Abstract
Maximal operators of Schrödinger type but with a complex parameter are considered. For these operators we obtain results which in a certain sense lie between the results for the corresponding maximal operators for solutions to the Schrödinger equation and for solutions to the heat equation.

1. Introduction
Letting \( f \) belong to the Schwartz space \( \mathcal{S}(\mathbb{R}) \) we set
\[
S_t f(x) = \int_{\mathbb{R}} e^{ix\xi} e^{it\xi^2} \hat{f}(\xi) \, d\xi, \quad x \in \mathbb{R}.
\]
Here \( t \) is a complex number with \( \text{Im} \, t \geq 0 \) and \( \hat{f} \) denotes the Fourier transform of \( f \), defined by
\[
\hat{f}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} f(x) \, dx.
\]
We also set \( U(x,t) = (2\pi)^{-1} S_t f(x) \) for \( x \in \mathbb{R} \) and \( t \in \mathbb{R} \). It then follows that \( U(x,0) = f(x) \) and \( U \) satisfies the Schrödinger equation \( i \frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} \).

We also introduce the maximal function \( S^* f \) defined by
\[
S^* f(x) = \sup_{0 < t < 1} |S_t f(x)|, \quad x \in \mathbb{R},
\]
and define Sobolev spaces \( H_s \) by setting
\[
H_s = \{ f \in \mathcal{S}' ; \| f \|_{H_s} < \infty \}, \quad s \in \mathbb{R},
\]
where
\[
\| f \|_{H_s} = \left( \int_{\mathbb{R}} (1 + \xi^2)^s |\hat{f}(\xi)|^2 \, d\xi \right)^{1/2}.
\]
It is well-known that the estimate
\[ \|S^*f\|_2 \leq C \|f\|_{H_s} \]
holds for \( s > 1/2 \) and does not hold for \( s < 1/2 \) (see [2]). Here \( \|S^*f\|_2 \) denotes the norm of \( S^*f \) in the space \( L^2(\mathbb{R}) \). We then set
\[ H_{uf}(x) = S_{iu}f(x) = \int_{\mathbb{R}} e^{ix\xi} e^{-u\xi^2} \widehat{f}(\xi) \, d\xi, \quad x \in \mathbb{R}, \]
for \( u \geq 0 \). If we set \( V(x, u) = (2\pi)^{-1} H_{uf}(x) \) for \( x \in \mathbb{R} \) and \( u \geq 0 \), then \( V(x, 0) = f(x) \) and \( V \) satisfies the heat equation \( \partial V / \partial u = \partial^2 V / \partial x^2 \). We also introduce the maximal function \( H^*f \) defined by
\[ H^*f(x) = \sup_{0 < u < 1} |H_{uf}(x)|, \quad x \in \mathbb{R}. \]
It is then well-known that the estimate \( \|H^*f\|_2 \leq C \|f\|_{H_s} \) holds if and only if \( s \geq 0 \).

We shall then introduce a class of maximal operators for which one has results lying between the above results for \( S^* \) and \( H^* \). For \( 0 < \gamma < \infty \) we set
\[ P_{uf}(x) = S_{iu+iu^\gamma} f(x) = \int_{\mathbb{R}} e^{ix\xi} e^{iu\xi^2} e^{-u\gamma\xi^2} \widehat{f}(\xi) \, d\xi, \quad x \in \mathbb{R}, \ 0 < u < 1, \]
and
\[ P^*f(x) = \sup_{0 < u < 1} |P_{uf}(x)|, \quad x \in \mathbb{R}. \]
We shall here study the inequality
\[ \|P^*f\|_2 \leq C \|f\|_{H_s} \]
for various values of \( \gamma \). We have the following results.

**Theorem 1.**
(i) For \( 0 < \gamma \leq 1 \) (1) holds if and only if \( s \geq 0 \).
(ii) For \( \gamma = 2 \) (1) holds if and only if \( s \geq 1/4 \).
(iii) If \( \gamma \geq 4 \) and (1) holds then \( s \geq 1/2 - 1/\gamma \).

For \( \gamma > 0 \) we let \( E_\gamma \) denote the set of all \( s \) with the property that (1) holds. Also set \( s(\gamma) = \inf E_\gamma \) for \( \gamma > 0 \). Using the fact that \( \lim_{u \to 0} P_{uf}(x) = 2\pi f(x) \) it is easy to see that \( s(\gamma) \geq 0 \). We shall use the following lemma.

