A RESULT ON FRACTIONAL $k$-DELETED GRAPHS

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Abstract
Let $k \geq 2$ be an integer, and let $G$ be a graph of order $n$ with $n \geq 4k - 5$. A graph $G$ is a fractional $k$-deleted graph if there exists a fractional $k$-factor after deleting any edge of $G$. The binding number of $G$ is defined as

$$\text{bind}(G) = \min \left\{ \frac{|N_G(X)|}{|X|} : \emptyset \neq X \subseteq V(G), N_G(X) \neq V(G) \right\}.$$

In this paper, it is proved that if $\text{bind}(G) > \frac{(2k-1)(n-1)}{k(n-2)},$ then $G$ is a fractional $k$-deleted graph. Furthermore, it is shown that the result in this paper is best possible in some sense.

1. Introduction

We consider only finite undirected simple graph $G$ with vertex set $V(G)$ and edge set $E(G)$. For $x \in V(G)$, we use $N_G(x)$ for the set of vertices of $V(G)$ adjacent to $x$, and $d_G(x)$ for the degree of $x$ in $G$. The minimum vertex degree of $G$ is denoted by $\delta(G)$. For any $S \subseteq V(G)$, we define $N_G(S) = \bigcup_{x \in S} N_G(x)$. We denote by $G[S]$ the subgraph of $G$ induced by $S$, by $G - S$ the subgraph obtained from $G$ by deleting vertices in $S$ together with the edges incident to vertices in $S$. A vertex set $S \subseteq V(G)$ is called independent if $G[S]$ has no edges. The binding number of $G$ is defined as

$$\text{bind}(G) = \min \left\{ \frac{|N_G(X)|}{|X|} : \emptyset \neq X \subseteq V(G), N_G(X) \neq V(G) \right\}.$$

Let $k$ be an integer such that $k \geq 1$. Then a spanning subgraph $F$ of $G$ is called a $k$-factor if $d_F(x) = k$ for all $x \in V(G)$. A fractional $k$-factor is a function $h$ that assigns to each edge of a graph $G$ a number in $[0, 1]$, so that for each vertex $x$ we have $d_G^h(x) = k$, where $d_G^h(x) = \sum_{e \ni x} h(e)$ (the sum is taken over all edges incident to $x$) is a fractional degree of $x$ in $G$. A graph $G$ is a fractional $k$-deleted graph if there exists a fractional $k$-factor after deleting any edge of $G$. The other terminologies and notations not given in this paper can be found in [1] and [10].

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Many authors have investigated factors [2], [3], [4], [7], [9], [12], and fractional factors [8]. Li, Yan and Zhang gave a necessary and sufficient condition for a graph to be a fractional \(k\)-deleted graph [5]. Li, Zhang and Yan showed a sufficient condition for a graph to be a fractional \(k\)-deleted graph [6]. Recently, Zhou and Duan obtained a sufficient condition for a graph to be a fractional \(k\)-deleted graph [13]. In this paper, we give a new sufficient condition for a graph to be a fractional \(k\)-deleted graph.

The following results on fractional \(k\)-deleted graphs are known.

**Theorem 1.1** ([6]). Let \(G\) be a graph, and let \(k \geq 2\) be an integer. If \(\delta(G) \geq k + 1\) and \(I(G) > k\), then \(G\) is a fractional \(k\)-deleted graph.

**Theorem 1.2** ([13]). Let \(G\) be a graph. Then \(G\) is a fractional \(2\)-deleted graph if \(\delta(G) \geq 3\) and \(\text{bind}(G) \geq 2\).

We prove the following theorem for a graph to be a fractional \(k\)-deleted graph, which is an extension of Theorem 1.2.

**Theorem 1.3.** Let \(k \geq 2\) be an integer, and let \(G\) be a graph of order \(n\) with \(n \geq 4k - 5\). If
\[
\text{bind}(G) > \frac{(2k - 1)(n - 1)}{k(n - 2)},
\]
then \(G\) is a fractional \(k\)-deleted graph.

The following two results are essential to the proof of Theorem 1.3.

