$

then the seminorms of the form

$$\max(p_1 \circ \rho_{\lambda_1}, \ldots, p_n \circ \rho_{\lambda_n})$$

for p_j in $S(A_{\lambda_j})$ are cofinal in $S(A)$.

PROOF. A collection of C^*-seminorms on A that is closed under the pairwise max operation is cofinal if and only if it determines the topology on A. In this case, the topology is component-wise convergence, and the seminorms $p \circ \rho_\lambda$, for $p \in S(A_\lambda)$, determine the topology.

Suppose $\mathscr{X}$ is any set. For each $l : \mathscr{X} \to [0, \infty)$ define

$$F_l\langle\mathscr{X}\rangle = C^* \langle \mathscr{X} \mid \forall x \in \mathscr{X}, \|x\| \leq l(x)\rangle$$

with ι_l the universal representation. Consider also the $*$-algebra of $*$-polynomials in noncommuting variables

$$\{x, x^* \mid x \in \mathscr{X}\}$$

(the x^* being some symbols not in $\mathscr{X}$) hereby denoted $\mathsf{C}[\mathscr{X} \cup \mathscr{X}^*]$.

Lemma 3.7 is by Goodearl and Menal [7, Proposition 2.2]. The proof is only a little modified from theirs.

LEMMA 3.7. *For any $l > 0$ the canonical $*$-homomorphism*

$$\mathsf{C}[\mathscr{X} \cup \mathscr{X}^*] \to C^* \langle \mathscr{X} \mid \|x\| \leq l(x)\rangle$$

is one-to-one.

PROOF. For two nonzero choices for l we get isomorphic C^*-algebras. It is then easy to reduce to the case $l(x) = 2$ for all x.

Let U denote the full group C^*-algebra of the free group generated by two copies of $\mathscr{X}$. Let the two disjoint copies of $x \in \mathscr{X}$ be $\dot{x}$ and $\bar{x}$. In terms of generators and relations in the category of unital C^*-algebras,

$$U = C_1^*\left\langle \dot{\mathscr{X}} \cup \bar{\mathscr{X}} \;\middle|\; \text{each } \dot{x} \text{ and } \bar{x} \text{ is unitary}\right\rangle.$$

We know that the group algebra embeds in U, and so it is safe to drop the inclusion map from our notation. Define

$$\varphi : \mathsf{C}[\mathscr{X} \cup \mathscr{X}^*] \to U$$

by $\varphi(x) = \dot{x} + \bar{x}$. Notice $\varphi(x^*) = \dot{x}^{-1} + \bar{x}^{-1}$. Given a $*$-polynomial p of degree n, consider the terms in $\varphi(p)$ that are in the alternating pattern "$\dot{}\,\bar{}\,\dot{}\,\bar{}\,\dot{}\,\bar{}$". These terms will not simplify in $\varphi(p)$, so $\varphi(p) = 0$ implies that all top-degree monomials have coefficient zero. This means φ is injective, and the result follows.

To illustrate the argument based on the pattern of decorations, suppose $\mathscr{X} = \{x\}$ and

$$p = x^*x + 2xx^* + 3x.$$

Then

$$\varphi(p) = \left(\dot{x}^{-1}\bar{x} + 2\dot{x}\bar{x}^{-1}\right) + \left(\bar{x}^{-1}\dot{x} + 2\bar{x}\dot{x}^{-1}\right) + 3\dot{x} + 3\bar{x} + 6$$

and so the dot-dash terms of length two reflect the coefficients of the terms of length two in p.

There are surjections between these "free" C^*-algebras. If $l \geq l'$ then sending x to x determines

$$\gamma_{l,l'} : F_l\langle\mathscr{X}\rangle \to F_{l'}\langle\mathscr{X}\rangle.$$

Finally let $\mathsf{F}\langle\mathscr{X}\rangle = \varprojlim F_l\langle\mathscr{X}\rangle$ and $\iota : \mathscr{X} \to \mathsf{F}\langle\mathscr{X}\rangle$ be defined so that $\iota(x)$ corresponds to the coherent family $\langle\iota_l(x)\rangle_l$. There are the obvious $*$-homomorphisms $\gamma_l : \mathsf{F}\langle\mathscr{X}\rangle \to F_l\langle\mathscr{X}\rangle$. These are in fact surjections, as each generator determines a coherent family that is then sent to the copy of that generator in $F_l\langle\mathscr{X}\rangle$. Notice $\iota(\mathscr{X})$ algebraically generates a dense copy of $\mathsf{C}[\mathscr{X} \cup \mathscr{X}^*]$.

THEOREM 3.8. *In the category of pro-C^*-algebras and continuous $*$-homomorphisms, $\iota : \mathscr{X} \to \mathsf{F}\langle\mathscr{X}\rangle$ is free.*

PROOF. First suppose A is a C^*-algebra. For any function $f : \mathscr{X} \to A$ we can set $l(x) = \|f(x)\|$ and there is a $*$-homomorphism $\varphi_l : F_l\langle\mathscr{X}\rangle \to A$ sending $\iota_l(x)$ to $f(a)$. Then $\varphi_l \circ \gamma_l$ is a continuous $*$-homomorphism that sends $\iota(x)$ to $f(a)$. This is the unique such map since $\iota(\mathscr{X})$ generates $\mathsf{F}\langle\mathscr{X}\rangle$.

Suppose A is a pro-C^*-algebra and $f : \mathscr{X} \to A$ is a function. For each p in $S(A)$ there is a unique continuous $*$-homomorphism $\varphi_p : \mathsf{F}\langle\mathscr{X}\rangle \to A_p$ for which $\varphi_p \circ \iota = \pi_p \circ f$. Since

$$\pi_{p,p'} \circ \varphi_p \circ \iota = \pi_{p,p'} \circ \pi_p \circ f = \pi_{p'} \circ f$$

we can conclude $\pi_{p,p'} \circ \varphi_p = \varphi_{p'}$. This means there is a continuous $*$-homomorphism $\varphi : \mathsf{F}\langle\mathscr{X}\rangle \to A$ so that $\pi_p \circ \varphi = \varphi_p$. Therefore

$$\pi_p(\varphi(\iota(x))) = \varphi_p(\iota(x)) = \pi_p(f(x))$$

and so $\varphi(\iota(x)) = f(x)$.

The uniqueness of φ again follows from the fact that $\iota(\mathscr{X})$ generates $\mathsf{F}\langle\mathscr{X}\rangle$.

LEMMA 3.9. *The pro-C^*-algebra $\mathsf{F}\langle\mathscr{X}\rangle$ is a σ-C^*-algebra if and only if $\mathscr{X}$ is finite.*

PROOF. Suppose $\mathscr{X}$ is the finite set $\{x_1, \ldots, x_n\}$. The functions l_k defined by $l_k(x_j) = k$ are cofinal among all functions from $\mathscr{X}$ to $[0, \infty)$. Therefore $\mathsf{F}\langle x_1, \ldots, x_n\rangle$ is an inverse limit of a sequence of C^*-algebras,

$$\mathsf{F}\langle x_1, \ldots, x_n\rangle = \varprojlim C^*\big\langle x_1, \ldots, x_n \mid \|x_1\| \leq k, \ldots, \|x_n\| \leq k\big\rangle.$$

For the converse it suffices to show that $\mathsf{F}\langle x_1, x_2, \ldots\rangle$ is not a σ-C^*-algebra.

