INEQUALITIES FOR PRODUCTS OF POLYNOMIALS I

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Abstract
We study inequalities connecting the product of uniform norms of polynomials with the norm of their product. This circle of problems include the Gelfond-Mahler inequality for the unit disk and the Kneser-Borwein inequality for the segment $[-1, 1]$. Furthermore, the asymptotically sharp constants are known for such inequalities over arbitrary compact sets in the complex plane. It is shown here that this best constant is smallest (namely: 2) for a disk. We also conjecture that it takes its largest value for a segment, among all compact connected sets in the plane.

1. The problem and its history
Let $E$ be a compact set in the complex plane $C$. For a function $f : E \to C$ define the uniform (sup) norm as follows:

$$\|f\|_E = \sup_{z \in E} |f(z)|.$$ 

Clearly $\|f_1 f_2\|_E \leq \|f_1\|_E \|f_2\|_E$, but this inequality is not reversible, in general, not even with a constant factor in front of the right hand side. Indeed, $\|f_1\|_E \|f_2\|_E \leq C \|f_1 f_2\|_E$ does not hold for functions with disjoint supports in $E$, for example. However, the situation is quite different for algebraic polynomials $\{p_k(z)\}_{k=1}^m$ and their product $p(z) := \prod_{k=1}^m p_k(z)$. Polynomial inequalities of the form

$$\prod_{k=1}^m \|p_k\|_E \leq C \|p\|_E,$$

exist and are readily available. One of the first results in this direction is due to Kneser [19], for $E = [-1, 1]$ and $m = 2$ (see also Aumann [1]), who proved that

$$\|p_1\|_{[-1, 1]} \|p_2\|_{[-1, 1]} \leq K_{\ell,n} \|p_1 p_2\|_{[-1, 1]}, \quad \deg p_1 = \ell, \quad \deg p_2 = n - \ell,$$

∗Research of I. P. was partially supported by the National Security Agency (grant H98230-06-1-0055), and by the Alexander von Humboldt Foundation. S. R. acknowledges partial support from the German-Israeli Foundation (grant G-809-234.6/2003), from FONDECYT (grants 1070269 and 7080064) and from DGIP-UTFSM (grant 240862).

Received August 14, 2006; in revised form May 18, 2007.
where

\[ K_{\ell,n} := 2^{n-1} \prod_{k=1}^{\ell} \left( 1 + \cos \frac{2k-1}{2n} \pi \right) \prod_{k=1}^{n-\ell} \left( 1 + \cos \frac{2k-1}{2n} \pi \right). \]

Note that equality holds in (1.2) for the Chebyshev polynomial

\[ t(z) = \cos n \arccos z = p_1(z) p_2(z), \]

with a proper choice of the factors \( p_1(z) \) and \( p_2(z) \). P. B. Borwein [7] generalized this to the multifactor inequality

\[ \prod_{k=1}^{m} \| p_k \|_{[-1,1]} \leq 2^{n-1} \prod_{k=1}^{\lfloor \frac{m}{2} \rfloor} \left( 1 + \cos \frac{2k-1}{2n} \pi \right)^2 \| p \|_{[-1,1]}. \]

He also showed that

\[ 2^{n-1} \prod_{k=1}^{\lfloor \frac{m}{2} \rfloor} \left( 1 + \cos \frac{2k-1}{2n} \pi \right)^2 \sim (3.20991 \ldots)^n \text{ as } n \to \infty. \]

A different version of inequality (1.1) for \( E = D \), where \( D := \{ w : |w| \leq 1 \} \) is the closed unit disk, was considered by Gelfond [15, p. 135] in connection with the theory of transcendental numbers:

\[ \prod_{k=1}^{m} \| p_k \|_D \leq e^n \| p \|_D. \]

The latter inequality was improved by Mahler [23], who replaced \( e \) by 2:

\[ \prod_{k=1}^{m} \| p_k \|_D \leq 2^n \| p \|_D. \]

It is easy to see that the base 2 cannot be decreased, if \( m = n \) and \( n \to \infty \). However, (1.7) has recently been further improved in two directions. D. W. Boyd [9], [10] showed that, given the number of factors \( m \) in (1.7), one has

\[ \prod_{k=1}^{m} \| p_k \|_D \leq (C_m)^n \| p \|_D, \]

where

\[ C_m := \exp \left( \frac{m}{\pi} \int_0^{\pi/m} \log \left( 2 \cos \frac{t}{2} \right) dt \right) \]
is asymptotically best possible for each fixed \( m \), as \( n \to \infty \). Kroó and Pritsker [20] showed that, for any \( m \leq n \),

\[
\prod_{k=1}^{m} \|p_k\|_D \leq 2^{n-1} \|p\|_D,
\]

where equality holds in (1.10) for each \( n \in \mathbb{N} \), with \( m = n \) and \( p(z) = z^n - 1 \).

