MATH. SCAND. 105 (2009), 161–170

COFINITENESS AND COASSOCIATED PRIMES OF LOCAL COHOMOLOGY MODULES

MOHARRAM AGHAPOURNAHR and LEIF MELKERSSON

Abstract

Let R be a noetherian ring, $\mathfrak{a}$ an ideal of R such that $\dim R/\mathfrak{a} = 1$ and M a finite R-module. We will study cofiniteness and some other properties of the local cohomology modules $\mathrm{H}^i_{\mathfrak{a}}(M)$. For an arbitrary ideal $\mathfrak{a}$ and an R-module M (not necessarily finite), we will characterize $\mathfrak{a}$-cofinite artinian local cohomology modules. Certain sets of coassociated primes of top local cohomology modules over local rings are characterized.

1. Introduction

Throughout R is a commutative noetherian ring. By a finite module we mean a finitely generated module. For basic facts about commutative algebra see [3] and [9] and for local cohomology we refer to [2].

Grothendieck [7] made the following conjecture:

Conjecture. *For every ideal $\mathfrak{a}$ and every finite R-module M, the module $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^n_{\mathfrak{a}}(M))$ is finite for all n.*

Hartshorne [8] showed that this is false in general. However, he defined an R-module M to be $\mathfrak{a}$-*cofinite* if $\mathrm{Supp}_R(M) \subset \mathrm{V}(\mathfrak{a})$ and $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ is finite (finitely generated) for each i and he asked the following question:

Question. If $\mathfrak{a}$ is an ideal of R and M is a finite R-module. When is $\mathrm{Ext}^i_R(R/\mathfrak{a}, \mathrm{H}^j_{\mathfrak{a}}(M))$ finite for every i and j?

Hartshorne [8] showed that if $(R, \mathfrak{m})$ is a complete regular local ring and M a finite R-module, then $\mathrm{H}^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite in two cases:

(a) If $\mathfrak{a}$ is a nonzero principal ideal, and

(b) If $\mathfrak{a}$ is a prime ideal with $\dim R/\mathfrak{a} = 1$.

Yoshida [14] and Delfino and Marley [4] extended (b) to all dimension one ideals $\mathfrak{a}$ of an arbitrary local ring R.

In Corollary 2.3, we give a characterization of the $\mathfrak{a}$-cofiniteness of these local cohomology modules when $\mathfrak{a}$ is a one-dimensional ideal in a non-local

Received September 19, 2007, in final form October 14, 2008.

ring. In this situation we also prove in Theorem 2.7, that these local cohohomology modules always belong to a class introduced by Zöschinger in [16].

Our main result in this paper is Theorem 2.10, where we for an arbitrary ideal $\mathfrak{a}$ and an R-module M (not necessarily finite), characterize the artinian $\mathfrak{a}$-cofinite local cohomology modules (in the range $i < n$). With the additional assumption that M is finitely generated, the characterization is also given by the existence of certain filter-regular sequences.

The second author has in [10, Theorem 5.5] previously characterized artinian local cohomology modules (in the same range). In case the module M is not supposed to be finite, the two notions differ. For example let $\mathfrak{a}$ be an ideal of a local ring R, such that $\dim(R/\mathfrak{a}) > 0$ and let M be the injective hull of the residue field of R. The module $\mathrm{H}^0_{\mathfrak{a}}(M)$, which is equal to M, is artinian. However it is not $\mathfrak{a}$-cofinite, since $0 \underset{M}{:} \mathfrak{a}$ does not have finite length.

An R-module M has *finite Goldie dimension* if M contains no infinite direct sum of submodules. For a commutative noetherian ring this can be expressed in two other ways, namely that the injective hull $\mathrm{E}(M)$ of M decomposes as a finite direct sum of indecomposable injective modules or that M is an essential extension of a finite submodule.

A prime ideal $\mathfrak{p}$ is said to be *coassociated* to M if $\mathfrak{p} = \mathrm{Ann}_R(M/N)$ for some $N \subset M$ such that M/N is artinian and is said to be *attached* to M if $\mathfrak{p} = \mathrm{Ann}_R(M/N)$ for some arbitrary submodule N of M, equivalently $\mathfrak{p} = \mathrm{Ann}_R(M/\mathfrak{p}M)$. The set of these prime ideals are denoted by $\mathrm{Coass}_R(M)$ and $\mathrm{Att}_R(M)$ respectively. Thus $\mathrm{Coass}_R(M) \subset \mathrm{Att}_R(M)$ and the two sets are equal when M is an artinian module. The two sets behave well with respect to exact sequences. If $0 \to M' \to M \to M'' \to 0$ is an exact sequence, then

$$\mathrm{Coass}_R(M'') \subset \mathrm{Coass}_R(M) \subset \mathrm{Coass}_R(M') \cup \mathrm{Coass}_R(M'')$$

and

$$\mathrm{Att}_R(M'') \subset \mathrm{Att}_R(M) \subset \mathrm{Att}_R(M') \cup \mathrm{Att}_R(M'').$$