Suppose $p_1, p_2, \ldots$ is an increasing sequence of C^*-seminorms determining the topology of $\mathsf{F}\langle x_1, x_2, \ldots\rangle$. By passing to a subsequence, we are able to assume $p_n(\iota(x_n)) \neq 0$ for all n. Define

$$\alpha_k = \min_{n \leq k} (kp_n(\iota(x_k)))^{-1}$$

and $y_n = \alpha_n \iota(x_n)$. For $k \geq n$ we have $p_n(y_k) \leq \frac{1}{k}$. Therefore $\lim_{k\to\infty} y_k = 0$. Take any sequence a_k in $B(\mathsf{H})$ so that $\|a_k\| = \alpha_k^{-1}$. There is a continuous $*$-homomorphism

$$\varphi : \mathsf{F}\langle x_1, x_2, \ldots\rangle \to B(\mathsf{H})$$

with $\varphi(\iota(x_k)) = a_k$. This means $\alpha_k a_k$ converges to zero, contradicting the fact that $\|\alpha_k a_k\|$ has norm 1.

DEFINITION 3.10. Given a set X, the *null pro-C^*-relation on $\mathscr{X}$* is the category $\mathscr{F}_{\mathscr{X}}^{\mathrm{pro}\,C^*}$ whose objects are of the form (j, A), where A is a pro-C^*-algebra and $j : \mathscr{X} \to A$ is a function from $\mathscr{X}$ to (the underlying set of) A. As morphisms from (j, A) to (k, B) it has all continuous $*$-homomorphisms $\varphi : A \to B$ for which $\varphi \circ j = k$.

DEFINITION 3.11. Given a set $\mathscr{X}$, a *pro-C^*-relation on $\mathscr{X}$* is full subcategory $\mathscr{R}$ of $\mathscr{F}_{\mathscr{X}}^{\mathrm{pro}\,C^*}$ such that:

PC1: the unique map $\mathscr{X} \to \{0\}$ is an object;

PC2: if $\varphi : A \hookrightarrow B$ is the inclusion of a closed $*$-subalgebra of a pro-C^*-algebra B and if $f : \mathscr{X} \to A$ is a function, then

$$f \text{ is an object} \Leftarrow \varphi \circ f \text{ is an object;}$$

PC3: if $\varphi : A \to B$ is a continuous $*$-homomorphism, and if $f : \mathscr{X} \to A$ is a function, then

$$f \text{ is an object} \Rightarrow \varphi \circ f \text{ is an object;}$$

PC4: if Λ is a nonempty set, and if $f_\lambda : \mathscr{X} \to A_\lambda$ is an object for each $\lambda \in \Lambda$, then

$$\prod f_\lambda : \mathscr{X} \to \prod_{\lambda\in\Lambda}^{\mathrm{pro}\,C^*} A_\lambda$$

is an object.

Again we will conflate statements with categories and representations with objects.

DEFINITION 3.12. Suppose $\mathscr{X}$ is a set and $\mathscr{R}$ is a pro-C^*-relation on $\mathscr{X}$. A function $\iota : \mathscr{X} \to U$ from $\mathscr{X}$ to a pro-C^*-algebra U is *universal for* $\mathscr{R}$ if:

- **PU1**: given a pro-C^*-algebra A, if $\varphi : U \to A$ is a continuous $*$-homomorphism then $\varphi \circ \iota : \mathscr{X} \to A$ is a representation of $\mathscr{R}$;
- **PU2**: given a pro-C^*-algebra A, if a function $f : \mathscr{X} \to A$ is a representation in $\mathscr{R}$ then there is a unique $*$-homomorphism $\varphi : U \to A$ so that $f = \varphi \circ \iota$.

It should be clear that ι and U are unique, up to isomorphism. Also notice that ι must be a representation.

The definition of a universal representation is again summarized by the bijection

$$\hom(U, A) \to \mathrm{rep}_{\mathscr{R}}(\mathscr{X}, A)$$

defined by $\varphi \mapsto \varphi \circ \iota$, but now for A any pro-C^*-algebra and $\hom(-, -)$ meaning the set of continuous $*$-homomorphisms.

THEOREM 3.13. *If $\mathscr{R}$ is a pro-C^*-relation on $\mathscr{X}$ then there exists a universal representation for $\mathscr{R}$.*

PROOF. Suppose $g : \mathscr{X} \to A$ is a representation of $\mathscr{R}$. Let B be the closed $*$-algebra generated by $g(\mathscr{X})$. There is a continuous $*$-homomorphism $\varphi : \mathsf{F}\langle\mathscr{X}\rangle \to B$ so that $\varphi(\iota(x)) = g(x)$. By **PC2**, we can corestrict g to a representation $f_1 : \mathscr{X} \to B$. Let κ be the inclusion of B in A, so $\kappa \circ f_1 = g$. There is an open, continuous *-algebra isomorphism

$$\psi : \overline{\mathsf{F}\langle\mathscr{X}\rangle / \ker(\varphi)} \to B$$

where the completion is with respect to the seminorms

$$\varphi(y) + \ker(\varphi) \mapsto p\,(\varphi(y)) \qquad (\text{for } \ p \in S(B)).$$

By **PC3**, $f_2 = \psi^{-1} \circ f_1$ is a representation and $f = \kappa \circ \psi \circ f_2$.

The algebraic quotients of $\mathsf{F}\langle\mathscr{X}\rangle$ by closed, two-sided self-adjoint ideals form a set. The collection of all C^*-seminorms on each quotient is a set, and so the collection of all possible completions of quotients of $\mathsf{F}\langle\mathscr{X}\rangle$ is a set. Therefore, we can index by a set Λ all representation into these particular pro-C^*-algebras $f_\lambda : \mathscr{X} \to A_\lambda$ so that a generic representation g as above factors as $g = \gamma \circ f_\lambda$ for some continuous $*$-homomorphism γ.

By **PC1** there are representations, so we know $\Lambda \neq \emptyset$.

Let

$$f = \prod_{\lambda \in \Lambda} f_\lambda : \mathscr{X} \to \prod_{\lambda \in \Lambda}^{\mathrm{pro}\, C^*} A_\lambda$$

This is well-defined and a representation by **PC4**. Let U denote the pro-C^*-algebra generated by the image of f and let $\iota : \mathscr{X} \to U$ be the corestriction of f. The inclusion of U in the product we call η, so $f = \eta \circ \iota$.

The proof that ι is universal for $\mathscr{R}$ is similar to the argument given in the proof of Theorem 2.10.

For notation, the universal pro-C^* algebra will be

$$\mathrm{pro}\, C^* \langle \mathscr{X} \mid \mathscr{R} \rangle .$$

If A is a pro-C^*-algebra and $\mathscr{R}_A$ is defined on the set A with $\varphi : A \to B$ considered a representation if and only if it is a continuous $*$-homomorphism, then $\mathscr{R}_A$ is a pro-C^*-relation and A is isomorphic to $\mathrm{pro}\, C^* \langle A \mid \mathscr{R}_A \rangle$. This can easily be made a bit more general.

