

THE GROUP LAW ON A TROPICAL ELLIPTIC CURVE

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Abstract

In analogy with the classical group law on a plane cubic curve, we define a group law on a smooth plane tropical cubic curve. We show that the resulting group is isomorphic to S^1 .

1. Introduction

Tropical geometry is a recent, but rapidly growing field of research in mathematics, in which one seeks to establish connections between complex algebraic geometry and the combinatorics of certain piecewise linear objects, called tropical varieties. Such connections has led to new insight in various areas, like enumerative geometry [3], mirror symmetry [1] and statistics [5].

A favorite subject among many tropical geometers is the study of plane tropical curves, and their many fascinating similarities with classical plane algebraic curves. The purpose of this paper is to give a contribution to the list of such analogies, by defining – in a manner resembling the classical case – a group law on a smooth plane tropical cubic curve.

We define the *Jacobian* as an abelian group associated to a tropical curve. Unlike the classical situation, the Jacobian of a tropical elliptic curve C is not equal as a set to the curve itself, but to a smaller part of it, namely the curve's unique cycle \bar{C} . For $P, Q \in \bar{C}$ we define $d_C(P, Q)$ to be the displacement from P to Q with respect to the Z-metric on \bar{C} (and a chosen orientation of \bar{C}). This plays a crucial role in the main results, which can be summarized as follows:

THEOREM 1.1. *Let C be a tropical elliptic curve, and let \bar{C} be its unique cycle. Let \mathcal{O} be a point on \bar{C} .*

- a) *We have a bijection of sets $\bar{C} \longrightarrow \text{Jac}(C)$, given by $P \longmapsto P - \mathcal{O}$.*
- b) *The induced group law on \bar{C} satisfies the relation*

$$d_C(\mathcal{O}, P + Q) = d_C(\mathcal{O}, P) + d_C(\mathcal{O}, Q).$$

- c) *As a group, \bar{C} is isomorphic to the circle group S^1 .*

2. Preliminaries

Let $\mathbf{R}_{\text{tr}} := (\mathbf{R}, \oplus, \odot)$ be the *tropical semiring*, where the binary operations are defined by $a \oplus b := \max\{a, b\}$ and $a \odot b := a + b$. The multiplicative identity element of \mathbf{R}_{tr} is 0, while there is no additive identity (since $-\infty$ is not included as an element in \mathbf{R}_{tr}).

REMARK 2.1. The operations \oplus and \odot can be extended to \mathbf{R}^n as follows:

$$(a_1, \dots, a_n) \oplus (b_1, \dots, b_n) := (\max\{a_1, b_1\}, \dots, \max\{a_n, b_n\}), \quad \text{and} \\ \lambda \odot (a_1, \dots, a_n) := (\lambda + a_1, \dots, \lambda + a_n), \quad \text{for } \lambda \in \mathbf{R}.$$

Moreover, we can define *tropical projective n -space* by setting $\mathbf{P}_{\text{tr}}^n := \mathbf{R}^{n+1}/\sim$, where $\mathbf{x} \sim \mathbf{y} \iff \mathbf{x} = \lambda \odot \mathbf{y}$ for some $\lambda \in \mathbf{R}$. Note that unlike the classical situation, \mathbf{P}_{tr}^n does not have more points than \mathbf{R}^n . For example, every equivalence class in \mathbf{P}_{tr}^n has a representative in \mathbf{R}^{n+1} with 0 as the last coordinate.

Let $\mathcal{A} \subseteq \mathbf{Z}^n$ be a finite set of vectors $\mathbf{a} = (a_1, \dots, a_n)$. A tropical (Laurent) polynomial in indeterminates x_1, \dots, x_n , with support \mathcal{A} , is an expression of the form

$$f = \bigoplus_{\mathbf{a} \in \mathcal{A}} \lambda_{\mathbf{a}} \odot x_1^{a_1} \odot \dots \odot x_n^{a_n} = \max_{\mathbf{a} \in \mathcal{A}} \left\{ \dots, \lambda_{\mathbf{a}} + \sum_{i=1}^n a_i x_i, \dots \right\},$$

where each $\lambda_{\mathbf{a}} \in \mathbf{R}_{\text{tr}}$. The convex hull of \mathcal{A} is called the *Newton polytope of f* and is denoted by Δ . When in danger of ambiguity, we use indices to indicate the polynomial, as in \mathcal{A}_f and Δ_f .

Notice that as a function $\mathbf{R}^n \rightarrow \mathbf{R}$, f is convex and piecewise linear.

DEFINITION 2.2. The *tropical hypersurface $V(f)$* defined by f is the set of points in \mathbf{R}^n where the function $f: \mathbf{R}^n \rightarrow \mathbf{R}$ is not linear.

REMARK 2.3. Note that if f consists of a single monomial, $V(f)$ is the empty set.

REMARK 2.4. Different tropical polynomials can define the same tropical hypersurface. In particular, it is easy to see that if $g = f \odot m$, where $m = x^a y^b$ is a tropical monomial, then $V(g) = V(f)$. Note that in this case \mathcal{A}_g (resp. Δ_g) is a translation of \mathcal{A}_f (resp. Δ_f) by the vector (a, b) .

3. Tropical curves

We now focus our attention to tropical hypersurfaces in \mathbf{R}^2 :

DEFINITION 3.1. Let $f(x, y)$ be a tropical polynomial in two indeterminates. The tropical hypersurface $V(f) \subseteq \mathbf{R}^2$ is called a *tropical curve* in \mathbf{R}^2 .

We recall some basic properties of tropical curves. For proofs and more details, see [6, Section 3], or [3, Sections 1-3] for a more exhaustive approach.

Given a tropical polynomial f , we can associate a lattice subdivision of the Newton polygon Δ of f in the following way: Let $\widehat{\Delta}$ be the convex hull of the set $\{(a, b, \lambda_{abc})\} \subseteq \mathbb{R}^2 \times \mathbb{R}$, where (a, b) runs through \mathcal{A} . Then define Subdiv_f to be the image under the projection to \mathbb{R}^2 of the top facets of $\widehat{\Delta}$, i.e., the facets whose outer normal unit vector has positive last coordinate.

The subdivision Subdiv_f is in a natural way dual to the tropical hypersurface $V(f)$. In particular, each edge of $V(f)$ corresponds to an edge of Subdiv_f , and corresponding edges are perpendicular to each other. The unbounded rays in $V(f)$ correspond to the edges of $\partial\Delta$. (Cf. [6, Proposition 3.5] and [3, Proposition 3.11].)