There are equalities $\mathrm{Coass}_R(M \otimes_R N) = \mathrm{Coass}_R(M) \cap \mathrm{Supp}_R(N)$ and $\mathrm{Att}_R(M \otimes_R N) = \mathrm{Att}_R(M) \cap \mathrm{Supp}_R(N)$, whenever the module N is required to be finite. We prove the second equality in Lemma 2.11. In particular $\mathrm{Coass}_R(M/\mathfrak{a}M) = \mathrm{Coass}_R(M) \cap \mathrm{V}(\mathfrak{a})$ and $\mathrm{Att}_R(M/\mathfrak{a}M) = \mathrm{Att}_R(M) \cap \mathrm{V}(\mathfrak{a})$ for every ideal $\mathfrak{a}$. Coassociated and attached prime ideals have been studied in particular by Zöschinger, [17] and [18].

In Corollary 2.13 we give a characterization of certain sets of coassociated primes of the highest nonvanishing local cohomology module $\mathrm{H}^t_{\mathfrak{a}}(M)$, where M is a finitely generated module over a complete local ring. In case it happens that $t = \dim M$, the characterization is given in [4, Lemma 3]. In that case the top local cohomology module is always artinian, but in general the top local cohomology module is not artinian if $t < \dim M$.

2. Main results

First we extend a result by Zöschinger [15, Lemma 1.3] with a much weaker condition. Our method of proof is also quite different.

PROPOSITION 2.1. *Let M be a module over the noetherian ring R. The following statements are equivalent:*

(i) *M is a finite R-module.*

(ii) *$M_{\mathfrak{m}}$ is a finite $R_{\mathfrak{m}}$-module for all $\mathfrak{m} \in \operatorname{Max} R$ and $\operatorname{Min}_R(M/N)$ is a finite set for all finite submodules $N \subset M$.*

PROOF. The only nontrivial part is (ii) $\Rightarrow$ (i).

Let $\mathscr{F}$ be the set of finite submodules of M. For each $N \in \mathscr{F}$ the set $\operatorname{Supp}_R(M/N)$ is closed in $\operatorname{Spec}(R)$, since $\operatorname{Min}_R(M/N)$ is a finite set. Also it follows from the hypothesis that, for each $\mathfrak{p} \in \operatorname{Spec}(R)$ there is $N \in \mathscr{F}$ such that $M_{\mathfrak{p}} = N_{\mathfrak{p}}$, that is $\mathfrak{p} \notin \operatorname{Supp}_R(M/N)$. This means that $\bigcap_{N \in \mathscr{F}} \operatorname{Supp}_R(M/N) = \varnothing$. Now $\operatorname{Spec}(R)$ is a quasi-compact topological space. Consequently $\bigcap_{i=1}^{r} \operatorname{Supp}_R(M/N_i) = \varnothing$ for some $N_1, \ldots, N_r \in \mathscr{F}$. We claim that $M = N$, where $N = \sum_{i=1}^{r} N_i$. Just observe that $\operatorname{Supp}_R(M/N) \subset \operatorname{Supp}_R(M/N_i)$ for each i, and therefore $\operatorname{Supp}_R(M/N) = \varnothing$.

COROLLARY 2.2. *Let M be an R-module such that $\operatorname{Supp} M \subset V(\mathfrak{a})$ and $M_{\mathfrak{m}}$ is $\mathfrak{a}R_{\mathfrak{m}}$-cofinite for each maximal ideal $\mathfrak{m}$. The following statements are equivalent:*

(i) *M is $\mathfrak{a}$-cofinite.*

(ii) *For all j, $\operatorname{Min}_R(\operatorname{Ext}^j_R(R/\mathfrak{a}, M)/T)$ is a finite set for each finite submodule T of $\operatorname{Ext}^j_R(R/\mathfrak{a}, M)$.*

PROOF. The only nontrivial part is (ii) $\Rightarrow$ (i).

Suppose $\mathfrak{m}$ is a maximal ideal of R. By hypothesis $M_{\mathfrak{m}}$ is $\mathfrak{a}R_{\mathfrak{m}}$-cofinite. Therefore $\operatorname{Ext}^j_R(R/\mathfrak{a}, M)_{\mathfrak{m}}$ is a finite $R_{\mathfrak{m}}$-module for all j. Hence by Proposition 2.1 $\operatorname{Ext}^j_R(R/\mathfrak{a}, M)$ is finite for all j. Thus M is $\mathfrak{a}$-cofinite.