LEMMA 3.14. *Suppose $f : \mathscr{X} \to A$ is a function whose image generates the pro-C^*-algebra A. Let $\mathscr{R}_f$ be the full subcategory of $\mathscr{F}_{\mathscr{X}}^{\mathrm{pro}\, C^*}$ for which*

$$\mathrm{rep}_{\mathscr{R}_f}(B) = \{\varphi \circ f \mid \varphi \in \hom(A, B)\} .$$

Then $\mathscr{R}$ is a pro-C^*-relation and

$$\mathrm{pro}\, C^* \left\langle \mathscr{X} \mid \mathscr{R}_{A_f} \right\rangle \cong A,$$

where the isomorphism sends $\iota(x)$ to $f(x)$.

PROOF. We know the zero function $A \to \{0\}$ is in $\hom(A, \{0\})$ and so the zero function $\mathscr{X} \to \{0\}$ is a representation.

Suppose $g : \mathscr{X} \to B$ is a function and $\psi : B \hookrightarrow C$ is an embedding of a closed $*$-subalgebra and $\psi \circ g$ is a representation. Then $\psi \circ g = \varphi \circ f$ for some φ in $\hom(A, C)$. Thus $\varphi(f(\mathscr{X})) \subseteq B$ and so $\varphi(A) \subseteq B$ and $\varphi = \psi \circ \varphi_0$ for some φ_0 in $\hom(A, B)$ and

$$\psi \circ \varphi_0 \circ f = \psi \circ g.$$

Since ψ is injective, $\varphi_0 \circ f = g$ and g is a representation.

If $g : \mathscr{X} \to B$ is a representation and ψ is in $\hom(B, C)$ then $g = \varphi \circ f$ for some φ in $\hom(A, B)$. Therefore $\psi \circ g = \psi \circ \varphi \circ f$ is a representation.

Suppose $g_\lambda : \mathscr{X} \to B_\lambda$ is a representation for all $\lambda \in \Lambda$. Then $g_\lambda = \varphi_\lambda \circ f$ for some φ_λ in $\hom(A, B_\lambda)$. Then

$$\prod g_\lambda = \left(\prod \varphi_\lambda\right) \circ f$$

is a representation.

For the second statement, we need to show that $f : \mathscr{X} \to A$ is universal. But that says there is a bijection

$$\hom(A, B) \to \operatorname{rep}_{\mathscr{R}_f}(\mathscr{X}, B)$$

defined by $\varphi \mapsto \varphi \circ f$, and this is true by definition.

LEMMA 3.15. *Every pro-C^*-relation is closed under inverse limits.*

PROOF. Suppose $\mathscr{R}$ is a pro-C^*-relation on $\mathscr{X}$. Suppose we have an inverse system. That is A_λ is a pro-C^*-algebra for each λ in a directed set Λ and there are bonding maps $\rho_{\lambda,\mu} : A_\lambda \to A_\mu$ that are continuous $*$-homomorphisms whenever $\mu \preceq \lambda$. Then the limit can be constructed as

$$\varprojlim A_\lambda = \left\{ \langle a_\lambda \rangle \in \prod_{\lambda \in \Lambda}^{\text{pro}\, C^*} A_\lambda \;\middle|\; \rho_{\lambda,\mu}(a_\lambda) = a_\mu \text{ if } \mu \preceq \lambda \right\}$$

and $\rho_\lambda : A \to A_\lambda$ defined by $\rho_\lambda(\langle a_\alpha \rangle) = a_\lambda$.

Given $f_\lambda : \mathscr{X} \to A_\lambda$, representations that are coherent in the sense that $\rho_{\lambda,\mu} \circ f_\lambda = f_\mu$ wherever $\mu \preceq \lambda$, we have a function $f : \mathscr{X} \to A$ define by corestricting the product,

$$f(x) = \langle f_\lambda(x) \rangle \in \varprojlim A_\lambda$$

and this is a representation by **PC2** and **PC4**.

PROPOSITION 3.16. *If $\mathscr{R}$ is a pro-C^*-relation, then its restriction to C^*-algebras is a C^*-relation. If two pro-C^*-relations on the same set have the same restriction to C^*-algebras then they are equal.*

PROOF. The first statement is clear, since the pro-C^* product of a finite number of C^*-algebras equals the C^* product.

As to the second, every pro-C^*-algebra is the inverse limit of C^*-algebras, so Lemma 3.15 applies.

PROPOSITION 3.17. *Suppose $\mathscr{R}$ is a C^*-relation on $\mathscr{X}$. If we define $\hat{\mathscr{R}}$ as the full subcategory of $\mathscr{F}_{\mathscr{X}}^{\text{pro}\, C^*}$, where $f : \mathscr{X} \to A$ is an object if $\pi_p \circ f$ is a* *representation of $\mathscr{R}$ for all p in $S(A)$, then $\hat{\mathscr{R}}$ is a pro-C^*-relation extending $\mathscr{R}$.*

PROOF. Since quotients take representations to representations, $\hat{\mathscr{R}}$ extends $\mathscr{R}$. Notice also that $f : \mathscr{X} \to A$ must be a representation for $\hat{\mathscr{R}}$ if $\pi_p \circ f$ is a representation of $\mathscr{R}$ for all p in a cofinal set in $S(A)$.

Since $0 : \mathscr{X} \to \{0\}$ is a representation in $\mathscr{R}$ it is also a representation in $\hat{\mathscr{R}}$.

Suppose $\varphi : A \hookrightarrow B$ is the inclusion of a closed $*$-subalgebra of a pro-C^*-algebra B and $f : \mathscr{X} \to A$ is a function for which $\varphi \circ f$ is a representation of $\hat{\mathscr{R}}$. By Lemma 3.2 there is a cofinal function $\theta : S(B) \to S(A)$ and injective $*$-homomorphisms $\varphi_p : A_{\theta(p)} \hookrightarrow B_p$ so that $\pi_p \circ \varphi = \varphi_p \circ \pi_{\theta(p)}$ for all p in $S(B)$. We know that

$$\pi_p \circ \varphi \circ f = \varphi_p \circ \pi_{\theta(p)} \circ f$$

is a representation of $\mathscr{R}$, and since φ_p is injective, also that $\pi_{\theta(p)} \circ f$ is a representation of $\mathscr{R}$. Since the image of θ is cofinal in $S(A)$, we conclude f is a representation of $\mathscr{R}$.

Suppose $\varphi : A \to B$ is a continuous $*$-homomorphism and $f : \mathscr{X} \to A$ is a representation of $\hat{\mathscr{R}}$. By Lemma 3.3 there is a function $\theta : S(B) \to S(A)$ and $*$-homomorphisms $\varphi_p : A_{\theta(p)} \to B_p$ so that $\pi_p \circ \varphi = \varphi_p \circ \pi_{\theta(p)}$ for all p in $S(B)$. Since f is a representation of $\hat{\mathscr{R}}$, we know $\pi_{\theta(p)} \circ f$ is a representation of $\mathscr{R}$, and so

$$\pi_p \circ \varphi \circ f = \varphi_p \circ \pi_{\theta(p)} \circ f$$

is a representation of $\mathscr{R}$. This being true for all p in $S(B)$, we conclude $\varphi \circ f$ is a representation of $\hat{\mathscr{R}}$.