**Lemma 1.** Assume that \( g \) and \( h \) are continuous functions on the interval \((0, 1)\) and that \( 0 \leq g(u) \leq h(u) \) for \( 0 < u < 1 \). Set
\[ P^*_g f(x) = \sup_{0 < u < 1} |S_{iu+ig(u)} f(x)| \]
and
\[ P_h^* f(x) = \sup_{0 < u < 1} |S_{u+i h(u)} f(x)|. \]

Then
\[ \| P_h^* f \|_2 \leq C \| P_g^* f \|_2. \]

It follows from the lemma that \( s(\gamma) \) is an increasing function of \( \gamma \) on the interval \((0, \infty)\). Also the result mentioned above for the operator \( S^* \) implies that \( s(\gamma) \leq 1/2 \) (take \( g(u) \equiv 0 \) and \( h(u) = u^\gamma \) in the lemma). The results in Theorem 1 can then be stated in the following way.

**Theorem 2.**
(i) For \( 0 < \gamma \leq 1 \) one has \( s(\gamma) = 0 \).
(ii) \( s(2) = 1/4 \).
(iii) For \( \gamma > 4 \) one has \( 1/2 - 1/\gamma < s(\gamma) \leq 1/2 \) and hence \( \lim_{\gamma \to \infty} s(\gamma) = 1/2 \).

In Section 2 we shall prove Lemma 1 and state and prove a second lemma. In Section 3 we shall prove Theorem 1.

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## 2. Lemmas

**Proof of Lemma 1.** We have

\[
S_{u+i h(u)} f(x) = \int e^{i x \xi} e^{i u \xi^2} e^{-h(u) \xi^2} \hat{f}(\xi) d\xi
\]
\[
= \int e^{i x \xi} e^{i u \xi^2} e^{-g(u) \xi^2} e^{-(h(u)-g(u)) \xi^2} \hat{f}(\xi) d\xi
\]
\[
= \int e^{i x \xi} e^{i u \xi^2} e^{-g(u) \xi^2} e^{-v \xi^2} \hat{f}(\xi) d\xi
\]

where \( v = h(u) - g(u) \geq 0 \).

For \( v = 0 \) one has

\[
S_{u+i h(u)} f(x) = S_{u+i g(u)} f(x).
\]

We then assume \( v > 0 \). It is well-known that

\[
e^{-v \xi^2} = \hat{K}_v(\xi) = \int e^{-i \xi y} K_v(y) dy,
\]
where
\[ K_v(y) = \frac{1}{v^{1/2}} \frac{1}{2\sqrt{\pi}} e^{-y^2/(4v)} = \frac{1}{v^{1/2}} K(y/v^{1/2}) \]
with \( K(y) = e^{-y^2/4}/(2\sqrt{\pi}) \).
It follows that
\[ S_{u+ih(u)} f(x) = \int e^{ix\xi} e^{iu\xi^2} e^{-g(u)\xi^2} \left( \int e^{-i\xi y} K_v(y) \, dy \right) \hat{f}(\xi) \, d\xi \]
\[ = \int \left( \int e^{i(x-y)\xi} e^{iu\xi^2} e^{-g(u)\xi^2} \hat{f}(\xi) \, d\xi \right) K_v(y) \, dy \]
\[ = \int S_{u+ig(u)} f(x-y) K_v(y) \, dy = K_v \ast S_{u+ig(u)} f(x) \]
and hence
\[ |S_{u+ih(u)} f(x)| \leq K_v \ast P_g^* f(x) \leq CM P_g^* f(x) \]
where \( M \) denotes the Hardy-Littlewood maximal operator.

We conclude that
\[ P_h^* f(x) \leq P_g^* f(x) + CM P_g^* f(x) \]
and since \( M \) is a bounded operator on \( L^2(\mathbb{R}) \) we obtain \( \|P_h^* f\|_2 \leq C \|P_g^* f\|_2 \).

The following lemma was proved in [3].

**Lemma 2.** Assume that \( a > 1, 1/2 \leq s < 1 \) and \( \mu \in C_0^\infty(\mathbb{R}) \). Then

\[ \left| \int_{\mathbb{R}} e^{ix\xi + it|\xi|^s} |\xi|^{-s} \mu(\xi/N) \, d\xi \right| \leq C \frac{1}{|x|^{1-s}} \]

for \( x \in \mathbb{R}, t \in \mathbb{R} \) and \( N = 1, 2, 3, \ldots \). Here the constant \( C \) may depend on \( s \) and \( a \) but not on \( x, t \) or \( N \).