**Theorem 1.4** ([5]). A graph \(G\) is a fractional \(k\)-deleted graph if and only if for any \(S \subseteq V(G)\) and \(T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq k\}\)
\[
\delta_G(S, T) = k|S| + d_{G-S}(T) - k|T| \geq \varepsilon(S, T),
\]
where \(\varepsilon(S, T)\) is defined as follows,
\[
\varepsilon(S, T) = \begin{cases} 
2, & \text{if } T \text{ is not independent}, \\
1, & \text{if } T \text{ is independent, and } e_G(T, V(G) \setminus (S \cup T)) \geq 1, \\
0, & \text{otherwise}.
\end{cases}
\]

**Theorem 1.5** ([11]). Let \(G\) be a graph of order \(n\) with \(\text{bind}(G) > c\). Then \(\delta(G) > n - \frac{n - 1}{c}\).

2. **Proof of Theorem 1.3**

**Proof.** Suppose that \(G\) satisfies the assumption of the theorem, but it is not a fractional \(k\)-deleted graph. Then by Theorem 1.4, there exist some \(S \subseteq V(G)\) and \(T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq k\}\) such that
\[
\delta_G(S, T) = k|S| + d_{G-S}(T) - k|T| \leq \varepsilon(S, T) - 1.
\]
Claim 1. \(|T| \geq k + 1\).

Proof. In view of Theorem 1.5, we have

\[
|S| + d_{G-S}(x) \geq d_G(x) \geq \delta(G) > n - \frac{n - 1}{(2k-1)(n-1)} \times \frac{k(n-2)}{k(n-2)}
\]

\[
\geq \frac{2k-1}{2k-1} = \frac{(k-1)n + 2k}{2k-1}
\]

If \(k \geq 3\), then according to the integrity of \(\delta(G)\) we obtain

\[
|S| + d_{G-S}(x) \geq \delta(G) \geq 2k - 2.
\]

If \(k = 2\), then by the integrity of \(\delta(G)\) we get

\[
|S| + d_{G-S}(x) \geq \delta(G) \geq 2k - 1.
\]

Let \(|T| \leq k\) and \(k \geq 3\), then by (1) and (2), we have

\[
\varepsilon(S, T) - 1 \geq \delta_G(S, T) = k|S| + d_{G-S}(T) - k|T| \geq |T||S| + d_{G-S}(T) - k|T|
\]

\[
= \sum_{x \in T} (|S| + d_{G-S}(x) - k) \geq \sum_{x \in T} (2k-2 - k)
\]

\[
= \sum_{x \in T} (k - 2) = (k - 2)|T| \geq |T| \geq \varepsilon(S, T),
\]

which is a contradiction.

Let \(|T| \leq k\) and \(k = 2\), then by (1) and (3), we have

\[
\varepsilon(S, T) - 1 \geq \delta_G(S, T) = k|S| + d_{G-S}(T) - k|T| \geq |T||S| + d_{G-S}(T) - k|T|
\]

\[
= \sum_{x \in T} (|S| + d_{G-S}(x) - k) \geq \sum_{x \in T} (2k-1 - k)
\]

\[
= \sum_{x \in T} (k - 1) = (k - 1)|T| = |T| \geq \varepsilon(S, T),
\]

a contradiction.

Claim 2. \(S \neq \emptyset\).
Proof. Let $S = \emptyset$. If $k \geq 3$, then by (1) and (2) we get that

$$
\varepsilon(S, T) - 1 \geq \delta_G(S, T) = k|S| + d_{G-S}(T) - k|T|
$$

$$
= d_G(T) - k|T| \geq (\delta(G) - k)|T|
$$

$$
\geq (2k - 2 - k)|T| = (k - 2)|T| \geq |T| \geq \varepsilon(S, T),
$$

this is a contradiction.

If $k = 2$, then by (1) and (3) we have

$$
\varepsilon(S, T) - 1 \geq \delta_G(S, T) = k|S| + d_{G-S}(T) - k|T|
$$

$$
= d_G(T) - k|T| \geq (\delta(G) - k)|T|
$$

$$
\geq (2k - 1 - k)|T| = (k - 1)|T| = |T| \geq \varepsilon(S, T),
$$

which is a contradiction.

Claim 3. There exists $x \in T$ such that $d_{G-S}(x) \leq k - 1$.