Inequalities (1.2)–(1.10) clearly indicate that the constant \( C \) in (1.1) grows exponentially fast with \( n \), with the base for the exponential depending on the set \( E \). A natural general problem arising here is to find the smallest constant \( M_E > 0 \), such that

\[
\prod_{k=1}^{m} \|p_k\|_E \leq M_E \|p\|_E
\]

for arbitrary algebraic polynomials \( \{p_k(z)\}_{k=1}^{m} \) with complex coefficients, where \( p(z) = \prod_{k=1}^{m} p_k(z) \) and \( n = \deg p \). The solution of this problem is based on the logarithmic potential theory (cf. [36] and [35]). Let \( \text{cap}(E) \) be the logarithmic capacity of a compact set \( E \subset \mathbb{C} \). For \( E \) with \( \text{cap}(E) > 0 \), denote the equilibrium measure of \( E \) by \( \mu_E \). We remark that \( \mu_E \) is a positive unit Borel measure supported on \( \partial E \) (see [36, p. 55]). Define

\[
d_E(z) := \max_{t \in E} |z - t|, \quad z \in \mathbb{C},
\]

which is clearly a positive and continuous function in \( \mathbb{C} \). It is easy to see that the logarithm of this distance function is subharmonic in \( \mathbb{C} \). Furthermore, it has the following integral representation

\[
\log d_E(z) = \int \log |z - t|d\sigma_E(t), \quad z \in \mathbb{C},
\]

where \( \sigma_E \) is a positive unit Borel measure in \( \mathbb{C} \) with unbounded support, see Lemma 5.1 of [31] and [22]. For further in-depth analysis of the representing measure \( \sigma_E \), we refer to the recent paper of Gardiner and Netuka [14]. This integral representation is the key fact used by the first author to prove the following result [31].

**Theorem 1.1.** Let \( E \subset \mathbb{C} \) be a compact set, \( \text{cap}(E) > 0 \). Then the best constant \( M_E \) in (1.11) is given by

\[
M_E = \exp \left( \frac{\int \log d_E(z)d\mu_E(z)}{\text{cap}(E)} \right).
\]
Theorem 1.1 is applicable to any compact set with a connected component consisting of more than one point (cf. [36, p. 56]). In particular, if $E$ is a continuum, i.e., a connected set, then we obtain a simple universal bound for $M_E$ [31]:

**Corollary 1.2.** Let $E \subseteq \mathbb{C}$ be a bounded continuum (not a single point). Then we have

\begin{equation}
M_E \leq \frac{\text{diam}(E)}{\text{cap}(E)} \leq 4,
\end{equation}

where $\text{diam}(E)$ is the Euclidean diameter of the set $E$.

On the other hand, for non-connected sets $E$ the constants $M_E$ can be arbitrarily large. For example, consider $E_k = [-\sqrt{k} + 4, -\sqrt{k}] \cup [\sqrt{k}, \sqrt{k} + 4]$, so that $\text{cap}(E_k) = 1$ [35] and

\begin{equation}
M_E = \exp \left( \int \log d_{E_k}(z) \, d\mu_{E_k}(z) \right) \geq e^{\log(2\sqrt{k})} \to \infty \quad \text{as } k \to \infty.
\end{equation}

For the closed unit disk $D$, we have that $\text{cap}(D) = 1$ [36, p. 84] and that

\begin{equation}
d\mu_D = \frac{d\theta}{2\pi},
\end{equation}

where $d\theta$ is the arclength on $\partial D$. Thus Theorem 1.1 yields

\begin{equation}
M_D = \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \log d_D(e^{i\theta}) \, d\theta \right) = \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \log 2 \, d\theta \right) = 2,
\end{equation}

so that we immediately obtain Mahler’s inequality (1.7).

If $E = [-1, 1]$ then $\text{cap}([-1, 1]) = 1/2$ and

\begin{equation}
d\mu_{[-1,1]} = \frac{dx}{\pi \sqrt{1 - x^2}}, \quad x \in [-1, 1],
\end{equation}

which is the Chebyshev (or arcsin) distribution (see [36, p. 84]). Using Theorem 1.1, we obtain

\begin{equation}
M_{[-1,1]} = 2 \exp \left( \frac{1}{\pi} \int_{-1}^{1} \frac{\log d_{[-1,1]}(x)}{\sqrt{1 - x^2}} \, dx \right) = 2 \exp \left( \frac{2}{\pi} \int_0^{1} \frac{\log(1 + x)}{\sqrt{1 - x^2}} \, dx \right)
\end{equation}

\begin{equation}
= 2 \exp \left( \frac{2}{\pi} \int_{0}^{\pi/2} \log(1 + \sin t) \, dt \right) \approx 3.2099123,
\end{equation}

which gives the asymptotic version of Borwein’s inequality (1.4)–(1.5).
Considering the above analysis of Theorem 1.1, it is natural to conjecture that the sharp universal bounds for $M_E$ are given by

$$2 = M_D \leq M_E \leq M_{[-1,1]} \approx 3.2099123,$$

for any bounded non-degenerate continuum $E$, see [33].