The next lemma will be used to prove that \( s(2) \leq 1/4 \).

**Lemma 3.** Assume that \( 1/2 \leq \alpha < 1, 0 < d_1 < 1, 0 < d_2 < 1 \), and \( \mu \in C_0^\infty(\mathbb{R}) \) and \( \mu \) even and real-valued. Then

\[ \left| \int_{\mathbb{R}} e^{i(d_1-x_1)\xi^2 - x_1^2} e^{-(d_2^2 + d_2^2)\xi^2} \mu(\xi/N) \, d\xi \right| \leq K(x) \]

for \( x \in \mathbb{R} \) and \( N = 1, 2, 3, \ldots \), where \( K \in L^1(\mathbb{R}) \). Here \( K \) is independent of \( d_1, d_2 \) and \( N \), and one may take \( K(x) = Cx^{-2} \) for \( |x| \geq C_0 \) and \( K(x) = C|x|^{\alpha-1} \) for \( |x| < C_0 \). Here \( C_0 \) denotes a large constant.
Proof of Lemma 3. The structure in the proof of the lemma will be the same as in the proof of Lemma 3 in [4]. Without loss of generality we may assume \( d_2 < d_1 \). We set \( d = d_1 - d_2 \) and \( \epsilon = d_1^2 + d_2^2 \) so that \( 0 < d < 1 \) and \( 0 < \epsilon < 2 \). First assume \( |x| \geq C_0 \) where \( C_0 \) denotes a large constant.

We choose an even function \( \varphi_0 \in C_0^\infty(\mathbb{R}) \) such that \( \varphi_0(\xi) = 1 \) for \( |\xi| \leq 1/2 \) and \( \varphi_0(\xi) = 0 \) for \( |\xi| \geq 1 \).

Set \( \psi(\xi) = (1 + \xi^2)^{-\alpha/2} e^{-\epsilon \xi^2} \mu(\xi/N) \) and \( \psi_0 = \psi \varphi_0 \) so that \( \text{supp} \psi_0 \subset [-1, 1] \).

We also set \( \rho = |x|/(2d) \) and let \( K \) denote a large constant. Then choose \( \varphi_2 \in C_0^\infty(\mathbb{R}) \) so that \( \text{supp} \varphi_2 \subset [\rho/4, 2K \rho] \) and \( \varphi_2(\xi) = 1 \) for \( \rho/2 \leq \xi \leq K \rho \). We may also assume that \( |\varphi''_2(\xi)| \leq C \xi^{-1} \) and \( |\varphi''_2(\xi)| \leq C \xi^{-2} \) for \( \xi > 0 \). We then set \( \varphi_3 = (1 - \varphi_2) \chi_{(K \rho, \infty)} \) and \( \varphi_1 = (1 - \varphi_2 - \varphi_0) \chi_{[0, \rho/2]} \).

Let \( \tilde{\varphi} \) be defined by \( \tilde{\varphi}(\xi) = \varphi(-\xi) \) and set \( \varphi_{-1} = \tilde{\varphi}_1, \varphi_{-2} = \tilde{\varphi}_2 \) and \( \varphi_{-3} = \tilde{\varphi}_3 \). Setting \( F(\xi) = d\xi^2 - x\xi \) we then have

\[
\int_{-\infty}^{\infty} e^{iF} \psi \varphi_j d\xi = \sum_{j=0}^{3} \int e^{iF} \psi \varphi_j d\xi + \sum_{j=1}^{3} \int e^{iF} \psi \varphi_{-j} d\xi.
\]

The integrals \( \int e^{iF} \psi \varphi_{-j} \) can be reduced to \( \int e^{iF} \psi \varphi_j \) for \( j = 1, 2, 3 \). Setting \( \psi_j = \psi \varphi_j, j = 0, 1, 2, 3 \), it is therefore sufficient to estimate the integrals

\[
J_j = \int e^{iF} \psi_j d\xi, \quad j = 0, 1, 2, 3.
\]