Let E be an edge of a tropical curve $C = V(f)$, and let E^\vee be the corresponding edge in Subdiv_f . We define the *weight* of E to be the lattice length of E^\vee , i.e. $1 + \#\{\text{interior lattice points of } E^\vee\}$.

LEMMA 3.2. *For any node V of a tropical curve, the following balancing condition holds: Let E_1, \dots, E_n be the edges adjacent to V . For each $i = 1, \dots, n$ let m_i be the weight of E_i , and \mathbf{v}_i the primitive integer vector pointing into E_i from V . Then*

$$(1) \quad m_1 \mathbf{v}_1 + \dots + m_n \mathbf{v}_n = \mathbf{0},$$

where $\mathbf{0} = (0, 0) \in \mathbb{R}^2$.

The balancing condition characterizes tropical curves: Assume C is a 1-dimensional polyhedral complex in \mathbb{R}^2 , consisting of rays and line segments with rational slopes, each assigned some positive integral weight. Then $C = V(f)$ for some tropical polynomial f if and only if (1) is satisfied at every vertex of C .

Next we define the degree of a tropical curve. For each $d \in \mathbb{N}_0$, let Γ_d be the triangle with vertices $(0, 0)$, $(d, 0)$, $(0, d)$. (When $d = 0$ we get the degenerated triangle $\Gamma_0 = \{(0, 0)\}$.)

DEFINITION 3.3. Let $C = V(f)$ be a tropical curve in \mathbb{R}^2 , and let Δ be the Newton polygon of f . If Δ fits inside Γ_d , but not inside Γ_{d-1} , then C has *degree d* . If $\Delta = \Gamma_d$, we say that C has *degree d with full support*.

REMARK 3.4. There seems to be no clear consensus in the literature on how to define the degree of a tropical curve. Definition 3.3 differs slightly from the ones in [6] and [3], but serves the purpose of this paper better. In particular, as we will see in the next section, Definition 3.3 gives room for an extended version of the tropical Bezout's theorem compared to that in [6].

EXAMPLE 3.5. A *tropical line* is a tropical curve of degree 1 with full support. For instance, if $f = ax \oplus by \oplus c$, then the tropical line $L = V(f)$ consists of three unbounded rays, emanating from the “center” $(c - a, c - b)$ in the directions $(-1, 0)$, $(0, -1)$ and $(1, 1)$ respectively.

EXAMPLE 3.6. If f is any monomial, then Δ consists of a single point. Hence $V(f)$ has degree 0. This is appropriate since $V(f)$ is an empty set. (Cf. Remark 2.3.)

A vertex V of a tropical curve is called *3-valent* if V has exactly 3 adjacent edges. Furthermore, if these edges have weights m_1, m_2, m_3 and primitive integer direction vectors $\mathbf{u} = (u_0, u_1)$, $\mathbf{v} = (v_0, v_1)$, $\mathbf{w} = (w_0, w_1)$ respectively, we define the *multiplicity* of V to be the absolute value of the number

$$m_1 m_2 \begin{vmatrix} u_0 & u_1 \\ v_0 & v_1 \end{vmatrix} = m_2 m_3 \begin{vmatrix} v_0 & v_1 \\ w_0 & w_1 \end{vmatrix} = m_1 m_3 \begin{vmatrix} w_0 & w_1 \\ u_0 & u_1 \end{vmatrix}.$$

DEFINITION 3.7. A tropical curve is called *smooth* if every vertex is 3-valent and has multiplicity 1.

Notice that in a smooth tropical curve, every edge has weight 1.

DEFINITION 3.8. The *genus* of a smooth tropical curve $C = V(f)$ is the number of vertices of Subdiv_f in the interior of the Newton polygon Δ_f .

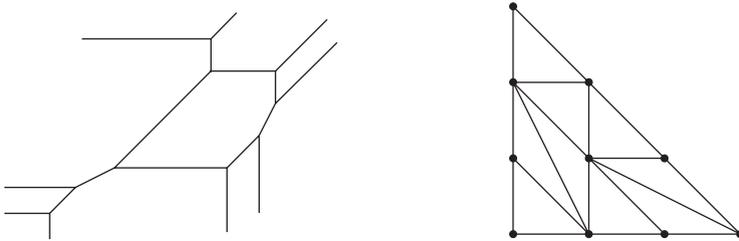


FIGURE 1. A tropical curve and its associated subdivision. The subdivision shows that the curve is smooth of degree 3 and genus 1.

Figure 1 shows a smooth curve of degree 3 and genus 1, and its associated subdivision.

3.1. The Z-metric

Let $C \subseteq \mathbb{R}^2$ be a smooth tropical curve. If E is any edge of C , we define a metric on E called the *Z-metric*, in the following way. For any two points $\mathbf{x}, \mathbf{y} \in E$, we set their distance in the Z-metric to be the number $\frac{\|\mathbf{x}-\mathbf{y}\|}{\|\mathbf{v}\|}$, where $\|\cdot\|$ denotes the Euclidean norm, and \mathbf{v} is a primitive integral direction vector of E . In particular, if E is a bounded edge, we define its *lattice length*, $l(E)$, to be

the distance (in the Z-metric) between its endpoints. Note that if both endpoints of E have integral coordinates, then $l(E) = 1 + \#\{\text{interior lattice points on } E\}$.

REMARK 3.9. By identifying each edge E of C with the real interval $[0, l(E)]$ (or $[0, \infty)$ if E is unbounded), C can be thought of as a “metric graph with possibly unbounded edges”. This is equivalent to giving C a *Z-affine structure*, or *tropical structure* as described e.g. in [4].

4. Intersections of tropical curves

We say that two tropical curves C and D intersect *transversally* if no vertex of C lies on D and vice versa. In a transversal intersection we define intersection multiplicities as follows: Let P be an intersection point of C and D , where the two edges meeting have weights m_1 and m_2 , and primitive direction vectors (v_0, v_1) and (w_0, w_1) respectively. The *intersection multiplicity* $\text{mult}_P(C \cap D)$ is then the absolute value of

$$m_1 m_2 \begin{vmatrix} v_0 & v_1 \\ w_0 & w_1 \end{vmatrix}.$$

Non-transversal intersections are dealt with in the following way: For *any* intersecting tropical curves C and D , let C_ε and D_ε be nearby translations of C and D such that C_ε and D_ε intersect transversally. We then have ([6, Theorem 4.3]):

THEOREM-DEFINITION 4.1. *Let the stable intersection of C and D , denoted $C \cap_{\text{st}} D$, be defined by*

$$C \cap_{\text{st}} D = \lim_{\varepsilon \rightarrow 0} (C_\varepsilon \cap D_\varepsilon).$$

This limit is independent of the choice of perturbations, and is a well-defined subset of points with multiplicities in $C \cap D$.