COROLLARY 2.3. *Let $\mathfrak{a}$ an ideal of R such that $\dim R/\mathfrak{a} = 1$, M a finite R-module and $i \geq 0$. The following statements are equivalent:*

(i) *$\mathrm{H}^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite.*

(ii) *For all j, $\operatorname{Min}_R(\operatorname{Ext}^j_R(R/\mathfrak{a}, \mathrm{H}^i_{\mathfrak{a}}(M))/T)$ is a finite set for each finite submodule T of $\operatorname{Ext}^j_R(R/\mathfrak{a}, \mathrm{H}^i_{\mathfrak{a}}(M))$.*

PROOF. For all maximal ideals $\mathfrak{m}$, $\mathrm{H}^i_{\mathfrak{a}}(M)_{\mathfrak{m}} \cong \mathrm{H}^i_{\mathfrak{a}R_{\mathfrak{m}}}(M_{\mathfrak{m}})$. By [4, Theorem 1] $\mathrm{H}^i_{\mathfrak{a}R_{\mathfrak{m}}}(M_{\mathfrak{m}})$ is $\mathfrak{a}R_{\mathfrak{m}}$-cofinite.

A module M is *weakly Laskerian*, when for each submodule N of M the quotient M/N has just finitely many associated primes, see [6]. A module M is $\mathfrak{a}$-*weakly cofinite* if $\mathrm{Supp}_R(M) \subset \mathrm{V}(\mathfrak{a})$ and $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ is weakly Laskerian for all i. Clearly each $\mathfrak{a}$-cofinite module is $\mathfrak{a}$-weakly cofinite but the converse is not true in general see [5, Example 3.5(i) and (ii)].

COROLLARY 2.4. *If* $\mathrm{H}^i_{\mathfrak{a}}(M)$ (*with* $\dim R/\mathfrak{a} = 1$) *is an* $\mathfrak{a}$*-weakly cofinite module, then it is also* $\mathfrak{a}$*-cofinite.*

Next we will introduce a subcategory of the category of R-modules that has been studied by Zöschinger in [16, Satz 1.6].

THEOREM 2.5 (Zöschinger). *For any* R*-module* M *the following are equivalent:*

(i) *M satisfies the minimal condition for submodules N such that M/N is soclefree.*

(ii) *For any descending chain $N_1 \supset N_2 \supset N_3 \supset \cdots$ of submodules of M, there is n such that the quotients N_i/N_{i+1} have support in* $\mathrm{Max}\, R$ *for all $i \geq n$.*

(iii) *With $L(M) = \bigoplus_{\mathfrak{m}\in \mathrm{Max}\, R} \Gamma_{\mathfrak{m}}(M)$, the module $M/L(M)$ has finite Goldie dimension, and* $\dim R/\mathfrak{p} \leq 1$ *for all* $\mathfrak{p} \in \mathrm{Ass}_R(M)$.

If they are fulfilled, then for each monomorphism $f : M \longrightarrow M$,

$$\mathrm{Supp}_R(\mathrm{Coker}\, f) \subset \mathrm{Max}\, R.$$

We will say that M is in the class $\mathscr{Z}$ if M satisfies the equivalent conditions in Theorem 2.5.

A module M is *soclefree* if it has no simple submodules, or in other terms $\mathrm{Ass}\, M \cap \mathrm{Max}\, R = \varnothing$. For example if M is a module over the local ring $(R, \mathfrak{m})$ then the module $M/\Gamma_{\mathfrak{m}}(M)$, where $\Gamma_{\mathfrak{m}}(M)$ is the submodule of M consisting of all elements of M annihilated by some high power $\mathfrak{m}^n$ of the maximal ideal $\mathfrak{m}$, is always soclefree.

PROPOSITION 2.6. *The class $\mathscr{Z}$ is a Serre subcategory of the category of R-modules, that is $\mathscr{Z}$ is closed under taking submodules, quotients and extensions.*

PROOF. The only difficult part is to show that $\mathscr{Z}$ is closed under taking extensions. To this end let $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ be an exact sequence with $M', M'' \in \mathscr{Z}$ and let $N_1 \supset N_2 \supset \cdots$ be a descending chain of submodules of M. Consider the descending chains $f^{-1}(N_1) \supset$

$f^{-1}(N_2) \supset \cdots$ and $g(N_1) \supset g(N_2) \supset \cdots$ of submodules of M' and M'' respectively. By (ii) there is n such that $\operatorname{Supp}_R(f^{-1}(N_i)/f^{-1}(N_{i+1})) \subset \operatorname{Max} R$ and $\operatorname{Supp}_R(g(N_i)/g(N_{i+1})) \subset \operatorname{Max} R$ for all $i \geq n$. We use the exact sequence

$$0 \longrightarrow f^{-1}(N_i)/f^{-1}(N_{i+1}) \longrightarrow N_i/N_{i+1} \longrightarrow g(N_i)/g(N_{i+1}) \longrightarrow 0.$$

to conclude that $\operatorname{Supp}_R(N_i/N_{i+1}) \subset \operatorname{Max} R$ for all $i \geq n$.