We claim that one has the following estimates for \( j = 1, 2, 3 \) and \( \xi \geq 1/2 \):

\[
|\psi_j(\xi)| \leq C \frac{1}{(1 + \xi^2)^{\alpha/2}}, \quad (2)
\]

\[
|\psi'_j(\xi)| \leq C \frac{1}{(1 + \xi^2)^{\alpha/2} \xi}, \quad (3)
\]

and

\[
|\psi''_j(\xi)| \leq C \frac{1}{(1 + \xi^2)^{\alpha/2} \xi^2}, \quad (4)
\]

We set \( h(\xi) = h_\epsilon(\xi) = e^{-\epsilon \xi^2} \) for \( \xi \geq 1/2 \) and \( 0 < \epsilon < 2 \). The above estimates for \( \psi_j, \psi'_j \) and \( \psi''_j \) will follow if we can prove that

\[
|h'(\xi)| \leq C \frac{1}{\xi}, \quad \xi \geq 1/2.
\]
\begin{align*}
\frac{\varrho \lambda}{\eta} & \leq \frac{1}{\xi^2}, \quad \xi \geq 1/2, \\
\text{with } C \text{ independent of } \varepsilon. \quad \text{(6)}
\end{align*}

We have
\begin{align*}
h'(\xi) &= -e^{-\varepsilon \xi^2} 2 \varepsilon \xi
\end{align*}
and
\begin{align*}
h''(\xi) &= -e^{-\varepsilon \xi^2} 2 \varepsilon + e^{-\varepsilon \xi^2} 4 \varepsilon^2 \xi^2.
\end{align*}

It follows that for \( \xi \geq 1/2 \) one has
\begin{align*}
|h'(\xi)| &\leq e^{-\varepsilon \xi^2} 2 \varepsilon \xi = e^{-\varepsilon \xi^2} 2 \varepsilon \xi^2 \frac{1}{\xi^2} \leq 2 (\max_{t>0} t e^{-t}) \frac{1}{\xi} = 2 A \frac{1}{\xi}, \\
\text{where } A &= \max_{t \geq 0} t e^{-t}. \text{ Also}
\end{align*}
\begin{align*}
|h''(\xi)| &\leq e^{-\varepsilon \xi^2} 2 \varepsilon + 4 e^{-\varepsilon \xi^2} \varepsilon^2 \xi^2 \\
&= e^{-\varepsilon \xi^2} 2 \varepsilon \xi^2 \frac{1}{\xi^2} + 4 e^{-\varepsilon \xi^2} \varepsilon^2 \xi^4 \frac{1}{\xi^2} \leq 2 A \frac{1}{\xi^2} + 4 B \frac{1}{\xi^2},
\end{align*}
where \( B = \max_{t \geq 0} t^2 e^{-t} \). Hence (5) and (6) are proved and (2), (3) and (4) follow.

We shall first estimate \( J_0 \). We have
\begin{align*}
J_0 &= \int e^{-i \xi} e^{i \xi} \psi_0(\xi) d\xi
\end{align*}
where \( \text{supp } \psi_0 \subset [-1, 1] \) and two integrations by parts give the estimate
\( |J_0| \leq C x^{-2} \).

We shall then estimate \( J_2 \). One has
\begin{align*}
J_2 &= \int e^{i F} \psi_2 d\xi
\end{align*}
where
\begin{align*}
\psi_2(\xi) &= (1 + \xi^2)^{-a/2} e^{-\varepsilon \xi^2} \mu(\xi/N) \varphi_2(\xi)
\end{align*}
and \( \text{supp } \psi_2 \subset [\rho/4, 2K \rho] \).

We have \( F''(\xi) = 2d \) and van der Corput’s Lemma with the second derivative (see Stein [5], p. 334) gives
\begin{align*}
[J_2] &\leq Cd^{-1/2} \left( \max |\psi_2| + \int |\varphi'_2| d\xi \right).
\end{align*}
It is clear that

\begin{equation}
\max |\psi_2| \leq C \rho^{-\alpha} e^{-\epsilon \rho^2}
\end{equation}

where \( c \) denotes a positive constant.