THEOREM 4.2 (Tropical Bezout). *Assume C and D are tropical curves of degrees c and d respectively. If both curves have full support, then their stable intersection consists of cd points, counting multiplicities.*

PROOF. See [6, Theorem 4.2 and Corollary 4.4]. The idea is to show that the number of (stable) intersection points is invariant under translations of the curves. Thus we can arrange the two curves such that for each of them, the intersection points lie on the unbounded rays in one of the three coordinate directions. It is then trivial to check that $\#\{C \cap_{\text{st}} D\} = cd$.

There is also a tropical version of Bernstein’s Theorem: Recall that the *mixed area* of two convex polygons R and S is defined as the number $\text{Area}(R + S) - \text{Area}(R) - \text{Area}(S)$, where $R + S$ is the *Minkowski sum* of R and S .

THEOREM 4.3 (Tropical Bernstein). *Let $C = V(f)$ and $D = V(g)$ be any tropical curves intersecting transversally, with Newton polygons Δ_f and Δ_g respectively. Then the number of intersection points, counting multiplicities, equals the mixed area of Δ_f and Δ_g .*

PROOF. See [7, Theorem 9.5].

Although perhaps not as enlightening as the homotopy argument given in [6], one can prove Theorem 4.2 as a special case of Theorem 4.3. In fact, we can get a stronger result:

THEOREM 4.4 (Strong version of Tropical Bezout). *Assume C and D are tropical curves of degrees c and d respectively. If at least one of the curves have full support, then their stable intersection consists of cd points, counting multiplicities.*

PROOF. Because of Theorem-Definition 4.1 we can assume that the intersection is transversal. Note that for any positive integers c and d , we have the Minkowski sum $\Gamma_c + \Gamma_d = \Gamma_{c+d}$. Hence the mixed area of $\Gamma_c + \Gamma_d$ equals $\frac{1}{2}(c+d)^2 - \frac{1}{2}c^2 - \frac{1}{2}d^2 = cd$. This proves Theorem 4.2.

Suppose now C has full support, i.e. $\Delta_f = \Gamma_c$, and that Δ_g is a convex polygon of the form $\Gamma_d \setminus Q$, where $Q \subseteq \Gamma_d$ is a lattice polygon containing exactly one of the corners of Γ_d , say $(d, 0)$. Then $\text{Area}(\Delta_f + \Delta_g) = \text{Area}(\Gamma_c + (\Gamma_d \setminus Q)) = \text{Area}(\Gamma_c + \Gamma_d) - \text{Area}(Q)$. Thus the mixed area of Δ_f and Δ_g is

$$\begin{aligned} & \text{Area}(\Delta_f + \Delta_g) - \text{Area}(\Delta_f) - \text{Area}(\Delta_g) \\ &= (\text{Area}(\Gamma_c + \Gamma_d)) - \text{Area}(Q) - \text{Area}(\Gamma_d) - (\text{Area}(\Gamma_d) - \text{Area}(Q)) \\ &= cd. \end{aligned}$$

The same argument shows that we can do the same at the other corners, without changing the mixed area. In this way we can form any Newton polygon Δ_g associated to a tropical curve of degree d . Hence $\sharp(C \cap_{\text{st}} D) = cd$ for any tropical curve D of degree d .

REMARK 4.5. If neither of the two curves have full support, the theorem will not hold in general. For example, if C and D are the quadric curves given by $C = V(x^2 \oplus y)$ and $D = V(x \oplus y^2)$, then $C \cap D$ consists of a single point with multiplicity 3. Another example is given by the non-intersecting lines $V(0 \oplus x)$ and $V(1 \oplus x)$.

An important special case of Theorem 4.4 is the following corollary:

COROLLARY 4.6. *Let D be any tropical curve of degree d . Then any tropical line meets D stably in exactly d points, counting multiplicities.*

5. Divisors on smooth tropical curves

Let C be a smooth tropical curve in \mathbb{R}^2 .

DEFINITION 5.1. We define the *group of divisors* on C , $\text{Div}(C)$, to be the free abelian group generated by the points on C . A *divisor* D on C is an element of $\text{Div}(C)$, i.e. a finite formal sum of the form $D = \sum \mu_P P$.

The number $\sum \mu_P$ is as usual called the *degree* of D . Observe that the elements of degree 0 in $\text{Div}(C)$ form a group, denoted by $\text{Div}^0(C)$.

To define principle divisors, we must first define rational functions. By a *tropical rational function* $h: \mathbb{R}^2 \rightarrow \mathbb{R}$ we mean a function of the form $h = f - g$, where f and g are tropical polynomials with equal Newton polygons.

DEFINITION 5.2. Given a tropical polynomial f , we define the divisor $\text{div}(f) \in \text{Div}(C)$ as the formal sum of points in $C \cap_{\text{st}} V(f)$, counted with their respective intersection multiplicities. Furthermore, if $h = f - g$ is a tropical rational function on \mathbb{R}^2 , we set $\text{div}(h) := \text{div}(f) - \text{div}(g)$. A divisor $D \in \text{Div}(C)$ is called a *principal divisor* if $D = \text{div}(h)$ for some tropical rational function h .

It follows from Theorem 4.3 that any principal divisor on C has degree 0.

REMARK 5.3. Suppose the Newton polygons of f and g differ by a translation. Then we would still have $\text{div}(f) - \text{div}(g) \in \text{Div}^0(C)$, because of Theorem 4.3. In fact, $\text{div}(f) - \text{div}(g)$ is a principle divisor. Indeed, if (a, b) is the translation vector from Δ_f to Δ_g , let $m = x^a y^b$ be the corresponding tropical monomial. Since $V(f) = V(f \odot m)$ (by Remark 2.4) it follows that $\text{div}(f) - \text{div}(g) = \text{div}(h)$, where h is the tropical rational function $(f \odot m) - g$.