Suppose $f_\lambda : \mathscr{X} \to A_\lambda$ is a representation of $\hat{\mathscr{R}}$ for each λ in a nonempty set Λ. To show $f = \prod f_\lambda$ is a representation of $\hat{\mathscr{R}}$, it suffices to show $\pi_q \circ f$ is a representation for

$$q = \max(p_1 \circ \rho_{\lambda_1}, \ldots, p_n \circ \rho_{\lambda_n}).$$

Let

$$A = \prod_{\lambda \in \Lambda}^{\mathrm{pro}\, C^*} A_\lambda .$$

Consider the continuous $*$-homomorphism

$$\gamma : A \to (A_{\lambda_1})_{p_n} \oplus \cdots \oplus (A_{\lambda_n})_{p_n}$$

defined as

$$\gamma = \pi_{p_1} \circ \rho_{\lambda_1} \oplus \cdots \oplus \pi_{p_n} \circ \rho_{\lambda_n} .$$

This corresponds to the seminorm q, as

$$\begin{aligned}\|\gamma(\langle a_\lambda\rangle_\lambda)\| &= \|\pi_{p_1}(a_{\lambda_1}) \oplus \cdots \oplus \pi_{p_n}(a_{\lambda_n})\| \\ &= \max(\|\pi_{p_1}(a_{\lambda_1})\|, \ldots, \|\pi_{p_1}(a_{\lambda_1})\|) \\ &= \max(p_1(a_{\lambda_1}), \ldots, p_n(a_{\lambda_1}))\end{aligned}$$

and so we have a $*$-isomorphism

$$\psi : A_q \to (A_{\lambda_1})_{p_n} \oplus \cdots \oplus (A_{\lambda_n})_{p_n}$$

satisfying $\psi \circ \pi_q = \gamma$. Finally

$$\begin{aligned}\pi_q \circ f &= \psi^{-1} \circ \gamma \circ f \\ &= \psi^{-1} \circ (\pi_{p_1} \circ \rho_{\lambda_1} \circ f \oplus \cdots \oplus \pi_{p_n} \circ \rho_{\lambda_n} \circ f) \\ &= \psi^{-1} \circ (\pi_{p_1} \circ f_{\lambda_1} \oplus \cdots \oplus \pi_{p_n} \circ f_{\lambda_n})\end{aligned}$$

which means $\pi_q \circ f$ is a representation of $\mathscr{R}$.

4. Pushouts of Pro-C^*-algebras

Recall that a diagram of pro-C^*-algebras and continuous $*$-homomorphisms

$$\begin{tikzcd} C \arrow[r, "\theta_2"] \arrow[d, "\theta_1"] & B \arrow[d, "\iota_2"] \\ A \arrow[r, "\iota_1"] & D \end{tikzcd}$$

is a pushout (and D an amalgamated free product) if $\varphi \mapsto (\varphi \circ \iota_1, \varphi \circ \iota_2)$ determines a bijection

$$\hom(D, E) \to \left\{(\varphi_1, \varphi_2) \in \hom(A, E) \times \hom(B, E) \mid \varphi_1 \circ \theta_1 = \varphi_2 \circ \theta_2\right\}.$$

By the usual category theory result we know that pushouts must be unique.

Lemma 4.1 extends [15, Proposition 1.5.3(1)], showing pushouts exist in full generality.

LEMMA 4.1. *Suppose A, B and C are pro-C^*-algebras and that $\theta_1 : C \to A$ and $\theta_2 : C \to B$ are continuous $*$-homomorphisms. Assume A and B are disjoint. Define $\mathscr{R}$ to have as representations each function $f : A \cup B \to E$ such that $f|_A : A \to E$ and $f|_B : B \to E$ are continuous $*$-homomorphisms* *and* $f \circ \theta_1 = f \circ \theta_2$. *Then* $\mathscr{R}$ *is a pro-*C^**-relation. The diagram*

$$\begin{tikzcd} C \arrow[r, "\theta_2"] \arrow[d, "\theta_1"] & B \arrow[d, "\iota|_B"] \\ A \arrow[r, "\iota|_A"] & C^*_{\mathrm{pro}}\langle A \cup B \mid \mathscr{R} \rangle \end{tikzcd}$$

is a pushout.

PROOF. The proof is routine.

LEMMA 4.2. *Suppose*

$$\begin{tikzcd} C \arrow[r, "\theta_2"] \arrow[d, "\theta_1"] & B \arrow[d, "\iota_2"] \\ A \arrow[r, "\iota_1"] & D \end{tikzcd}$$

a diagram of pro-C^**-algebras and continuous* $*$*-homomorphisms. This is a pushout if and only if* $\iota_1(A) \cup \iota_2(B)$ *generates* D *and for every pair*

$$(\varphi_1, \varphi_2) \in \hom(A, E) \times \hom(B, E)$$

such that $\varphi_1 \circ \theta_1 = \varphi_2 \circ \theta_2$ *there exists* φ *in* $\hom(D, E)$ *with* $\varphi \circ \iota_j = \varphi_j$.

PROOF. Without loss of generality, A and B are disjoint.

Pushouts are unique. If the diagram is a pushout then up to isomorphism D is given by generators $A \cup B$ and the relations as in Lemma 4.1. Therefore

$$\iota(A \cup B) = \iota_1(A) \cup \iota_2(B)$$

must generate.

For the converse, we are given the existence of φ for compatible φ_1 and φ_2 and need only show uniqueness. However, if $\iota_1(A) \cup \iota_2(B)$ generates, then φ is uniquely determined by $\varphi(\iota_1(a)) = \varphi_1(a)$ and $\varphi(\iota_2(b)) = \varphi_2(b)$.

LEMMA 4.3. *Suppose*

$$\begin{tikzcd} C \arrow[r, "\theta_2"] \arrow[d, "\theta_1"] & B \arrow[d, "\iota_2"] \\ A \arrow[r, "\iota_1"] & D \end{tikzcd}$$

a diagram of pro-C^**-algebras and continuous* $*$*-homomorphisms. The diagram is a pushout if for every* C^**-algebra* E *and every pair*

$$(\varphi_1, \varphi_2) \in \hom(A, E) \times \hom(B, E)$$

such that $\varphi_1 \circ \theta_1 = \varphi_2 \circ \theta_2$ *there exists a unique* φ *in* $\hom(D, E)$ *with* $\varphi \circ \iota_j = \varphi_j$.

PROOF. This follows easily using the universal properties of pushouts and inverse limits.

5. Pushouts in two categories

First a look at an easy example of a pushout diagram in the category of C^*-algebras. Then a method to create pushout diagrams in the pro-C^* category out of a sequence of pushouts in the C^* category.