THEOREM 2.7. *Let N be a module over a noetherian ring R and $\mathfrak{a}$ an ideal of R such that* $\dim R/\mathfrak{a} = 1$. *If $N_{\mathfrak{m}}$ is $\mathfrak{a}R_{\mathfrak{m}}$-cofinite for all $\mathfrak{m} \in \operatorname{Max} R$, then N is in the class $\mathscr{X}$. In particular, if M is a finite R-module, then $\mathrm{H}^i_{\mathfrak{a}}(M)$ is in the class $\mathscr{X}$ for all i.*

PROOF. Let $X = N/L(N)$. Note that $\operatorname{Ass}_R(X) \subset \operatorname{Min} \mathfrak{a}$ and therefore is a finite set. Since

$$\mathrm{E}(X) = \bigoplus_{\mathfrak{p} \in \operatorname{Ass}_R(X)} \mathrm{E}(R/\mathfrak{p})^{\mu^i(\mathfrak{p}, X)},$$

it is enough to prove that $\mu^i(\mathfrak{p}, X)$ is finite for all $\mathfrak{p} \in \operatorname{Ass}_R(X)$. This is clear, since each $\mathfrak{p} \in \operatorname{Ass}_R(X)$ is minimal over $\mathfrak{a}$ and therefore $X_{\mathfrak{p}} \cong N_{\mathfrak{p}}$ which is, $\mathfrak{a}R_{\mathfrak{p}}$-cofinite, i.e. artinian over $R_{\mathfrak{p}}$.

Given elements $x_1, \ldots, x_r$ in R, we denote by $\mathrm{H}^i(x_1, \ldots, x_r; M)$ the i'th Koszul cohomology module of the R-module M. The following lemma is used in the proof of Theorem 2.10.

LEMMA 2.8. *Let E be an injective module. If* $\mathrm{H}^0(x_1, \ldots, x_r; E) = 0$, *then* $\mathrm{H}^i(x_1, \ldots, x_r; E) = 0$ *for all i.*

PROOF. We may assume that $E = \mathrm{E}(R/\mathfrak{p})$ for some prime ideal $\mathfrak{p}$, since E is a direct sum of modules of this form, and Koszul cohomology preserves (arbitrary) direct sums.

Put $\mathfrak{a} = (x_1, \ldots, x_r)$. By hypothesis $0 :_E \mathfrak{a} = 0$, which means that $\mathfrak{a} \not\subset \mathfrak{p}$. Take an element $s \in \mathfrak{a} \setminus \mathfrak{p}$. It acts bijectively on E, hence also on $\mathrm{H}^i(x_1, \ldots, x_r; E)$ for each i. But $\mathfrak{a} \subset \operatorname{Ann}_R(\mathrm{H}^i(x_1, \ldots, x_r; E))$ for all i, so the element s therefore acts as the zero homomorphism on each $\mathrm{H}^i(x_1, \ldots, x_r; E)$. The conclusion follows.

First we state the definition, given in [10], of the notion of filter regularity on modules (not necessarily finite) over any noetherian ring. When $(R, \mathfrak{m})$ is local and M is finite, it yields the ordinary notion of filter-regularity, see [12].

DEFINITION 2.9. Let M be a module over the noetherian ring R. An element x of R is called filter-regular on M if the module $0 :_M x$ has finite length.

A sequence $x_1, \ldots, x_s$ is said to be filter regular on M if x_j is filter-regular on $M/(x_1, \ldots, x_{j-1})M$ for $j = 1, \ldots, s$.

The following theorem yields a characterization of artinian cofinite local cohomology modules.