DEFINITION 5.4. Two divisors D_1 and D_2 are *linearly equivalent*, denoted as $D_1 \sim D_2$, if $D_1 - D_2$ is principal.

Linear equivalence is an equivalence relation, and as in the classical case one can show that it restricts to an equivalence relation on the subgroup $\text{Div}^0(C)$. Hence we can make the following definition:

DEFINITION 5.5. The group $\text{Div}^0(C)/\sim$ is called the *Jacobian* of C , $\text{Jac}(C)$.

5.1. A formula for the divisor of a tropical rational function

The purpose of this section is to develop a formula for the divisor of a tropical rational function $h: \mathbb{R}^2 \rightarrow \mathbb{R}$, using only the properties of h restricted to C . We begin with some easy observations:

LEMMA 5.6. *Let $h: \mathbb{R}^2 \rightarrow \mathbb{R}$ be a tropical rational function, and $C \subseteq \mathbb{R}^2$ a tropical curve. The restriction of h to C is then*

- a) *continuous on C ,*
- b) *piecewise linear on each edge of C , with integer slopes (with respect to the Z -metric on the edge),*
- c) *eventually constant on each unbounded ray of C .*

PROOF. a) Note that h is the difference of tropical polynomials, which are continuous.

b) It is enough to prove this for tropical polynomials. Let E be an edge of C , with primitive integer direction vector \mathbf{v} , and let $f(\mathbf{x}) = \max_{\mathbf{a} \in \mathcal{A}} \{\alpha_{\mathbf{a}} + \mathbf{a} \cdot \mathbf{x}\}$ be a tropical polynomial function. Clearly, $f|_E$ is piecewise linear on E . Furthermore, consider any point $P \in E$ such that f is linear in an open interval $I \subseteq E$ containing P . Then $f(\mathbf{x}) = \alpha_{\mathbf{a}} + \mathbf{a} \cdot \mathbf{x}$ for all $\mathbf{x} \in I$, for some $\mathbf{a} \in \mathcal{A}$, and the slope of f at P in the direction of \mathbf{v} (w.r.t the Z -metric on E), is $f(P + \mathbf{v}) - f(P) = \mathbf{a} \cdot \mathbf{v} \in \mathbb{Z}$.

c) Suppose $h(\mathbf{x}) = f(\mathbf{x}) - g(\mathbf{x})$, where $f(\mathbf{x}) = \max_{\mathbf{a} \in \mathcal{A}_f} \{\alpha_{\mathbf{a}} + \mathbf{a} \cdot \mathbf{x}\}$ and $g(\mathbf{x}) = \max_{\mathbf{b} \in \mathcal{A}_g} \{\beta_{\mathbf{b}} + \mathbf{b} \cdot \mathbf{x}\}$ are tropical polynomials with $\Delta_f = \Delta_g$. Let $\ell = \{V + t\mathbf{u} \mid t \geq 0\}$ be an unbounded ray of C , starting at the vertex V and with primitive direction vector \mathbf{u} . Then $f(V + t\mathbf{u}) = \max_{\mathbf{a} \in \mathcal{A}_f} \{\alpha_{\mathbf{a}} + \mathbf{a} \cdot V + t(\mathbf{a} \cdot \mathbf{u})\}$. For all $t \gg 0$ this maximum is achieved for some $\mathbf{a} = \mathbf{a}_\ell$ with the property that $\mathbf{a}_\ell \cdot \mathbf{u} \geq \mathbf{a} \cdot \mathbf{u}$ for all $\mathbf{a} \in \mathcal{A}_f$. In particular this implies that $\mathbf{a}_\ell \in \partial \Delta_f$. Similarly, when $t \gg 0$, we have $g(V + t\mathbf{u}) = \beta_{\mathbf{b}_\ell} + \mathbf{b}_\ell \cdot V + t(\mathbf{b}_\ell \cdot \mathbf{u})$, for some $\mathbf{b}_\ell \in \partial \Delta_g$ such that $\mathbf{b}_\ell \cdot \mathbf{u} \geq \mathbf{b} \cdot \mathbf{u}$ for all $\mathbf{b} \in \mathcal{A}_g$. Since $\Delta_f = \Delta_g$ this implies that $\mathbf{a}_\ell \cdot \mathbf{u} = \mathbf{b}_\ell \cdot \mathbf{u}$, and we conclude that for $t \gg 0$ we have $h(V + t\mathbf{u}) = \alpha_{\mathbf{a}_\ell} - \beta_{\mathbf{b}_\ell} + (\mathbf{a}_\ell - \mathbf{b}_\ell) \cdot V$, which is constant.

For any function $r: C \rightarrow \mathbb{R}$ satisfying a) and b) above, we associate to each point $P \in C$ an integer $\text{ord}_P(r)$ as follows: If P is a vertex of C , then $\text{ord}_P(r)$ is the sum of the outgoing slopes of r along the edges adjacent to P . If $P \in C$ is not a vertex, we use the same definition, after first having inserted a (2-valent) vertex at P (but otherwise keeping C unchanged).

We say that r is locally linear at $P \in C$ if there exists an open neighborhood $U \subseteq \mathbb{R}^2$ containing P , and an affine-linear function $s: \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $r|_{C \cap U} = s|_{C \cap U}$. It is easy to see that $\text{ord}_P(r) = 0$ if r is locally linear at P . Note however, that the converse is not true if P is a vertex of C .

We are now ready to prove the following:

LEMMA 5.7. *For any tropical rational function $h: \mathbb{R}^2 \rightarrow \mathbb{R}$ we have*

$$\text{div}(h) = \sum_{P \in C} \text{ord}_P(h|_C)P.$$

PROOF. Let $f = \max_{\mathbf{a} \in \mathcal{A}_f} \{\alpha_{\mathbf{a}} + \mathbf{a} \cdot \mathbf{x}\}$ be a tropical polynomial, and let $\overline{f} := f|_C$. It is clear that \overline{f} is locally linear at any $P \in C \setminus (C \cap_{\text{st}} V(f))$,

and therefore $\text{ord}_P(\bar{f}) = 0$ for such P . We show below that for each $P \in C \cap_{\text{st}} V(f)$, the intersection multiplicity at P equals $\text{ord}_P(\bar{f})$. This implies $\text{div}(f) = \sum_{P \in C} \text{ord}_P(\bar{f}) P$. By the obvious extension from tropical polynomials to tropical rational functions, the lemma follows from this.