LEMMA 5.1. *Consider the commutative diagram of* C^**-algebras and* $*$*-homomorphisms*

$$\begin{tikzcd} C \arrow[r, "\beta"] \arrow[d, "\alpha"] & B \arrow[d, "\gamma"] \\ A \arrow[r, "\delta"] & X \end{tikzcd}$$

If α *and* β *are onto and the square is a pushout then:*

(1) γ *and* δ *are surjections;*

(2) $\alpha(\ker(\beta)) = \ker(\delta)$*;*

(3) *given* a *in* A *and* b *in* B *with* $\delta(a) = \gamma(b)$*, there exists* c *in* C *with* $\alpha(c) = a$ *and* $\beta(c) = b$.

PROOF. Without loss of generality, $B = C/J$ and $A = C/K$ for some ideals J and K of C. Since

$$\begin{tikzcd} C \arrow[r] \arrow[d] & C/J \arrow[d] \\ A \arrow[r] & C/(J+K) \end{tikzcd}$$

is a pushout, and pushouts are unique, we can also assume

$$X = C/(J + K).$$

That shows (1).

Notice (2) is a special case of (3).

As to (3), we can assume we have c and c' in C with $c - c'$ in $J + K$. There are elements j in J and k in K with $c - k = c' + j$. Taking $c'' = c - k$ we have c'' in C with $c'' + K = c + K$ and $c'' + J = c' + J$.

THEOREM 5.2. *Suppose*

$$\begin{array}{ccc} A_{n+1} & \xrightarrow{\rho_{n+1}} & B_{n+1} \\ \downarrow{\scriptstyle \alpha_{n+1,n}} & & \downarrow{\scriptstyle \beta_{n+1,n}} \\ A & \xrightarrow{\rho_n} & B_n \end{array}$$

is a pushout in the category of C^*-algebras for all n. Let $A = \varprojlim A_n$ and $B = \varprojlim B_n$ with associated maps $\alpha_n : A \to A_n$ and $\beta_n : B \to B_n$. Define $\rho : A \to B$ by $\beta_n \circ \rho = \rho_n \circ \alpha_n$.

(1) *If $\alpha_{n+1,n}$ and $\beta_{n+1,n}$ are surjective for all n then the diagram*

$$\begin{array}{ccc} A & \xrightarrow{\rho} & B \\ \downarrow{\scriptstyle \alpha_n} & & \downarrow{\scriptstyle \beta_n} \\ A_n & \xrightarrow{\rho_n} & B_n \end{array}$$

is a pushout in the category of pro-C^*-algebras.

(2) *If $\alpha_{n+1,n}$, $\beta_{n+1,n}$ and ρ_n are surjective for all n then ρ is a surjection.*

PROOF. (1) It suffices to show that

$$\begin{array}{ccc} A & \xrightarrow{\rho} & B \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \beta_1} \\ A_1 & \xrightarrow{\rho_1} & B_1 \end{array}$$

is a pushout. By Lemma 4.3 we need only consider a C^*-algebra E and $\varphi : A_1 \to E$ and $\psi : B \to E$ such that $\varphi \circ \alpha_1 = \psi \circ \rho$. By Lemma 3.4 there is some n and a map $\psi_n : B_n \to E$ so that $\psi = \psi_n \circ \beta_n$. We have

$$\begin{aligned} \varphi \circ \alpha_{n-1,1} \circ \alpha_{n,n-1} \circ \alpha_n &= \varphi \circ \alpha_1 = \psi \circ \rho \\ &= \psi_n \circ \beta_n \circ \rho \\ &= \psi_n \circ \rho_n \circ \alpha_n, \end{aligned}$$

and since α_n is onto,

$$\varphi \circ \alpha_{n-1,1} \circ \alpha_{n,n-1} = \psi_n \circ \rho_n.$$

The pushout property of the square involving ρ_n and $\alpha_{n,n-1}$ tells us there is a $\psi_{n-1} : B_{n-1} \to E$ so that $\psi_n = \psi_{n-1} \circ \beta_{n,n-1}$. Thus $\psi = \psi_{n-1} \circ \beta_{n-1}$ and we are where we were before, but with n decreased by one.

By induction, there is a continuous $*$-homomorphism $\psi_1 : B_1 \to E$ with $\psi = \psi_1 \circ \beta_1$. Also

$$\varphi \circ \alpha_1 = \psi \circ \rho = \psi_1 \circ \beta_1 \circ \rho = \psi_1 \circ \rho_1 \circ \alpha_1$$

and α_1 is onto so $\varphi = \psi_1 \circ \rho_1$. That takes care of existence.

As to uniqueness, notice that $\rho_1(A_1)$ equals B_1 so the equation $\varphi = \psi_1 \circ \rho_1$ makes φ unique.

(2) Given a coherent sequence $b_1, b_2, \ldots$ in $B_1, B_2, \ldots$, we choose any a_1 with $\rho_1(a_1) = b_1$. Now we repeatedly apply Lemma 5.1 to find a coherent sequence $a_1, a_2, \ldots$ that is mapped to $b_1, b_2, \ldots$, proving the surjectivity of ρ.

6. Closed Relations

DEFINITION 6.1. For a set $\mathscr{X}$, and given functions $f_\lambda : \mathscr{X} \to A_\lambda$ into C^*-algebras A_λ for each λ in a nonempty set Λ, if

$$\sup_\lambda \| f_\lambda(x) \| < \infty$$

for all x then we call $\langle f_\lambda \rangle$ a *bounded* family of functions and define

$$\prod f_\lambda : \mathscr{X} \to \prod_{\lambda\in\Lambda}^{C^*} A_\lambda$$

by

$$\prod f_\lambda(x) = \langle f_\lambda(x) \rangle_{\lambda\in\Lambda} .$$

DEFINITION 6.2. A C^*-relation on $\mathscr{X}$ is called *closed* if

C4b: if Λ is a nonempty set, and if $f_\lambda : \mathscr{X} \to A_\lambda$ form a bounded family of objects, then

$$\prod f_\lambda : \mathscr{X} \to \prod_{\lambda\in\Lambda} A_\lambda$$

is an object.

Of course, compact implies closed. The intersection of a closed C^*-relation with a compact C^*-relation is compact. An arbitrary intersection of closed C^*-relations is closed.

Next we offer a sweepingly general functional calculus, as considered in [8, §4].

DEFINITION 6.3. If g is an element of $\mathsf{F}\langle x_1, \ldots, x_n \rangle$ then we can define $g(a_1, \ldots, a_n)$ for $a_j \in A$, where A is a pro-C^*-algebra, by

$$g(a_1, \ldots, a_n) = \varphi(g)$$

where
$$\varphi : \mathsf{F}\langle x_1, \dots, x_n\rangle \to A$$
is the unique continuous $*$-homomorphism defined by $\varphi(x_j) = a_j$. For example, if
$$g = \sqrt{\iota(x_1)^*\iota(x_1)} + \iota(x_2)$$
then
$$g(a_1, a_2) = \sqrt{a_1^* a_1} + a_2.$$
This is clearly natural.

THEOREM 6.4. *If g is an element of $\mathsf{F}\langle x_1, \dots, x_n\rangle$ then*
$$g(x_1, \dots, x_n) = 0$$
is a closed C^*-relation.

PROOF. The only $*$-homomorphism from $\mathsf{F}\langle x_1, \dots, x_n\rangle$ to $\{0\}$ is the zero map ζ, and so $\zeta(g) = 0$ and so the zero map from $\{x_1, \dots, x_2\}$ to $\{0\}$ is a representation.