THEOREM 2.10. *Let $\mathfrak{a} = (x_1, \ldots, x_r)$ be an ideal of a noetherian ring R and let n be a positive integer. For each R-module M the following conditions are equivalent:*

(i) $\mathrm{H}^i_{\mathfrak{a}}(M)$ *is artinian and $\mathfrak{a}$-cofinite for all $i < n$.*

(ii) $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ *has finite length for all $i < n$.*

(iii) *The Koszul cohomology module $\mathrm{H}^i(x_1, \ldots, x_r; M)$ has finite length for all $i < n$.*

When M is finite these conditions are also equivalent to:

(iv) $\mathrm{H}^i_{\mathfrak{a}}(M)$ *is artinian for all $i < n$.*

(v) *There is a sequence of length n in $\mathfrak{a}$ that is filter-regular on M.*

PROOF. We use induction on n. When $n = 1$ the conditions (ii) and (iii) both say that $0 :_M \mathfrak{a}$ has finite length, and they are therefore equivalent to (i) [10, Proposition 4.1].

Let $n > 1$ and assume that the conditions are equivalent when n is replaced by $n - 1$. Put $L = \Gamma_{\mathfrak{a}}(M)$ and $\overline{M} = M/L$ and form the exact sequence $0 \longrightarrow L \longrightarrow M \longrightarrow \overline{M} \longrightarrow 0$. We have $\Gamma_{\mathfrak{a}}(\overline{M}) = 0$ and $\mathrm{H}^i_{\mathfrak{a}}(\overline{M}) \cong \mathrm{H}^i_{\mathfrak{a}}(M)$ for all $i > 0$. There are exact sequences

$$\mathrm{Ext}^i_R(R/\mathfrak{a}, L) \to \mathrm{Ext}^i_R(R/\mathfrak{a}, M) \to \mathrm{Ext}^i_R(R/\mathfrak{a}, \overline{M}) \to \mathrm{Ext}^{i+1}_R(R/\mathfrak{a}, L)$$

and

$$\mathrm{H}^i(x_1, \ldots, x_r; L) \to \mathrm{H}^i(x_1, \ldots, x_r; M) \to \mathrm{H}^i(x_1, \ldots, x_r; \overline{M}) \to \mathrm{H}^{i+1}(x_1, \ldots, x_r; L)$$

Because L is artinian and $\mathfrak{a}$-cofinite the outer terms of both exact sequences have finite length. Hence M satisfies one of the conditions if and only if $\overline{M}$ satisfies the same condition. We may therefore assume that $\Gamma_{\mathfrak{a}}(M) = 0$.

Let E be the injective hull of M and put $N = E/M$. Consider the exact sequence $0 \longrightarrow M \longrightarrow E \longrightarrow N \longrightarrow 0$. We know that $0 :_M \mathfrak{a} = 0$. Therefore $0 :_E \mathfrak{a} = 0$ and $\Gamma_{\mathfrak{a}}(E) = 0$. Consequently there are isomorphisms for all $i \geq 0$:

$$\mathrm{H}^{i+1}_{\mathfrak{a}}(M) \cong \mathrm{H}^i_{\mathfrak{a}}(N), \qquad \mathrm{Ext}^{i+1}_R(R/\mathfrak{a}, M) \cong \mathrm{Ext}^i_R(R/\mathfrak{a}, N)$$

and

$$\mathrm{H}^{i+1}(x_1, \ldots, x_r; M) \cong \mathrm{H}^i(x_1, \ldots, x_r; N).$$

In order to get the third isomorphism, we used that $\mathrm{H}^i(x_1,\ldots,x_r;E)=0$ for all $i\geq 0$ (Lemma 2.8). Hence M satisfies one of the three conditions if and only if N satisfies the same condition, with n replaced by $n-1$. By induction, we may therefore conclude that the module M satisfies all three conditions if it satisfies one of them.

Let now M be a finite module.

(ii) $\Leftrightarrow$ (iv) Use [10, Theorem 5.5(i) $\Leftrightarrow$ (ii)].

(v) $\Rightarrow$ (i) Use [10, Theorem 6.4].

(i) $\Rightarrow$ (v) We give a proof by induction on n. Put $L=\Gamma_{\mathfrak{a}}(M)$ and $\overline{M}=M/L$. Then $\operatorname{Ass}_R L=\operatorname{Ass}_R M\cap \mathrm{V}(\mathfrak{a})$ and $\operatorname{Ass}_R \overline{M}=\operatorname{Ass}_R M\setminus \mathrm{V}(\mathfrak{a})$. The module L has finite length and therefore $\operatorname{Ass}_R L\subset \operatorname{Max} R$. By prime avoidance take an element $y_1\in\mathfrak{a}\setminus\bigcup_{\mathfrak{p}\in\operatorname{Ass}_R(\overline{M})}\mathfrak{p}$. Then $\operatorname{Ass}_R(0:_M y_1)=\operatorname{Ass}_R(M)\cap\mathrm{V}(y_1)=(\operatorname{Ass}_R L\cap \mathrm{V}(y_1))\cup(\operatorname{Ass}_R\overline{M}\cap\mathrm{V}(y_1))\subset\operatorname{Max} R$. Hence $0:_M y_1$ has finite length, so the element $y_1\in\mathfrak{a}$ is filter regular on M.