We also set

\[ v(\xi) = (1 + \xi^2)^{-\alpha/2} \mu(\xi/N) \varphi_2(\xi) \]

so that

\[ \psi_2(\xi) = v(\xi) e^{-\epsilon \xi^2} \]

and

\[ \psi_2'(\xi) = v(\xi) \left( \frac{d}{d\xi} e^{-\epsilon \xi^2} \right) + v'(\xi) e^{-\epsilon \xi^2}. \]

It follows that

\[ \int |\psi_2'(\xi)| d\xi \leq \int v(\xi) \left( \frac{d}{d\xi} e^{-\epsilon \xi^2} \right) d\xi + \int |v'(\xi) e^{-\epsilon \xi^2}| d\xi \]

\[ \leq C \rho^{-\alpha} \int_{\rho/4}^{2K\rho} \left| \frac{d}{d\xi} e^{-\epsilon \xi^2} \right| d\xi + \int_{\rho/4}^{2K\rho} \rho^{-\alpha-1} d\xi e^{-\epsilon \rho^2} \]

\[ \leq C \rho^{-\alpha} \int_{\rho/4}^{2K\rho} - \left( \frac{d}{d\xi} e^{-\epsilon \xi^2} \right) d\xi + C \rho^{-\alpha} e^{-\epsilon \rho^2} \]

\[ = -C \rho^{-\alpha} \left[ e^{-\epsilon \xi^2} \right]_{\rho/4}^{2K\rho} + C \rho^{-\alpha} e^{-\epsilon \rho^2} \leq C \rho^{-\alpha} e^{-\epsilon \rho^2}. \]

Combining this estimate with (7) and (8) we obtain

\[ |J_2| \leq C d^{-1/2} \rho^{-\alpha} e^{-\epsilon \rho^2} = C d^{-1/2} \left( \frac{|x|}{d} \right)^{-\alpha} e^{-\epsilon x^2/d^2} \]

\[ \leq C d^{\alpha-1/2} |x|^{-\alpha} e^{-\epsilon x^2} \leq C e^{-\epsilon x^2}, \]

since \( \alpha \geq 1/2 \) and \( d^2 \leq \epsilon \). In fact

\[ d^2 = (d_1 - d_2)^2 = d_1^2 + d_2^2 - 2 d_1 d_2 \leq d_1^2 + d_2^2 = \epsilon. \]

This concludes the estimate of \( J_2 \) and we shall then estimate \( J_1 \). One has

\[ J_1 = \int e^{iF} \psi_1 d\xi \]

where \( \text{supp } \psi_1 \subset [1/2, \rho/2] \) and \( F' = 2 d\xi - x \).

On the interval \([1/2, \rho/2]\) one has

\[ 2 d\xi \leq 2 d\rho/2 = d \rho = d \frac{|x|}{2d} = \frac{|x|}{2} \quad \text{and} \quad |F'| \geq |x|/2 \geq 2 d\xi. \]
Also $F'' = 2d$ and $F^{(3)} = 0$. It follows that $|F''|/|F'| \leq C \xi^{-1}$ for $1/2 \leq \xi \leq \rho/2$. Integrating by parts twice we obtain

$$J_1 = \int e^{iF} \psi_1 \, d\xi = \int i F' e^{iF} \frac{\psi_1}{i F'} \, d\xi$$

$$= -\int e^{iF} \left( \frac{1}{i F'} \psi_1 - \frac{1}{i} \frac{F''}{(F')^2} \psi_1 \right) \, d\xi$$

$$= -\int i F' e^{iF} \left( \frac{1}{(i F')^2} \psi_1' + \frac{F''}{(F')^3} \psi_1 \right) \, d\xi$$

$$= \int e^{iF} \left( \frac{1}{i^2 (F')^2} \psi_1'' - \frac{1}{i^2 (F')^3} \psi_1' \right.$$

$$+ \frac{F''}{(F')^3} \psi_1' + \frac{F^{(3)}}{(F')^3} \psi_1 - \frac{3(F'')^2}{(F')^4} \psi_1 \left.) \, d\xi \right.$$ and hence

$$|J_1| \leq C \int \frac{1}{|F'|^2} \left( |\psi_1''| + \frac{|F''|}{|F'|} |\psi_1'| + \frac{|F''|^2}{|F'|^2} |\psi_1| \right) \, d\xi$$

(9)

$$\leq C \frac{1}{x^2} \int_{1/2}^\infty \xi^{-\alpha-2} \, d\xi = C \frac{1}{x^2}.$$

It remains to estimate $J_3 = \int e^{iF} \psi_3 \, d\xi$. Here supp $\psi_3 \subset [K \rho, \infty)$. For

$$\xi \geq K \rho = K \frac{|x|}{2d}$$

we have

$$2d\xi \geq 2dK \frac{|x|}{2d} = K |x|$$

and hence $|F'| \geq c|x|$ and $|F'| \geq c \, d \, \xi$.