Proof. If $d_{G-S}(x) \geq k$ for all $x \in T$, then we get from Claim 2

$$
\delta_G(S, T) = k|S| + d_{G-S}(T) - k|T| \geq k|S| \geq k \geq 2 \geq \varepsilon(S, T),
$$

which contradicts (1).

Define

$$
h = \min\{d_{G-S}(x) \mid x \in T\}.
$$

Then by Claim 3, we obtain

$$
0 \leq h \leq k - 1.
$$

By Theorem 1.5 and $\delta(G) \leq |S| + h$, we get

$$
|S| \geq \delta(G) - h > n - \frac{k(n - 2)}{2k - 1} - h = \frac{(k - 1)n + 2k}{2k - 1} - h.
$$

The proof splits into two cases.

Case 1. $h = 0$.

First, we prove the following claim.

Claim 4. $\frac{k(n-2)}{n-1} \geq 1$.

Proof. In view of $k \geq 2$ and $n \geq 4k - 5$, we get

$$
k(n - 2) - (n - 1) = (k - 1)(n - 2) - 1 \geq 0.
$$
Thus, we obtain
\[ \frac{k(n - 2)}{n - 1} \geq 1. \]

Let \( m \) be the number of vertices \( x \) in \( T \) such that \( d_{G - S}(x) = 0 \), and let \( Y = V(G) \setminus S \). Then \( N_G(Y) \neq V(G) \) since \( h = 0 \), and \( Y \neq \emptyset \) by Claim 1, and so \( |N_G(Y)| \geq \text{bind}(G)|Y| \). Thus
\[ n - m \geq |N_G(Y)| \geq \text{bind}(G)|Y| = \text{bind}(G)(n - |S|), \]
that is,
\[ |S| \geq n - \frac{n - m}{\text{bind}(G)} > n - \frac{k(n - 2)(n - m)}{(2k - 1)(n - 1)}. \]

According to (1), (5), Claim 4 and \( |T| \leq n - |S| \), we have
\[ \varepsilon(S, T) - 1 \geq \delta_G(S, T) = k|S| + d_{G - S}(T) - k|T| \geq k|S| - k|T| + |T| - m \geq k|S| - (k - 1)(n - |S|) - m = (2k - 1)|S| - kn + n - m > (2k - 1)\left(n - \frac{k(n - 2)(n - m)}{(2k - 1)(n - 1)}\right) - kn + n - m = kn - \frac{k(n - 2)(n - m)}{n - 1} - m \geq kn - \frac{k(n - 2)(n - 1)}{n - 1} - 1 = kn - k(n - 2) - 1 = 2k - 1 > 2 \geq \varepsilon(S, T). \]

This is a contradiction.

**Case 2.** \( 1 \leq h \leq k - 1 \).

In view of Claim 1, we obtain
\[ |T| \geq k + 1 > h + 1. \]

Let \( v \) be a vertex in \( T \) such that \( d_{G - S}(v) = h \), and put \( Y = T - N_{G - S}(v) \). Then \( |Y| \geq |T| - h > 1 \) and \( N_G(Y) \neq V(G) \). Thus, we get
\[ \frac{n - 1}{|T| - h} \geq \frac{|N_G(Y)|}{|Y|} \geq \text{bind}(G) > \frac{(2k - 1)(n - 1)}{k(n - 2)}. \]
that is,

\[
|T| < \frac{k(n-2)}{2k-1} + h.
\]

By (4) and (6), we have

\[
\delta_G(S, T) = k|S| + d_{G-S}(T) - k|T|
\]
\[
\geq k|S| - k|T| + h|T| = k|S| - (k-h)|T|
\]
\[
> k \left( \frac{(k-1)n + 2k}{2k-1} - h \right) - (k-h) \left( \frac{k(n-2)}{2k-1} + h \right).
\]

### Subcase 2.1. \( h = 1 \).

Obviously, we obtain

\[
\delta_G(S, T) > k \left( \frac{(k-1)n + 2k}{2k-1} - 1 \right) - (k-1) \left( \frac{k(n-2)}{2k-1} + 1 \right)
\]
\[
= k \cdot \frac{(k-1)n + 1}{2k-1} - (k-1) \cdot \frac{kn - 1}{2k-1} = \frac{2k - 1}{2k-1} = 1.
\]

According to the integrity of \( \delta_G(S, T) \), we get that

\[
\delta_G(S, T) \geq 2 \geq \varepsilon(S, T),
\]

this contradicts (1).