Consider first the case where P is a transversal intersection point, between an edge E_C of C and an edge E_f of $V(f)$. We can choose primitive direction vectors $\mathbf{v} = (v_1, v_2)$ and $\mathbf{u} = (u_1, u_2)$ of E_C and E_f respectively, such that if m is the weight of E_f , the intersection multiplicity is $\text{mult}_P(C \cap V(f)) = m(v_1u_2 - v_2u_1)$.

To find $\text{ord}_P(\bar{f})$, suppose $\mathbf{a}, \mathbf{b} \in \mathcal{A}_f$ are such that f equals $\alpha_a + \mathbf{a} \cdot \mathbf{x}$ on one side of P , and $\alpha_b + \mathbf{b} \cdot \mathbf{x}$ on the other side. Then $\mathbf{b} - \mathbf{a}$ is orthogonal to E_f . Moreover, by the definition of weight, we have (possibly after swapping \mathbf{a} and \mathbf{b}) that $\mathbf{b} - \mathbf{a} = m(u_2, -u_1)$. This implies that $\text{ord}_P(\bar{f}) = \mathbf{v} \cdot \mathbf{b} + (-\mathbf{v}) \cdot \mathbf{a} = \mathbf{v} \cdot (\mathbf{b} - \mathbf{a}) = m(v_1u_2 - v_2u_1)$, which equals the intersection multiplicity found above.

Next, suppose $P \in C \cap_{\text{st}} V(f)$ is a non-transversal intersection point, i.e., that P is a vertex of either C or $V(f)$. In either case, consider $f_\varepsilon = \max_{\mathbf{a} \in \mathcal{A}_f} \{\alpha_a^\varepsilon + \mathbf{a} \cdot \mathbf{x}\}$ such that $V(f_\varepsilon)$ is a small translation of $V(f)$ intersecting C transversally, and $P \notin V(f_\varepsilon)$. Let $P_1, \dots, P_k \in C \cap V(f_\varepsilon)$ be the intersection points close to P (i.e. the points tending to P when $f_\varepsilon \rightarrow f$). Then we have

$$(2) \quad \text{mult}_P(C \cap_{\text{st}} V(f)) = \sum_{i=1}^k \text{mult}_{P_i}(C \cap V(f_\varepsilon)).$$

We proceed to show that $\text{ord}_P(\bar{f})$ shows a similar, stable behavior. Let ℓ_1, \dots, ℓ_s be the edges of C emanating from P , with primitive direction vectors $\mathbf{v}_1, \dots, \mathbf{v}_s$. (If P is not a vertex of C , we insert a vertex at P , making $s = 2$, and $v_1 = -v_2$.) Furthermore, let $\mathbf{a}_1, \dots, \mathbf{a}_s \in \mathcal{A}_f$ be such that for $\mathbf{x} \in \ell_i$ close to P , we have $f(\mathbf{x}) = \lambda_{\mathbf{a}_i} + \mathbf{a}_i \cdot \mathbf{x}$. In particular, with this notation, we have $\text{ord}_P(\bar{f}) = \sum_{i=1}^s \mathbf{a}_i \cdot \mathbf{v}_i$.

Because $P \notin V(f_\varepsilon)$, f_ε is locally linear at P , and we can assume w.l.o.g. that $f_\varepsilon(\mathbf{x}) = \alpha_{\mathbf{a}_1}^\varepsilon + \mathbf{a}_1 \cdot \mathbf{x}$ in a neighborhood of P . For $j = 1, \dots, s$, let $B_j \subseteq \{P_1, \dots, P_k\}$ be the subset whose elements lies on ℓ_j . It is not hard to see that if $B_j \neq \emptyset$, then $\sum_{Q \in B_j} \text{ord}_Q(\bar{f}_\varepsilon) = \mathbf{a}_j \cdot \mathbf{v}_j - \mathbf{a}_1 \cdot \mathbf{v}_j$. Hence,

$$(3) \quad \begin{aligned} \sum_{i=1}^k \text{ord}_{P_i}(\bar{f}_\varepsilon) &= \sum_{B_j \neq \emptyset} (\mathbf{a}_j \cdot \mathbf{v}_j - \mathbf{a}_1 \cdot \mathbf{v}_j) \\ &= \sum_{B_j \neq \emptyset} \mathbf{a}_j \cdot \mathbf{v}_j + \sum_{B_j = \emptyset} \mathbf{a}_1 \cdot \mathbf{v}_j = \text{ord}_P(\bar{f}), \end{aligned}$$

where in the second to last transition we used the balancing condition, and in the final transition the easily proved observation that if $B_j = \emptyset$ then $\mathbf{a}_j = \mathbf{a}_1$.

From (2) and (3) we deduce that $\text{mult}_P(C \cap_{\text{st}} V(f)) = \text{ord}_P(\bar{f})$ also when P is non-transversal, and hence that $\text{div}(f) = \sum \text{ord}_P(\bar{f})P$. This proves the lemma.

REMARK 5.8. A consequence of the above two lemmas is that our definitions of tropical rational functions and their divisors are in agreement with those used by Gathmann and Kerber in [2]. The set $\mathcal{R} = \{h|_C \mid h \text{ is a tropical rational function}\}$ is a subset of what they call *rational functions on C*, i.e. functions $r : C \rightarrow \mathbb{R}$ which satisfy parts a) and b) of Lemma 5.6. Moreover, Lemma 5.7 implies that for any function in \mathcal{R} , the definition of its associated divisor given in [2] is equivalent to our Definition 5.2. In particular, the endpoints at infinity of unbounded rays (these are included as part of the curve in [2]) are avoided because of Lemma 5.6c).

6. Tropical elliptic curves

In the remainder of the paper C will denote a *tropical elliptic curve*, by which we mean a smooth tropical curve of degree 3 and genus 1. We assume that $C = V(f)$, where $f(x, y)$ has Newton polygon $\Delta_f \subseteq \Gamma_3$. Since $(1, 1)$ is the only lattice point in the interior of Γ_3 , the definition of genus requires that $(1, 1)$ is a vertex of Subdiv_f lying in the interior of Δ_f . Hence C contains a unique cycle, which we will denote by \bar{C} . Finally, each connected component of $C \setminus \bar{C}$ is called a *tentacle* of C .

6.1. An explicit homeomorphism $\bar{C} \rightarrow S^1$

Obviously, as a topological space, \bar{C} is homeomorphic to the circle group S^1 . We will now construct one such homeomorphism, based on the Z-metric on the edges of C .