We can estimate $J_3$ in the same way as we estimated $J_1$. We can use the inequality (9) with $\psi_1$ replaces by $\psi_3$ and $J_1$ replaced by $J_3$. One obtains $|J_3| \leq C|x|^{-2}$.

We have proved Lemma 3 in the case $|x| \geq C_0$. It remains to study the case $|x| < C_0$. The estimate in this case follows from the proof in [3] of our Lemma 2. The proof of Lemma 3 is complete.
3. Proof of Theorem 1

Proof of Theorem 1. We shall first study the case $0 < \gamma \leq 1$. It is well-known that $e^{-ax^2}$ has Fourier transform $\frac{\sqrt{\pi}}{z^{1/2}} e^{-\xi^2/4z}$ for $a > 0$. For $Re z > 0$ we then set $z^{1/2} = |z|^{1/2} e^{i(\arg z)/2}$ where $-\pi/2 < \arg z < \pi/2$. By use of elementary properties of analytic functions it is then easy to prove that $e^{-zx^2}$ has Fourier transform

$$\frac{\sqrt{\pi}}{z^{1/2}} e^{-\xi^2/4z}$$

for $Re z > 0$. It also follows that $e^{-z\xi^2}$ is the Fourier transform of

$$(10) \quad \frac{1}{2\sqrt{\pi}z^{1/2}} e^{-x^2/4z}$$

for $Re z > 0$.

Setting $t = u + iv$ with $u > 0$, $v > 0$ we then have $e^{it\xi^2} = e^{i(u+iv)\xi^2} = e^{-(v-iu)\xi^2}$. Taking $z = v - iu$ in (10) it then follows that $e^{-(v-iv)\xi^2}$ is the Fourier transform of

$$K(x) = \frac{1}{2\sqrt{\pi}(v-iu)^{1/2}} e^{-x^2/(4(v-iu))}.$$

It is clear that

$$|K(x)| \leq \frac{1}{(v^2 + u^2)^{1/4}} |e^{-x^2/(4(v-iu))}|.$$

Since

$$\frac{1}{v - iu} = \frac{v}{v^2 + u^2} + i \frac{u}{v^2 + u^2}$$

we conclude that

$$|K(x)| \leq \frac{1}{(v^2 + u^2)^{1/4}} e^{-x^2v/(4(v^2+u^2))}.$$

Letting $0 < u < 1$ and $v = u^\gamma$ with $0 < \gamma \leq 1$ we then obtain

$$|K(x)| \leq \frac{1}{(u^{2\gamma} + u^2)^{1/4}} \exp\left(-\frac{x^2}{4} \frac{u^\gamma}{u^{2\gamma} + u^2}\right) \leq \frac{1}{u^{\gamma/2}} e^{-c x^2/u^\gamma} = \frac{1}{u^{\gamma/2}} L(x/u^{\gamma/2}),$$

where $c > 0$ and $L(x) = e^{-c x^2}$. It follows that $P^* f(x) \leq CMf(x)$ and hence the first part of Theorem 1 follows.
We shall then study the case $\gamma = 2$. We let $u(x)$ denote a measurable function on $\mathbb{R}$ with $0 < u(x) < 1$ and set

$$Pf(x) = \int_{\mathbb{R}} e^{ix\xi} e^{iu(x)\xi^2} e^{-u(x)\xi^2} \hat{f}(\xi) d\xi, \quad x \in \mathbb{R}.$$  

We have to prove that

$$(11) \quad \|Pf\|_2 \leq C \|f\|_{H_s} = C \left( \int_{\mathbb{R}} |\hat{f}(\xi)|^2 (1 + \xi^2)^s d\xi \right)^{1/2}$$

for $s \geq 1/4$. We may also assume $s < 1/2$.