If $\varphi : A \to B$ is an injective $*$-homomorphism, and if $f : \mathscr{X} \to A$ is a function so that $\varphi \circ f$ is a representation, then
$$\varphi(g(f(x_1), \dots, f(x_n))) = g(\varphi(f(x_1)), \dots, \varphi(f(x_n))) = 0$$
so
$$g(f(x_1), \dots, f(x_n)) = 0$$
and f is also a representation.

If $\varphi : A \to B$ is a $*$-homomorphism, and if $f : \mathscr{X} \to A$ is a representation, then
$$g(\varphi(f(x_1)), \dots, \varphi(f(x_n))) = \varphi(g(f(x_1), \dots, f(x_n))) = 0$$
and so $\varphi \circ f$ is a representation.

Suppose Λ is a nonempty set and that $f_\lambda : \mathscr{X} \to A_\lambda$ form a bounded family of relations. Let
$$f = \prod f_\lambda : \mathscr{X} \to \prod_{\lambda\in\Lambda}^{C^*} A_\lambda.$$
Let
$$\varphi_\lambda : \mathsf{F}\langle x_1, \dots, x_n\rangle \to A_\lambda$$
and
$$\Phi : \mathsf{F}\langle x_1, \dots, x_n\rangle \to \prod_{\lambda\in\Lambda}^{C^*} A_\lambda$$

be the associated continuous $*$-homomorphisms. Let ρ_λ be the coordinate morphism, so that $\rho_\lambda \circ \Phi = \varphi_\lambda$. In particular,

$$\rho_\lambda \circ \Phi(g) = \varphi_\lambda(g) = 0$$

and so

$$g(f(x_1), \ldots, f(x_n)) = \Phi(g) = 0.$$

Therefore f is a representation.

Not all closed relations are best described by setting an element of $\mathsf{F}\langle x_1, \ldots, x_n\rangle$ to zero.

EXAMPLE 6.5. If p is a noncommutative $*$-polynomial in n variables with zero constant term and C is a positive constant then

$$\|p(x_1, \ldots, x_n)\| \leq C$$

is a closed C^*-relation.

EXAMPLE 6.6. The inequality

$$\|x\| < 1$$

is a C^*-relation that is not closed.

EXAMPLE 6.7. Let $\mathscr{X}$ denote a copy of $[0, 1]$,

$$\mathscr{X} = \{x_t \mid t \in [0, 1]\}$$

The statement

$$t \mapsto x_t \text{ is continuous}$$

is a C^*-relation that is not closed. This example and variations are discussed in [15, §1.3].

We want something like a universal representation, but technically not a representation since the function ι might not take $\mathscr{X}$ into a C^*-algebra.

DEFINITION 6.8. If $\mathscr{X}$ is a set and $\mathscr{R}$ is a full subcategory of $\mathscr{F}_X$, then a function $\iota : \mathscr{X} \to U$ from $\mathscr{X}$ to a pro-C^*-algebra U is *ubiquitous for* $\mathscr{R}$ if:

- **UB1**: given a C^*-algebra A, if $\varphi : U \to A$ is a continuous $*$-homomorphism then $\varphi \circ \iota : \mathscr{X} \to A$ is a representation in $\mathscr{R}$;
- **UB2**: given a C^*-algebra A, if a function $f : \mathscr{X} \to A$ is a representation in $\mathscr{R}$ then there is a unique continuous $*$-homomorphism $\varphi : U \to A$ so that $f = \varphi \circ \iota$.

LEMMA 6.9. *Every C^*-relation $\mathscr{R}$ has an ubiquitous function, namely the universal representation of the extension $\hat{\mathscr{R}}$ of $\mathscr{R}$ to a pro-C^*-relation.*

PROOF. Proposition 3.17 assures us that $\hat{\mathscr{R}}$ exists. Consider the universal representation $\iota : \mathscr{X} \to U$ of $\hat{\mathscr{R}}$. Suppose A is a C^*-algebra. If $\varphi : U \to A$ is a continuous $*$-homomorphism then $\varphi \circ \iota$ is in $\hat{\mathscr{R}}$ and so in $\mathscr{R}$. If a function $f : \mathscr{X} \to A$ is a representation in $\mathscr{R}$ then it is a representation in $\hat{\mathscr{R}}$, so there is a unique continuous $*$-homomorphism $\varphi : U \to A$ so that $f = \varphi \circ \iota$.

LEMMA 6.10. *The ubiquitous function for a C^*-relation is unique.*

PROOF. We will show that a function $\iota : \mathscr{X} \to U$ that is ubiquitous for $\mathscr{R}$ is universal for $\hat{\mathscr{R}}$.

Suppose $f : \mathscr{X} \to A$ is a representation of $\hat{\mathscr{R}}$. Then for all p in $S(A)$, the composition $\pi_p \circ f$ is a representation of $\mathscr{R}$. For each p there is a unique continuous $*$-homomorphism $\varphi_p : U \to A_p$ so that $\varphi_p \circ \iota = \pi_p \circ f$. If $p' \geq p$ then

$$\pi_{p',p} \circ \varphi_{p'} \circ \iota = \pi_{p'} \circ f$$

and so, by uniqueness, $\pi_{p',p} \circ \varphi_{p'} = \pi_p$. There is, therefore, a unique continuous $*$-homomorphism $\varphi : U \to A$ such that $\pi_p \circ \varphi = \varphi_p$. Therefore $\pi_p \circ \varphi \circ \iota = \pi_p \circ f$ for all p, and so $\pi_p \circ \varphi = f$.

If $\varphi' \circ \iota = f$ then $\pi_p \circ \varphi' \circ \iota = \pi_p \circ f$, and so by the uniqueness of the φ_p we have $\pi_p \circ \varphi' = \varphi_p$. Therefore $\pi_p \circ \varphi' = \pi_p \circ \varphi$ for all p, and so $\varphi' = \varphi$.

THEOREM 6.11. *Suppose $\mathscr{X}$ is finite. If $\mathscr{R}$ is a closed C^*-relation on $\mathscr{X}$ then there exists a function $\iota : \mathscr{X} \to U$ such that:*

(1) *ι is ubiquitous for $\mathscr{R}$ and U is a σ-C^*-algebra;*
(2) *the induced continuous $*$-homomorphism $\bar{\iota} : \mathsf{F}\langle\mathscr{X}\rangle \to U$ is onto and induces an isomorphism $U \cong \mathsf{F}\langle\mathscr{X}\rangle / I$ for $I = \ker(\bar{\iota})$;*
(3) *there is a single element g of $\mathsf{F}\langle\mathscr{X}\rangle$ so that*

$$U \cong \operatorname{pro} C^* \langle \mathscr{X} \mid g(x_1, \ldots, x_n) = 0 \rangle .$$