Choose any fixed point $\mathcal{O} \in \bar{C}$. Let V_1, \dots, V_n be the vertices of \bar{C} in counter-clockwise direction, such that if \mathcal{O} is a vertex then $V_1 = \mathcal{O}$, otherwise \mathcal{O} lies between V_1 and V_n . Let E_1, \dots, E_n be the edges of \bar{C} , such that $E_1 = V_1 V_2$ and so on. Recall that for each i , $l(E_i)$ denotes the length of E_i in the Z-metric on E_i . Let l be the *cycle length* of \bar{C} , i.e., $l = l(E_1) + \dots + l(E_n)$.

We now define a homeomorphism $\mu: \bar{C} \rightarrow \mathbb{R}/l\mathbb{Z} \approx S^1$, linear in the Euclidean metric of each edge E_i . It is then enough to specify the images in $\mathbb{R}/l\mathbb{Z}$ of the points $\mathcal{O}, V_1, \dots, V_n$, which we do recursively:

$$\begin{aligned}
 \mu(\mathcal{O}) &= 0 \\
 \mu(V_1) &= l(\mathcal{O}V_1) \\
 \mu(V_{i+1}) &= \mu(V_i) + l(E_i), \quad i = 1, \dots, n - 1.
 \end{aligned}
 \tag{4}$$

Finally, identifying $\mathbb{R}/l\mathbb{Z}$ with the interval $[0, l)$, we define the (signed) displacement function $d_C: \bar{C} \times \bar{C} \rightarrow \mathbb{R}$ by the formula

$$(5) \quad d_C(P, Q) = \mu(Q) - \mu(P).$$

Note that $d_C(Q, P) = -d_C(P, Q)$ for any $P, Q \in \bar{C}$. Moreover, for any three points $P, Q, R \in \bar{C}$ we have $d_C(P, Q) + d_C(Q, R) = d_C(P, R)$.

6.2. *When are two points on C linearly equivalent?*

In this section we give two propositions, which together give a complete answer to the question in the title. Namely, we prove that any two points on the same tentacle are linearly equivalent, while two distinct points on \bar{C} are never linearly equivalent.

PROPOSITION 6.1. *Let P and Q be points on the same tentacle of C . Then $P \sim Q$.*

PROOF. We begin by showing that the points on any unbounded ray are equivalent. By symmetry, it is enough to prove this for the rays that are unbounded in, say, the x -coordinate. Figure 2 shows a typical situation with three such rays, ℓ_1, ℓ_2 and ℓ_3 .

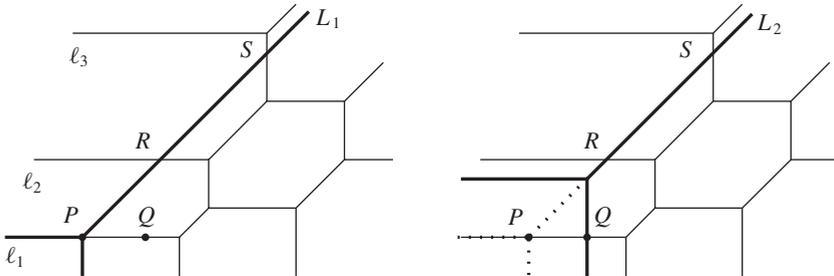


FIGURE 2. Sufficiently close points P and Q on ℓ_1 are linearly equivalent.

The following argument shows that any two sufficiently close points P and Q on ℓ_1 are equivalent: Assume P is further away from \bar{C} than Q . Let $h = f_1 - f_2$ be the tropical rational function where f_1 and f_2 are tropical linear polynomials such that $L_1 = V(f_1)$ is the tropical line with center in P , and $L_2 = V(f_2)$ is the line passing through Q and with center on the ray of L_1 with direction vector $(1, 1)$. Denote this ray by ρ . Then $\text{div}(f_1) = P + R + S$, where R and S lies on ρ , and $\text{div}(f_2) = Q + R + S$ (as long as P and Q are close enough). It follows that $\text{div}(h) = P - Q$, in other words $P \sim Q$.

To show that *any* two points P and Q on ℓ_1 are equivalent, we can choose a finite sequence of points $P = P_1, P_2, \dots, P_m = Q$ on ℓ_1 such that each pair

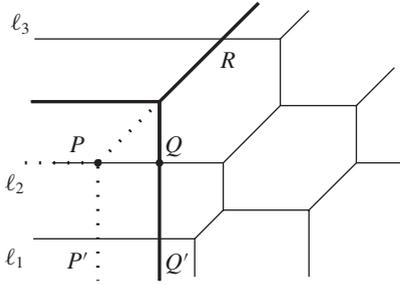


FIGURE 3. $P \sim Q$ on ℓ_2 .

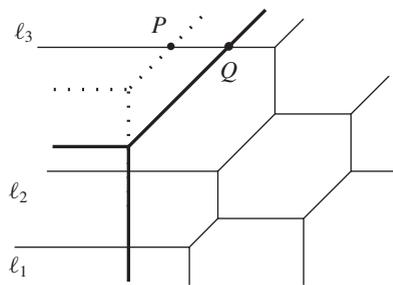


FIGURE 4. $P \sim Q$ on ℓ_3 .

(P_i, P_{i+1}) is close enough for the above technique to work. Then $P = P_1 \sim \dots \sim P_m = Q$.

A similar argument shows that the points on ℓ_2 are equivalent. The idea is sketched in Figure 3. To show that P and Q are equivalent, take the tropical line L_1 with center in P and slide it along the ray with direction $(1, 1)$ (i.e. keeping R as intersection point with C) until it passes through Q . With the notation on Figure 3, we see that $P + P' + R \sim Q + Q' + R$. But $P' \sim Q'$, since they are on ℓ_1 , thus $P \sim Q$.

The same technique works for ℓ_3 (see Figure 4) and also for the bounded line segments of the tentacles. Any tentacle of a tropical elliptic curve can be handled in this way.

PROPOSITION 6.2. *If $P, Q \in \bar{C}$ and $P \sim Q$, then $P = Q$.*

PROOF. Suppose otherwise that $P \neq Q$, and that there exists a tropical rational function h such that $\text{div}(h) = P - Q$. We will apply Lemma 5.7 to show that this leads to a contradiction.