Setting $g(\xi) = \hat{f}(\xi) (1 + \xi^2)^{s/2}$ and defining $T$ by

$$Tg(x) = \int_{\mathbb{R}} e^{ix\xi} e^{iu(x)\xi^2} e^{-u(x)\xi^2} (1 + \xi^2)^{-s/2} g(\xi) d\xi,$$

we have $Pf(x) = Tg(x)$. It is therefore sufficient to prove that $\|Tg\|_2 \leq C\|g\|_2$. For $N = 1, 2, 3, \ldots$ set

$$T_N g(x) = \chi_N(x) \int_{\mathbb{R}} e^{ix\xi} e^{iu(x)\xi^2} e^{-u(x)\xi^2} (1 + \xi^2)^{-s/2} \rho_N(\xi) g(\xi) d\xi.$$

Here $\chi_N(x) = \chi(x/N)$ and $\rho_N(\xi) = \rho(\xi/N)$ and $\chi$ and $\rho \in C_0^\infty(\mathbb{R})$ and have the property that $\chi(x) = \rho(x) = 1$ for $|x| \leq 1$ and $\chi(x) = \rho(x) = 0$ for $|x| \geq 2$. We also assume that $\chi$ and $\rho$ are even and real-valued. It is sufficient to prove that

$$\|T_N g\|_2 \leq C\|g\|_2, \quad N = 1, 2, 3, \ldots.$$

It is clear that

$$T_N^* h(\xi) = \rho_N(\xi)(1 + \xi^2)^{-s/2} \int_{\mathbb{R}} e^{-ix\xi} e^{-iu(x)\xi^2} e^{u(x)^2\xi^2} \chi_N(x) h(x) dx$$

and it is sufficient to prove that

$$(12) \quad \|T_N^* h\|_2 \leq C\|h\|_2, \quad N = 1, 2, 3, \ldots.$$
Invoking Lemma 3 we now have

\[ \| T_N^* h \|_2^2 = \int T_N^* h(\xi) \overline{T_N^* h(\xi)} d\xi \]

\[ = \int \rho_N(\xi)^2 (1 + \xi^2)^{-s} \left( \int e^{-ix\xi} e^{-iu(x)\xi^2} \chi_N(x) h(x) dx \right) \times \left( \int e^{iy\xi} e^{iu(y)\xi^2} \chi_N(y) \overline{h(y)} dy \right) \]

\[ = \int \int \left( \int (1 + \xi^2)^{-s} e^{i(y-x)\xi} e^{i(u(y)-u(x))\xi^2} e^{-(u(x)^2 + u(y)^2)\xi^2} \mu(\xi/N) d\xi \right) \times \chi_N(x) \chi_N(y) h(x) \overline{h(y)} dxdy \]

\[ \leq C \int \int K(x-y) |h(x)| |h(y)| dxdy \leq C \| h \|_2^2. \]

Here we have set \( \mu = \rho^2 \) and according to Lemma 3 we have \( K \in L^1(\mathbb{R}) \) since \( 1/4 \leq s < 1/2 \). Hence (12) and (11) are proved.

We shall then prove that if \( \gamma = 2 \) and \( s < 1/4 \) then (1) does not hold. First choose \( g \in C^\infty_0(\mathbb{R}) \) such that \( \text{supp } g \subset (-1, 1) \), \( g(\xi) \geq 0 \) and \( g(\xi) = 1 \) for \( |\xi| \leq 1/2 \). Then let \( v > 0 \) denote a small number and define a function \( f_v \) by setting

\[ \hat{f}_v(\xi) = v g(v\xi + 1/v). \]

It is well-known and easy to prove that \( \| f_v \|_{H_s} \to 0 \) as \( v \to 0 \) if \( s < 1/4 \) (see [1]). Setting \( u = u(x) = xv^2/2 \) and assuming \( 0 < x < 1/100 \) we have

\[ P_u f_v(x) = \int v g(v\xi + 1/v) e^{ix\xi} e^{ixv^2\xi^2/2} e^{-(x^2v^4\xi^2/4)} d\xi. \]

In this integral we make a change of variable \( \eta = v\xi + 1/v \), so that \( d\eta = v d\xi \) and \( \xi = \eta/v - 1/v^2 \). One obtains

\[ P_u f_v(x) = \int g(\eta) e^{iF(\eta)} e^{G(\eta)} d\eta \]

where

\[ F(\eta) = x \left( \frac{\eta}{v} - \frac{1}{v^2} \right) + \frac{xv^2}{2} \left( \frac{\eta}{v} - \frac{1}{v^2} \right)^2 \]

and

\[ G(\eta) = -\frac{x^2v^4}{4} \left( \frac{\eta}{v} - \frac{1}{v^2} \right)^2. \]
Hence

\[ F(\eta) = \frac{x \eta}{v} - \frac{x}{v^2} + \frac{x v^2}{2} \left( \frac{\eta^2}{v^2} + \frac{1}{v^4} - \frac{2\eta}{v^3} \right) \]

\[ = \frac{x \eta}{v} - \frac{x}{v^2} + \frac{x \eta^2}{2} + \frac{x}{2v^2} - \frac{x \eta}{v} = \frac{x \eta^2}{2} - \frac{x}{2v^2} \]

and it follows that

\[ |P_u f_v(x)| = \left| \int_{-1}^{1} g(\eta) e^{ix \eta^2/2} e^{G(\eta)} d\eta \right|. \]