PROOF. Let $\mathscr{S}_n$ denote the C^*-relations

$$\|x\| \leq n \qquad (\forall x \in \mathscr{X}).$$

Then $\mathscr{S}_n$ and $\mathscr{S}_n \cap \mathscr{R}$ are compact. We get a commutative diagram

$$\begin{array}{ccc}
C^* \langle \mathscr{X} \mid \mathscr{S}_{n+1} \rangle & \longrightarrow & C^* \langle \mathscr{X} \mid \mathscr{S}_{n+1} \cap \mathscr{R} \rangle \\
\big\downarrow & & \big\downarrow \\
C^* \langle \mathscr{X} \mid \mathscr{S}_{n} \rangle & \longrightarrow & C^* \langle \mathscr{X} \mid \mathscr{S}_{n} \cap \mathscr{R} \rangle
\end{array}$$

where all the maps are induced by the identity on the generators. This is clearly a pushout with surjective $*$-homomorphisms. Let $\mathscr{U}$ be the σ-C^*-algebra

$$U = \varprojlim C^* \langle \mathscr{X} \mid \mathscr{S}_n \cap \mathscr{R} \rangle$$

and let $\iota : \mathscr{X} \to U$ denote the limit of the $\iota_n = \iota_{\mathscr{S}_n \cap \mathscr{R}}$. Theorem 5.2 applies, telling us that $\bar{\iota} : \mathsf{F}\langle \mathscr{X} \rangle \to U$ is onto.

Suppose A is a C^*-algebra and $\varphi : U \to A$ is a continuous $*$-homomorphism. By Lemma 3.4, for some n there is a $*$-homomorphism

$$\bar{\varphi} : C^* \langle \mathscr{X} \mid \mathscr{S}_n \cap R \rangle \to A$$

so that $\varphi = \bar{\varphi} \circ \rho_n$. This means that $\varphi \circ \iota = \bar{\varphi} \circ \iota_n$ is a representation of $\mathscr{R}$.

Given a C^*-algebra A and a representation $f : \mathscr{X} \to A$, for some n we have $\| f(x) \| \le n$ for all x in $\mathscr{X}$ and so have a $*$-homomorphism

$$\varphi_n : C^* \langle \mathscr{X} \mid \mathscr{S}_n \cap R \rangle \to A$$

for which $f = \varphi_n \circ \iota_{\mathscr{S}_n \cap R}$. Therefore $f = (\varphi_n \circ \rho_n) \circ \iota$. Uniqueness follows since $\iota(\mathscr{X})$ generates U.

By [14, Corollary 5.4] we have an isomorphism $U \cong \mathsf{F}\langle \mathscr{X} \rangle / I$ for $I = \ker(\bar{\iota})$.

To prove (3) we modify a technique from [5, Theorem 2.1] and [8, Proposition 41]. Certainly

$$U \cong \operatorname{pro} C^* \langle \mathscr{X} \mid g(x_1, \ldots, x_n) = 0 \ (\forall g \in I) \rangle .$$

By the separability of $\mathscr{F}\langle \mathscr{X} \rangle$ we may replace all the elements of I with a sequence so that

$$U \cong \operatorname{pro} C^* \langle \mathscr{X} \mid g_k(x_1, \ldots, x_n) = 0 \ (\forall k \in \mathsf{N}) \rangle .$$

The fact that $y^* y = 0$ in a C^*-algebra if and only if $y = 0$ allows us to replace the g_k as needed to ensure the g_k are positive elements in I. Let p_n be a sequence of C^*-seminorms defining the topology on I. Taking a sequence of positive scalars α_k so that $\alpha_k \le \left(2^k p_r(g_k) \right)^{-1}$ for $1 \le r \le k$ we can ensure that $g = \sum \alpha_k g_k$ exists, and then

$$U \cong \operatorname{pro} C^* \langle \mathscr{X} \mid g(x_1, \ldots, x_n) = 0 \rangle .$$

7. Matrices having C^*-Relations

We will use $\tilde{A}$ to denote the unitization of a C^*-algebra A, where a unit is adjoined even when A is unital. The adjoined unit is denoted $\mathbf{1}$, and 1 denotes the original unit, when it exists.

In studying the boundary maps in K-theory ([11], [12]) we proved the projectivity of the C^*-algebras

$$C^*\left\langle h, k, x \;\middle|\; P^2 = P^* = P \text{ for } P = \begin{bmatrix} \mathbf{1} - h & x^* \\ x & k \end{bmatrix} \right\rangle \tag{1}$$

and

$$C^*\left\langle h, k, x \;\middle|\; hk = 0 \text{ and } 0 \leq P \leq 1 \text{ for } P = \begin{bmatrix} \mathbf{1} - h & x^* \\ x & k \end{bmatrix} \right\rangle \tag{2}$$

and, implicitly at least, also

$$C^*\left\langle h, k, x \;\middle|\; 0 \leq P \leq 1 \text{ for } P = \begin{bmatrix} \mathbf{1} - h & x^* \\ x & k \end{bmatrix} \right\rangle. \tag{3}$$

It may not be obvious these C^*-algebras exist. They do, and there is a general method to reinterpret C^*-relations in $\mathbf{M}_n(\tilde{B})$ as C^*-relations in B.

We are adding a chapter to an old story whose beginnings include [2, §7] by Bergman and [4] by Larry Brown. In the nonunital case, we cannot use a trick with free products and relative commutants. We must face the universal nonsense.

In this section n is a positive integer.

Notation 7.1. Let $\overline{n} = \{1, 2, \ldots, n\}$.

Definition 7.2. Suppose $\mathscr{R}$ is a C^*-relation on $\mathscr{X}$ and that $\alpha : \mathscr{X} \to \mathbf{M}_n(\mathbb{C})$ is a representation of $\mathscr{R}$. Define $\mathscr{R}_\alpha$ as the full subcategory of $\mathscr{F}_{\mathscr{X}\times\overline{n}\times\overline{n}}$ whose objects are the functions

$$f : \mathscr{X} \times \overline{n} \times \overline{n} \to B$$

for which $f_\alpha : \mathscr{X} \to \mathbf{M}_n(\widetilde{B})$ is a representation of $\mathscr{R}$, where

$$f_\alpha(x) = \sum_{i,j} \left(\alpha_{ij}\mathbf{1} + f(x, i, j)\right) \otimes e_{ij}.$$

For example, in (1) $\mathscr{R}$ is the relation $p^2 = p^* = p$ and

$$\alpha(p) = \begin{bmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{bmatrix}.$$

The generator $(p, 1, 2)$ is redundant, $h = -(p, 1, 1)$, $k = (p, 2, 2)$ and $x = (p, 2, 1)$.

LEMMA 7.3. *With $\mathscr{R}$ and α as in Definition 7.2, $\mathscr{R}_\alpha$ is a C^*-relation on $\mathscr{X} \times \overline{n} \times \overline{n}$. It is compact when $\mathscr{R}$ is compact. It is closed when $\mathscr{R}$ is closed.*

PROOF. Suppose first that $\mathscr{R}$ is any C^*-relation on $\mathscr{X}$.

If f is the zero map

$$f : \mathscr{X} \times \overline{n} \times \overline{n} \to \{0\}$$

then $f_\alpha = \alpha$ is a representation of $\mathscr{R}$ so f is a representation of $\mathscr{R}_\alpha$.