Let $\bar{h} = h|_C$. As a first observation, note that \bar{h} is constant on each tentacle of C . Indeed, this follows from Lemma 5.6c) and the fact that $\text{ord}_R(\bar{h}) = 0$ for all points $R \in C \setminus \bar{C}$. (Note in particular that if \bar{h} is constant on two edges adjacent to a 3-valent vertex V , then $\text{ord}_V(\bar{h}) = 0$ implies that \bar{h} is constant on the third edge as well.)

Now, let c_1 and c_2 be the two directed polygonal arcs of \bar{C} from P to Q . We claim that for each $i = 1, 2$, \bar{h} has constant slope along c_i , w.r.t. the \mathbb{Z} -metric. To see this, observe that \bar{h} is clearly linear along any edge of c_i . Furthermore, suppose two edges of c_i intersect in a vertex $V \in \bar{C}$, and that the slopes of \bar{h} along these edges (directed from P to Q) are s_1 and s_2 . Because \bar{h} is constant on the tentacle adjacent to V , $\text{ord}_V(\bar{h})$ is of the form $\pm(s_1 - s_2 + 0)$. This equals 0, hence $s_1 = s_2$. This proves the claim. Since $\text{ord}_P(\bar{h}) = 1$, the slopes of \bar{h} along the paths c_1 and c_2 must be s and $1 - s$ for some $s \in \mathbb{Z}$. But this contradicts the assumption of continuity of \bar{h} at Q , since for any choice of s ,

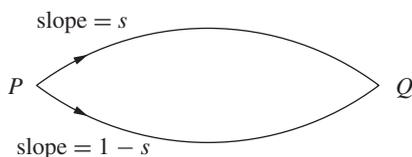


FIGURE 5. Slope properties of a function h satisfying $\operatorname{div}(h) = P - Q$.

one of the numbers s and $1 - s$ is positive, while the other is non-positive. (See Figure 5.)

6.3. The group law

In this final section we show that the Jacobian $\operatorname{Jac}(C)$ is set-theoretically equal to \bar{C} , and describe the resulting group structure on \bar{C} . A crucial step towards this goal is to determine when divisors of the form $P + Q$ are linearly equivalent. When trying to imitate the techniques from the classical case, we stumble across the following problem: Given two points P and Q on \bar{C} , we cannot always find a tropical line L that intersects C stably in P and Q . (Recall that a stable intersection is defined as a limit of transversal intersections.) If there exists such a tropical line, we call (P, Q) a *good pair*.

We fix the notation $\mathbf{p}_1 = (-1, 0)$, $\mathbf{p}_2 = (0, -1)$ and $\mathbf{p}_3 = (1, 1)$ for the primitive integer direction vectors of a tropical line.

LEMMA 6.3. *Let P, Q, P', Q' be any points on \bar{C} . Then*

$$P + Q \sim P' + Q' \iff d_C(P, P') = -d_C(Q, Q').$$

PROOF. We proceed in two steps. First, we prove the result when (P, Q) and (P', Q') are good pairs. Using this, we then generalize to any pairs.

• *Step 1.* Assume (P, Q) and (P', Q') are good pairs, and that $P + Q \sim P' + Q'$. Then there exists (unique) tropical lines L and L' , and a point $R \in \bar{C}$ such that $L \cap_{\text{st}} C = P + Q + R$ and $L' \cap_{\text{st}} C = P' + Q' + R$. (Note that the existence of R follows from Proposition 6.2.) Consider a homotopy L_t of lines containing R such that $L_0 = L$ and $L_1 = L'$. It is enough to consider the case where P and P' are on the same edge, Q and Q' are on the same edge, and L' is a parallel displacement of L along one of the axes. Indeed, in more complex cases, the homotopy can be broken down into parts with the above properties.

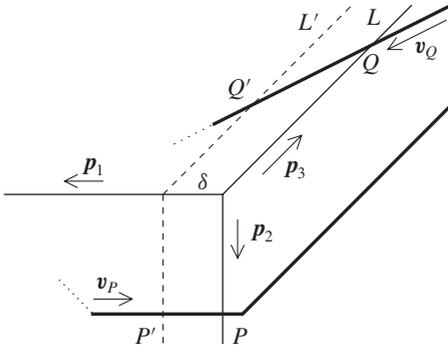


FIGURE 6. Step 1.

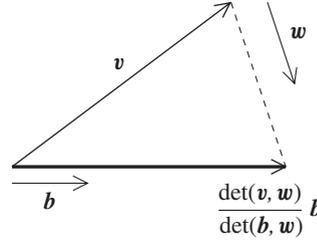


FIGURE 7. Non-orthogonal projection in \mathbb{R}^2 .

Let v_P and v_Q be primitive integer direction vectors of the edges of \bar{C} containing P, P' and Q, Q' respectively, and assume that L' equals the shifting of L δ units in the direction of, say, p_1 (see Figure 6). Then from the formula for non-orthogonal vector projection (Figure 7), we find the Z-metric displacements of P and Q :

$$|d_C(P, P')| = \frac{\|P' - P\|}{\|v_P\|} = \frac{\left\| \frac{\det(\delta p_1, p_2)}{\det(v_P, p_2)} v_P \right\|}{\|v_P\|} = |\delta|, \quad \text{and}$$

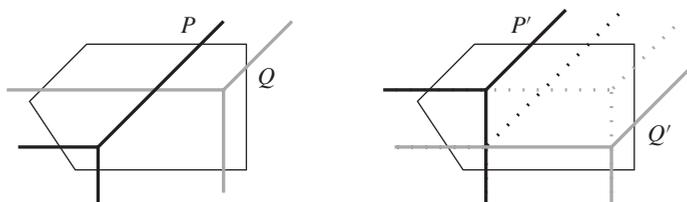
$$|d_C(Q, Q')| = \frac{\|Q' - Q\|}{\|v_Q\|} = \frac{\left\| \frac{\det(\delta p_1, p_3)}{\det(v_Q, p_3)} v_Q \right\|}{\|v_Q\|} = |\delta|$$

(Note that $|\det(v_P, p_2)| = |\det(v_Q, p_3)| = 1$ since the corresponding intersections have multiplicity 1.) Finally, it is clear that P and Q are moved in opposite directions on \bar{C} . Hence $d_C(P, P') = -d_C(Q, Q')$ as claimed.

The implication \Leftarrow follows by a similar argument.