We have

\[ |G(\eta)| \leq \frac{x^2 v^4}{4} \left( \frac{2}{v^2} \right)^2 = x^2 \leq 1 \]

for \(|\eta| \leq 1\) and we conclude that

\[ |P_u f_v(x)| \geq \int_{-1}^{1} g(\eta) \cos(x \eta^2/2) e^{G(\eta)} d\eta \geq \int_{-1}^{1} g(\eta) e^{-1/2} d\eta \geq \frac{1}{2e}. \]

Hence \(P^* f_v(x) \geq 1/(2e)\) for \(0 < x < 1/100\) and \(\|P^* f_v\|_2 \geq c > 0\) for small \(v\). It follows that the estimate \(\|P^* f_v\|_2 \leq C \|f_v\|_{H_s}\) does not hold for \(s < 1/4\).

Hence the statement in Theorem 1 in the case \(\gamma = 2\) has been proved.

It remains to study the case \(\gamma \geq 4\). Take \(g\) as above and choose \(f\) so that \(\hat{f}(\xi) = g(\xi + N)\) where \(N\) denotes a large positive number. Also set \(u = u(x) = x/(2N)\) and assume \(2^{-1} N^{1-2/\gamma} \leq x \leq N^{1-2/\gamma}\).

Setting \(\eta = \xi + N\) we obtain

\[ P_u f(x) = \int e^{ix \xi} e^{iu \xi^2} e^{-u \xi^2} g(\xi + N) d\xi \]

\[ = \int e^{ix(\eta - N)} e^{iu(\eta - N)^2} e^{-u \gamma (\eta - N)^2} g(\eta) d\eta \]

\[ = \int g(\eta) e^{iF(\eta)} e^{G(\eta)} d\eta \]

where \(F(\eta) = x(\eta - N) + u(\eta - N)^2\) and \(G(\eta) = -u \gamma (\eta - N)^2\). Hence

\[ F(\eta) = x \eta - x N + u \eta^2 + u N^2 - u 2 \eta N \]

\[ = x \eta - x N + \frac{x}{2N} \eta^2 + \frac{x}{2N} N^2 - 2 \eta \frac{x N}{2N} N \]

\[ = x \eta - x N + \frac{x}{2N} \eta^2 + \frac{x N}{2} - x \eta \]

\[ = \frac{x}{2N} \eta^2 - \frac{x N}{2} \]
and it follows that
\[ |P^*_uf(x)| = \left| \int_{-1}^{1} g(\eta)e^{ix\eta^2/(2N)}e^{G(\eta)}d\eta \right|. \]

We have
\[ |G(\eta)| \leq u^\gamma (2N)^2 = \frac{x^\gamma}{2^\gamma N^\gamma} \leq N^{(1-2/\gamma)\gamma} N^{2-\gamma} = 1 \]
for $|\eta| \leq 1$ and we conclude that
\[ |P^*_uf(x)| \geq \int_{-1}^{1} g(\eta) \cos(x\eta^2/(2N))e^{G(\eta)}d\eta \geq \int_{-1}^{1} g(\eta) \frac{1}{2} e^{-1}d\eta \geq \frac{1}{2e}. \]

Hence $P^*f(x) \geq 1/(2e)$ for $2^{-1}N^{1-2/\gamma} \leq x \leq N^{1-2/\gamma}$ and it follows that
\[ \|P^*f\|_2 \geq c N^{(1-2/\gamma)/2} = c N^{1-1/\gamma}. \]

On the other hand it is easy to see that $\|f\|_{H^s} \leq CN^s$, and if (1) holds, one obtains $N^{1/2-1/\gamma} \leq CN^s$. We conclude that $s \geq 1/2 - 1/\gamma$ and the proof of Theorem 1 is complete.

REFERENCES