Suppose $n>1$ and take y_1 as above.

Note that $\mathrm{H}^i_{\mathfrak{a}}(M)\cong\mathrm{H}^i_{\mathfrak{a}}(\overline{M})$ for all $i\geq 1$. Thus we may replace M by $\overline{M}$, [10, Proposition 6.3(b)], and we may assume that y_1 is a non-zerodivisor on M.

The exact sequence $0\to M\xrightarrow{y_1}M\to M/y_1M\to 0$ yields the long exact sequence

$$\cdots\longrightarrow \mathrm{H}^{i-1}_{\mathfrak{a}}(M)\longrightarrow \mathrm{H}^{i-1}_{\mathfrak{a}}(M/y_1M)\longrightarrow \mathrm{H}^{i}_{\mathfrak{a}}(M)\longrightarrow\cdots.$$

Hence $\mathrm{H}^i_{\mathfrak{a}}(M/y_1M)$ is $\mathfrak{a}$-cofinite and artinian for all $i<n-1$, by [11, Corollary 1.7]. Therefore by the induction hypothesis there exists $y_2,\ldots,y_n$ in $\mathfrak{a}$, which is filter-regular on M/y_1M. Thus $y_1,\ldots,y_n$ is filter-regular on M.

Remark. In [1] we studied the kernel and cokernel of the natural homomorphism $f:\operatorname{Ext}^n_R(R/\mathfrak{a},M)\to\operatorname{Hom}_R(R/\mathfrak{a},\mathrm{H}^n_{\mathfrak{a}}(M))$. Applying the criterion of Theorem 2.10 we get that if $\operatorname{Ext}^{t-j}_R(R/\mathfrak{a},\mathrm{H}^j_{\mathfrak{a}}(M))$ has finite length for $t=n,n+1$ and for all $j<n$, then $\operatorname{Ext}^n_R(R/\mathfrak{a},M)$ has finite length if and only if $\mathrm{H}^n_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite artinian.

Next we will study attached and coassociated prime ideals for the last non-vanishing local cohomology module. First we prove a lemma used in Corollary 2.13

Lemma 2.11. *For all R-modules M and for every finite R-module N,*

$$\operatorname{Att}_R(M\otimes_R N)=\operatorname{Att}_R(M)\cap\operatorname{Supp}_R(N).$$

Proof. Let $\mathfrak{p}\in\operatorname{Att}_R(M\otimes_R N)$, so $\mathfrak{p}=\operatorname{Ann}_R((M\otimes_R N)\otimes_R R/\mathfrak{p})$. However this ideal contains both $\operatorname{Ann}_R(M/\mathfrak{p}M)$ and $\operatorname{Ann}_R(N)$ and therefore $\mathfrak{p}=\operatorname{Ann}_R(M/\mathfrak{p}M)$ and $\mathfrak{p}\in\operatorname{Supp}_R(N)$.

Conversely let $\mathfrak{p} \in \operatorname{Att}_R(M) \cap \operatorname{Supp}_R(N)$. Then $\mathfrak{p} = \operatorname{Ann} M/\mathfrak{p}M$ and we want to show that $\mathfrak{p} = \operatorname{Ann}_R((M \otimes_R N) \otimes_R R/\mathfrak{p})$. Since

$$(M \otimes_R N) \otimes_R R/\mathfrak{p} \cong M/\mathfrak{p}M \otimes_{R/\mathfrak{p}} N/\mathfrak{p}N,$$

we may assume that R is a domain and $\mathfrak{p} = (0)$. Let K be the field of fractions of R. Then $\operatorname{Ann} M = 0$ and $N \otimes_R K \neq 0$. Therefore the natural homomorphism $f : R \longrightarrow \operatorname{End}_R(M)$ is injective and we have the following exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(N, R) \longrightarrow \operatorname{Hom}_R(N, \operatorname{End}_R(M)).$$

But $\operatorname{Hom}_R(N, \operatorname{End}_R(M)) \cong \operatorname{Hom}_R(M \otimes_R N, M)$. Hence we get

$$\begin{aligned}\operatorname{Ann}_R(M \otimes_R N) &\subset \operatorname{Ann}_R \operatorname{Hom}_R(M \otimes_R N, M)\\ &\subset \operatorname{Ann}_R \operatorname{Hom}_R(N, R) \subset \operatorname{Ann}_R(\operatorname{Hom}_R(N, R) \otimes_R K).\end{aligned}$$

On the other hand $\operatorname{Hom}_R(N, R) \otimes_R K \cong \operatorname{Hom}_R(N \otimes_R K, K)$, which is a nonzero vector space over K. Consequently $\operatorname{Ann}_R(M \otimes_R N) = 0$.