• *Step 2.* Now assume (P, Q) is not a good pair. Let L_1 and L_2 be tropical lines through P and Q respectively, and let R_1, S_1, R_2, S_2 be the other intersection points. The idea is to move L_1 and L_2 into new lines L'_1 and L'_2 in such a way that R_1, S_1, R_2, S_2 are preserved as intersection points. P and Q are not preserved; they move to the two new intersection points P' and Q' (see Figure 8). By construction, these points satisfy $P' + Q' \sim P + Q$. Using our results in Step 1 on each of the lines L_1 and L_2 , it follows that $d_C(P, P') = -d_C(Q, Q')$. Conversely, it is not hard to see that in this way one can reach any nearby pair (P', Q') satisfying $d_C(P, P') = -d_C(Q, Q')$.

Moving P and Q far enough apart (sometimes more than one move is needed) one obtains a good pair (P', Q') . Since we proved in Step 1 that the

FIGURE 8. Moving a bad pair (P, Q) .

lemma holds for good pairs, it follows that the lemma is true for any pairs (P, Q) and (P', Q') .

PROPOSITION 6.4. *For any fixed point $\mathcal{O} \in \bar{C}$, the map $\tau_{\mathcal{O}}: \bar{C} \rightarrow \text{Jac}(C)$ given by $P \mapsto P - \mathcal{O}$ is a bijection of sets.*

PROOF. Injectivity follows immediately from Lemma 6.3, since

$$P - \mathcal{O} \sim Q - \mathcal{O} \implies P + \mathcal{O} \sim Q + \mathcal{O} \implies d_C(P, Q) = 0 \implies P = Q.$$

To prove surjectivity, let D be any divisor of degree 0. We must show that there exists $P \in \bar{C}$ such that $D \sim P - \mathcal{O}$. Assume first that $D = P_1 - Q_1$, where $P_1, Q_1 \in \bar{C}$. Choose P such that $d_C(P, P_1) = d_C(\mathcal{O}, Q_1)$, then Lemma 6.3 gives $P + Q_1 \sim P_1 + \mathcal{O}$. Thus we have $D = P_1 - Q_1 \sim P - \mathcal{O}$.

Now assume $D = D_1 - D_2$, where $D_1 = P_1 + \dots + P_n$ and $D_2 = Q_1 + \dots + Q_n$ are any effective divisors of degree $n > 1$. Let P_{12} and Q_{12} be points such that $P_1 + P_2 \sim \mathcal{O} + P_{12}$ and $Q_1 + Q_2 \sim \mathcal{O} + Q_{12}$. Then

$$\begin{aligned} D &\sim \mathcal{O} + P_{12} + \dots + P_n - (\mathcal{O} + Q_{12} + \dots + Q_n) \\ &= P_{12} + \dots + P_n - (Q_{12} + \dots + Q_n). \end{aligned}$$

Hence $D \sim D'_1 - D'_2$, where D'_1 and D'_2 are effective of degree $n - 1$. This way we can reduce to the case $n = 1$, which we already proved.

Because of Proposition 6.4, \bar{C} has a natural group structure:

DEFINITION 6.5. Define (\bar{C}, \mathcal{O}) to be the group consisting of points on \bar{C} , with the group structure induced from $\text{Jac}(C)$ such that the bijection $\tau_{\mathcal{O}}$ is an isomorphism of groups.

The next theorem and its corollary are the main results of this paper.

THEOREM 6.6. *Let P and Q be any points on \bar{C} , and let $+$ denote addition in the group (\bar{C}, \mathcal{O}) . Then the point $P+Q$ satisfies the relation*

$$d_C(\mathcal{O}, P+Q) = d_C(\mathcal{O}, P) + d_C(\mathcal{O}, Q).$$

PROOF. Because $\tau_{\mathcal{O}}$ is a group isomorphism, the following equalities hold in $\text{Jac}(C)$:

$$(P+Q) - \mathcal{O} = \tau_{\mathcal{O}}(P+Q) = \tau_{\mathcal{O}}(P) + \tau_{\mathcal{O}}(Q) = P - \mathcal{O} + Q - \mathcal{O}.$$

Thus in $\text{Jac}(C)$ we have $(P+Q) + \mathcal{O} = P + Q$. This means that the divisors $(P+Q) + \mathcal{O}$ and $P + Q$ are equivalent, which by Lemma 6.3 implies the relation

$$d_C(P, P+Q) = d_C(\mathcal{O}, Q).$$

Adding $d_C(\mathcal{O}, P)$ on each side then gives $d_C(\mathcal{O}, P+Q) = d_C(\mathcal{O}, P) + d_C(\mathcal{O}, Q)$ as wanted.

REMARK 6.7. We can describe the group law geometrically just as in the classical case of elliptic curves: To add P and Q we do the following. If (P, Q) is a good pair, consider the tropical line L through P and Q , and let R be the third intersection point of L and \bar{C} . Now if (R, \mathcal{O}) is a good pair, let L' be the through R and \mathcal{O} . Then $P+Q$ is the third intersection point of L' and \bar{C} . (See Figure 9 for an example.)

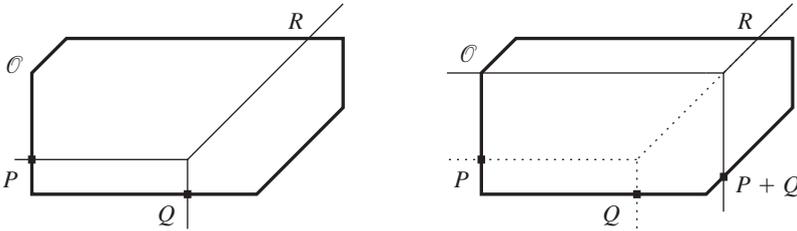


FIGURE 9. Adding points on a tropical elliptic curve.

If any of the pairs (P, Q) and (R, \mathcal{O}) fail to be good, then move the two points involved equally far (in the Z -metric) in opposite directions until they form a good pair, and use this new pair as described above.

COROLLARY 6.8. *The map $\mu: (\bar{C}, \mathcal{O}) \rightarrow \mathbb{R}/I\mathbb{Z} \approx S^1$ defined in (4) is a group isomorphism.*

PROOF. It follows from the relation (5) that for any P we have $\mu(P) = d_C(\mathcal{O}, P)$. Thus

$$\mu(P+Q) = d_C(\mathcal{O}, P+Q) = d_C(\mathcal{O}, P) + d_C(\mathcal{O}, Q) = \mu(P) + \mu(Q).$$

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