Suppose $\varphi : A \to B$ is an injective $*$-homomorphism and

$$\varphi \circ f : \mathscr{X} \times \overline{n} \times \overline{n} \to B$$

is a representation in $\mathscr{R}_\alpha$. Then $\mathbf{M}_n(\widetilde{\varphi}) = \widetilde{\varphi} \otimes \mathrm{id}$ is also an injective $*$-homomorphism,

$$\mathbf{M}_n(\widetilde{\varphi}) : \mathbf{M}_n\left(\widetilde{A}\right) \to \mathbf{M}_n\left(\widetilde{B}\right)$$

and

$$\begin{aligned}(\varphi \circ f)_\alpha(x) &= \sum_{i,j} \left(\alpha_{ij}\mathbf{1} + \varphi(f(x, i, j))\right) \otimes e_{ij} \\ &= \mathbf{M}_n(\widetilde{\varphi}) \left(\sum_{i,j} \left(\alpha_{ij}\mathbf{1} + f(x, i, j)\right) \otimes e_{ij} \right) \\ &= \left(\mathbf{M}_n(\widetilde{\varphi}) \circ f_\alpha\right)(x).\end{aligned}$$

Since $\varphi \circ f$ is a representation of $\mathscr{R}$, we know $\mathbf{M}_n(\widetilde{\varphi}) \circ f_\alpha$ is a representation of $\mathscr{R}_\alpha$. Therefore f_α is a representation of $\mathscr{R}_\alpha$ and so f is a representation of $\mathscr{R}$.

Suppose $\varphi : A \to B$ is a $*$-homomorphism and

$$f : \mathscr{X} \times \overline{n} \times \overline{n} \to A$$

is a representation in $\mathscr{R}_\alpha$. Then we still have that $\mathbf{M}_n(\widetilde{\varphi})$ is a $*$-homomorphism and

$$(\varphi \circ f)_\alpha = \mathbf{M}_n(\widetilde{\varphi}) \circ f_\alpha.$$

Since f is a representation, so is f_α. Therefore $(\varphi \circ f)_\alpha$ is a representation, and so $\varphi \circ f$ is a representation.

Now suppose

$$f_\lambda : \mathscr{X} \times \overline{n} \times \overline{n} \to A_\lambda$$

is a representation for each λ in a nonempty, finite set Λ. Each $(f_\lambda)_\alpha$ is a representation. Let

$$\Phi : \mathbf{M}_n\left(\left(\prod_\lambda A_\lambda\right)^{\sim}\right) \to \prod_\lambda \mathbf{M}_n\left(\widetilde{A_\lambda}\right)$$

be the injective $*$-homomorphism defined by

$$\Phi\left((\beta\mathbf{1} + \langle a_\lambda\rangle_\lambda) \otimes e_{ij}\right) = \left\langle(\beta\mathbf{1} + a_\lambda) \otimes e_{ij}\right\rangle_\lambda .$$

Since

$$\Phi \circ \left(\prod_\lambda f_\lambda\right)_\alpha = \prod_\lambda (f_\lambda)_\alpha$$

we know that

$$\left(\prod_\lambda f_\lambda\right)_\alpha$$

is a representation. This means $\prod_\lambda f_\lambda$ is a representation.

If $\mathscr{R}$ is compact, then the above argument works for infinite sets Λ. If $\mathscr{R}$ is only closed, we need to add the assumptions

$$\sup_\lambda \|f_\lambda(x, i, j)\| < \infty$$

for each x and each i and j. This forces, for each x,

$$\sup_\lambda \left\|(f_\lambda)_\alpha(x)\right\| = \sup_\lambda \left\| \sum_{i,j} \left(\alpha_{ij}\mathbf{1} + f_\lambda(x, i, j)\right) \otimes e_{ij} \right\| < \infty$$

and the above argument is still fine.

This is helpful even when n is 1. For example there is

$$C_0(0, 1) = C^*\left\langle x \mid (\mathbf{1} + x)^* = (\mathbf{1} + x)^{-1}\right\rangle .$$

For an example that does not produce a C^*-algebra, there is

$$\operatorname{pro} C^*\left\langle a, b, c, d \,\middle|\, P^2 = P \ \text{ for } \ P = \begin{bmatrix} \mathbf{1} + a & b \\ c & d \end{bmatrix}\right\rangle .$$

In these two examples it is easy rewrite the relations as $*$-polynomials not involving matrices. Such a reduction is not always practical, as illustrated by

$$C^*\left\langle a, b, c, d \,\middle|\, 0 \le P \le 1 \ \text{ for } \ P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right\rangle .$$

Define $\lambda : \widetilde{A} \to \mathsf{C}$ by $\lambda(\alpha\mathbf{1} + a) = \alpha$.

DEFINITION 7.4. If A is a C^*-algebra and $\alpha : A \to \mathbf{M}_n(\mathsf{C})$ is a $*$-homomorphism, define $\mathbf{W}_\alpha(A)$ as

$$C^*\left\langle A \times \overline{n} \times \overline{n} \;\middle|\; a \mapsto \left[\alpha_{ij}\mathbf{1} + (a, i, j)\right] \text{ is a } *\text{-homomorphism}\right\rangle.$$

That is, $\mathbf{W}_\alpha(A)$ has a $*$-homomorphism $\iota : A \to \mathbf{M}_n(\widetilde{A})$ so that $\mathbf{M}_n(\lambda) \circ \iota = \alpha$ that is universal for all $*$-homomorphisms $\varphi : A \to \mathbf{M}_n(\widetilde{B})$ such that $\mathbf{M}_n(\lambda) \circ \varphi = \alpha$. If $\alpha = 0$ then

$$\hom(\mathbf{W}_\alpha(A), B) \cong \hom(A, \mathbf{M}_n(B))$$

and $\mathbf{W}_\alpha = \mathbf{W}_n$ is the left-adjoint to the functor $\mathbf{M}_n$. This was investigated by Phillips in [16].

THEOREM 7.5. *If A is projective and $\alpha : A \to \mathbf{M}_n(\mathsf{C})$ is a representation then $\mathbf{W}_\alpha(A)$ is projective.*

PROOF. Suppose we have a diagram

$$\begin{tikzcd} & B \arrow[d, two heads, "\rho"] \\ \mathbf{W}_\alpha(A) \arrow[r, "\varphi"] \arrow[ur, dashed, "\bar{\varphi}"] & D \end{tikzcd}$$

in which ρ is surjective, φ is given and we want to find $\bar{\varphi}$ making the diagram commute. This translates to the lifting problem

$$\begin{tikzcd} & \mathbf{M}_n(\widetilde{B}) \arrow[d, two heads] \arrow[ddr, bend left, "\mathbf{M}_n(\lambda)"] & \\ A \arrow[r] \arrow[ur, dashed] \arrow[drr, bend right, "\alpha"] & \mathbf{M}_n(\widetilde{D}) \arrow[dr, "\mathbf{M}_n(\lambda)"] & \\ & & \mathbf{M}_n(\mathsf{C}) \end{tikzcd}$$

which is easily solved.

For example, Theorem 7.5 tells us that

$$C^*\left\langle a, b, c, d \;\middle|\; \left\| \begin{bmatrix} a & \mathbf{1} + b \\ c & d \end{bmatrix} \right\| \leq 1 \right\rangle$$

is projective.

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