THEOREM 2.12. *Let $(R, \mathfrak{m})$ be a complete local ring and let $\mathfrak{a}$ be an ideal of R. Let t be a nonnegative integer such that $\mathrm{H}^i_{\mathfrak{a}}(R) = 0$ for all $i > t$.*

(a) *If $\mathfrak{p} \in \operatorname{Att}_R(\mathrm{H}^t_{\mathfrak{a}}(R))$ then $\dim R/\mathfrak{p} \geq t$.*

(b) *If $\mathfrak{p}$ is a prime ideal such that $\dim R/\mathfrak{p} = t$, then the following conditions are equivalent:*

(i) $\mathfrak{p} \in \operatorname{Coass}_R(\mathrm{H}^t_{\mathfrak{a}}(R))$.

(ii) $\mathfrak{p} \in \operatorname{Att}_R(\mathrm{H}^t_{\mathfrak{a}}(R))$.

(iii) $\mathrm{H}^t_{\mathfrak{a}}(R/\mathfrak{p}) \neq 0$.

(iv) $\sqrt{\mathfrak{a} + \mathfrak{p}} = \mathfrak{m}$.

PROOF. (a) By the right exactness of the functor $\mathrm{H}^t_{\mathfrak{a}}(-)$ we have

$$\mathrm{H}^t_{\mathfrak{a}}(R/\mathfrak{p}) \cong \mathrm{H}^t_{\mathfrak{a}}(R)/\mathfrak{p}\,\mathrm{H}^t_{\mathfrak{a}}(R) \tag{1}$$

If $\mathfrak{p} \in \operatorname{Att}_R(\mathrm{H}^t_{\mathfrak{a}}(R))$, then $\mathrm{H}^t_{\mathfrak{a}}(R)/\mathfrak{p}\,\mathrm{H}^t_{\mathfrak{a}}(R) \neq 0$. Hence $\mathrm{H}^t_{\mathfrak{a}}(R/\mathfrak{p}) \neq 0$ and $\dim R/\mathfrak{p} \geq t$.

(b) Since $R/\mathfrak{p}$ is a complete local domain of dimension t, the equivalence of (iii) and (iv) follows from the local Lichtenbaum Hartshorne vanishing theorem.

If $\mathrm{H}^t_{\mathfrak{a}}(R/\mathfrak{p}) \neq 0$, then by (1) $\mathrm{H}^t_{\mathfrak{a}}(R)/\mathfrak{p}\,\mathrm{H}^t_{\mathfrak{a}}(R) \neq 0$. Therefore $\mathfrak{p} \subset \mathfrak{q}$ for some $\mathfrak{q} \in \operatorname{Coass}_R(\mathrm{H}^t_{\mathfrak{a}}(R)) \subset \operatorname{Att}_R(\mathrm{H}^t_{\mathfrak{a}}(R))$. By (a) $\dim R/\mathfrak{q} \geq t = \dim R/\mathfrak{p}$, so we must have $\mathfrak{p} = \mathfrak{q}$. Thus (iii) implies (i) and since always $\operatorname{Coass}_R(\mathrm{H}^t_{\mathfrak{a}}(R)) \subset \operatorname{Att}_R(\mathrm{H}^t_{\mathfrak{a}}(R))$, (i) implies (ii).

If (ii) holds then the module $\mathrm{H}^t_{\mathfrak{a}}(R)/\mathfrak{p}\,\mathrm{H}^t_{\mathfrak{a}}(R) \neq 0$, since its annihilator is $\mathfrak{p}$. Hence, using again the isomorphism (1), (ii) implies (iii).

COROLLARY 2.13. *Let $(R, \mathfrak{m})$ be a complete local ring, $\mathfrak{a}$ an ideal of R and M a finite R-module and t a nonnegative integer such that $\mathrm{H}^i_{\mathfrak{a}}(M) = 0$ for all $i > t$.*

(a) *If $\mathfrak{p} \in \mathrm{Att}_R(\mathrm{H}^t_{\mathfrak{a}}(M))$ then $\dim R/\mathfrak{p} \geq t$.*

(b) *If $\mathfrak{p}$ is a prime ideal in $\mathrm{Supp}_R(M)$ such that $\dim R/\mathfrak{p} = t$, then the following conditions are equivalent:*

(i) $\mathfrak{p} \in \mathrm{Coass}_R(\mathrm{H}^t_{\mathfrak{a}}(M))$.