### Subcase 2.2. \( 2 \leq h \leq k-1 \).

Clearly, \( k \geq 3 \). Let \( f(h) = k \left( \frac{(k-1)n + 2k}{2k-1} - h \right) - (k-h)\left( \frac{k(n-2)}{2k-1} + h \right) \). Then

\[
\delta_G(S, T) > f(h),
\]

and

\[
f'(h) = -2k + 2h + \frac{k(n-2)}{2k-1}.
\]

Since \( 2 \leq h \leq k-1 \) and \( n \geq 4k-5 \), we have

\[
f'(h) \geq -2k + 4 + \frac{k(n-2)}{2k-1} = \frac{-4k^2 + 2k + 8k - 4 + kn - 2k}{2k-1}
\]
\[
= \frac{kn - 4k^2 + 8k - 4}{2k-1} \geq \frac{k(4k-5) - 4k^2 + 8k - 4}{2k-1} = \frac{3k - 4}{2k-1} > 0.
\]

Thus, we get

\[
f(h) \geq f(2).
\]
From (7), (8) and \( k \geq 3 \), we obtain

\[
\delta_G(S, T) > f(h) \geq f(2)
\]

\[
= k \left( \frac{(k-1)n+2k}{2k-1} - 2 \right) - (k-2) \left( \frac{k(n-2)}{2k-1} + 2 \right)
\]

\[
= \frac{k(k-1)n+2k^2-4k^2+2k-k(k-2)n-2k^2+6k-4}{2k-1}
\]

\[
= \frac{kn-4k^2+8k-4}{2k-1} \geq \frac{k(4k-5)-4k^2+8k-4}{2k-1}
\]

\[
= \frac{3k-4}{2k-1} = 1 + \frac{k-3}{2k-1} \geq 1.
\]

By the integrity of \( \delta_G(S, T) \), we have

\[
\delta_G(S, T) \geq 2 \geq \varepsilon(S, T),
\]

which contradicts (1).

From all the cases above, we deduced the contradiction. Hence, \( G \) is a fractional \( k \)-deleted graph. This completes the proof of Theorem 1.3.

**Remark 1.** Let us show that the condition \( \text{bind}(G) > \frac{(2k-1)(n-1)}{k(n-2)} \) in Theorem 1.3 cannot be replaced by \( \text{bind}(G) \geq \frac{(2k-1)(n-1)}{k(n-2)} \). Let \( r \geq 1, k \geq 3 \) be two odd positive integer and let \( l = \frac{5kr+1}{2} \) and \( m = 5kr - 5r + 1 \), so that \( n = m + 2l = 10kr - 5r + 2 \). Let \( H = K_m \cup lK_2 \) and \( X = V(lK_2) \). Then for any \( x \in X \), \( |N_H(X \setminus x)| = n - 1 \). By the definition of \( \text{bind}(H) \),

\[
\text{bind}(H) = \left| \frac{|N_H(X\setminus x)|}{|X\setminus x|} \right| = \frac{n-1}{2l-1} = \frac{n-1}{5kr} = \frac{(2k-1)(n-1)}{k(n-2)}.
\]

Let \( S = V(K_m) \subseteq V(H) \), \( T = V(lK_2) \subseteq V(H) \). Then \( |S| = m, |T| = 2l \). Obviously, \( T \) is not independent, then \( \varepsilon(S, T) = 2 \). Thus, we obtain

\[
\delta_H(S, T) = k|S| - k|T| + d_{H-S}(T)
\]

\[
= k|S| - k|T| + |T| = k|S| - (k-1)|T|
\]

\[
= km - 2(k-1)l = k(5kr - 5r + 1) - (k-1)(5kr + 1)
\]

\[
= 1 < 2 = \varepsilon(S, T).
\]

By Theorem 1.4, \( H \) is not a fractional \( k \)-deleted graph. In the above sense, the result in Theorem 1.3 is best possible.

**Remark 2.** We don’t know whether the result can be strengthened to the form that if \( \text{bind}(G) > \frac{(2k-1)(n-1)}{k(n-2)} \) then \( G \) is \( k \)-deleted. We guess that the above result can hold for \( kn \) even.
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