(ii) $\mathfrak{p} \in \mathrm{Att}_R(\mathrm{H}^t_{\mathfrak{a}}(M))$.

(iii) $\mathrm{H}^t_{\mathfrak{a}}(R/\mathfrak{p}) \neq 0$.

(iv) $\sqrt{\mathfrak{a} + \mathfrak{p}} = \mathfrak{m}$.

PROOF. Passing from R to $R/\operatorname{Ann} M$, we may assume that $\operatorname{Ann} M = 0$ and therefore using Gruson's theorem, see [13, Theorem 4.1], $\mathrm{H}^i_{\mathfrak{a}}(N) = 0$ for all $i > t$ and every R-module N. Hence the functor $\mathrm{H}^t_{\mathfrak{a}}(-)$ is right exact and therefore, since it preserves direct limits, we get

$$\mathrm{H}^t_{\mathfrak{a}}(M) \cong M \otimes_R \mathrm{H}^t_{\mathfrak{a}}(R).$$

The claims follow from Theorem 2.12 using the following equalities

$$\mathrm{Coass}_R(\mathrm{H}^t_{\mathfrak{a}}(M)) = \mathrm{Coass}_R(\mathrm{H}^t_{\mathfrak{a}}(R)) \cap \mathrm{Supp}_R(M)$$

by [16, Folgerung 3.2] and

$$\mathrm{Att}_R(\mathrm{H}^t_{\mathfrak{a}}(M)) = \mathrm{Att}_R(\mathrm{H}^t_{\mathfrak{a}}(R)) \cap \mathrm{Supp}_R(M)$$

by Lemma 2.11.

REFERENCES

1. Aghapournahr, M., Melkersson, L., *A natural map in local cohomology*, preprint.
2. Brodmann, M. P., Sharp, R. Y., *Local Cohomology: an Algebraic Introduction with Geometric Applications*, Cambridge Studies on Advanced Math. 60, Cambridge University Press, Cambridge 1998.
3. Bruns, W., Herzog, J., *Cohen-Macaulay Rings*, revised ed., Cambridge Studies on Advanced Math. 39, Cambridge University Press, Cambridge 1998.
4. Delfino, D., and Marley, T., *Cofinite modules and local cohomology*, J. Pure Appl. Algebra 121 (1997), 45–52.
5. Divaani-Aazar, K., Mafi, A., *Associated primes of local cohomology modules of weakly Laskerian modules*, Comm. Algebra 34 (2006), 681–690.

6. Divaani-Aazar, K., Mafi, A., *Associated primes of local cohomology modules*, Proc. Amer. Math. Soc. 133 (2005), 655–660.
7. Grothendieck, A., *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux* (*SGA 2*), North-Holland, Amsterdam 1968.
8. Hartshorne, R., *Affine duality and cofiniteness*, Invent. Math. 9 (1970), 145–164.
9. Matsumura, H., *Commutative Ring Theory*, Cambridge Studies on Advanced Math. 8, Cambridge University Press, Cambridge 1986.
10. Melkersson, L., *Modules cofinite with respect to an ideal*, J. Algebra 285 (2005), 649–668.
11. Melkersson, L., *Properties of cofinite modules and applications to local cohomology*, Math. Proc. Cambridge Phil. Soc. 125 (1999), 417–423.
12. Schenzel, P., Trung, N. V., Cuong, N. T., *Verallgemeinerte Cohen-Macaulay-Moduln*, Math. Nachr. 85 (1978), 57–73.
13. Vasconcelos, W., *Divisor Theory in Module Categories*, North-Holland Math. Studies 14, North-Holland, Amsterdam 1974.
14. Yoshida, K. I., *Cofiniteness of local cohomology modules for ideals of dimension one*, Nagoya Math. J. 147 (1997), 179–191.
15. Zöschinger, H., *Koatomare Moduln*, Math. Z. 170 (1980), 221–232.
16. Zöschinger, H., *Minimax-Moduln*, J. Algebra 102 (1986), 1–32.
17. Zöschinger, H., *Über koassoziierte Primideale*, Math Scand. 63 (1988), 196–211.
18. Zöschinger, H., *Linear-kompakte Moduln über noetherschen Ringen*, Arch. Math. 41 (1983), 121–130.

ARAK UNIVERSITY
BEHESHTI ST
P.O. BOX: 879
ARAK
IRAN
E-mail: m-aghapour@araku.ac.ir
m.aghapour@gmail.com

DEPARTMENT OF MATHEMATICS
LINKÖPING UNIVERSITY
SE-581 83 LINKÖPING
SWEDEN
E-mail: lemel@mai.liu.se