It follows directly from the definition that $M_E$ is invariant with respect to the similarity transformations of the plane. Thus we can normalize the problem by setting $\text{cap}(E) = 1$. Thus, equivalently, we want to find the maximum and the minimum of the functional

$$\tau(E) := \int \log d_E(z) d\mu_E(z)$$

over all compact connected sets $E$ in the plane satisfying the above normalization. These questions are addressed in Section 2 of the paper. Section 3 discusses a more refined version of our problem on the best constant in (1.1). All proofs are given in Section 4.

In the forthcoming paper [34], we consider various improved bounds of the constant $M_E$, e.g., bounds for rotationally symmetric sets. From a different perspective, the results of Boyd (1.8)–(1.9) suggest that for some sets the constant $M_E$ can be replaced by a smaller one, if the number of factors is fixed. We characterize such sets in [34], and find the improved constant.

The problems considered in this paper have many applications in analysis, number theory and computational mathematics. We mention specifically applications in transcendence theory (see Gelfond [15]), and in designing algorithms for factoring polynomials (see Boyd [11] and Landau [21]). A survey of the results involving norms different from the sup norm (e.g., Bombieri norms) can be found in [11]. For polynomials in several variables, see the results of Mahler [24] for the polydisk, of Avanissian and Mignotte [2] for the unit ball in $\mathbb{C}^k$. Also, see Beauzamy and Enflo [5], and Beauzamy, Bombieri, Enflo and Montgomery [4] for multivariate polynomials in different norms.

ACKNOWLEDGEMENTS. The authors wish to express their gratitude to Richard Laugesen for several helpful discussions about these problems. Alexander Solynin communicated to the first author a sketch of proof for the inequality $M_E \geq 2$ for connected sets. We would like to thank him for the kind permission to use his argument in the proof of Theorem 2.5. This paper was written while the first author was visiting the University of Würzburg as a Humboldt Foundation Fellow. He would like to thank the Department of Mathematics and the Function Theory research group for their hospitality.
2. Sharp bounds for the constant $M_E$

We study bounds for the constant $M_E$ in this section, where $E \subset \mathbb{C}$ is a compact set satisfying $\text{cap}(E) > 0$. Our main goal here is to prove (1.19). It is convenient to first give some general observations on the properties of $M_E$.

**Theorem 2.1.** Let $I \subset E$ be compact sets in $\mathbb{C}$, $\text{cap}(I) > 0$. Denote the unbounded components of $\mathbb{C} \setminus E$ and $\mathbb{C} \setminus I$ by $\Omega_E$ and $\Omega_I$. If $d_E(z) = d_I(z)$ for all $z \in \partial \Omega_I$ then $M_E \leq M_I$, with equality holding only when $\text{cap}(\Omega_I \setminus \Omega_E) = 0$.

This theorem gives several interesting consequences. In particular, we show that if the set $E$ is contained in a disk whose diameter coincides with the diameter of $E$ then its constant $M_E$ does not exceed that of a segment. Thus segments indeed maximize $M_E$ among such sets. Denote the closed disk of radius $r$ centered at $z$ by $D(z, r)$.

**Corollary 2.2.** Let $z, w \in E$ satisfy $\text{diam } E = |z - w|$ and $[z, w] \subset E$. If $E \subset D \left( \frac{z+w}{2}, \frac{\text{diam } E}{2} \right)$ then $M_E \leq M_{[z,w]} = M_{[-2,2]}$.

The next result shows that the constant decreases when the set is enlarged in a certain way.

**Corollary 2.3.** Let $E^* := \bigcap_{z \in \partial \Omega_E} D(z, d_E(z))$, where $E \subset \mathbb{C}$ is compact, $\text{cap}(E) > 0$. If $H$ is a compact set such that $E \subset H \subset E^*$, then $M_H \leq M_E$. Equality holds if and only if $\text{cap}(\Omega_E \setminus \Omega_H) = 0$.

Let $\text{conv}(H)$ be the convex hull of $H$. The operation of taking the convex hull of a set satisfies the assumption of Corollary 2.3 (or Theorem 2.1), which gives

**Corollary 2.4.** Let $V \subset \mathbb{C}$ be a compact set, $\text{cap}(V) > 0$. If $H := \mathbb{C} \setminus \Omega_V$ is not convex, then $M_{\text{conv}(H)} < M_H$.

The above results help us to show that the minimum of $M_E$ is attained for the closed unit disk $D$, among all sets of positive capacity (connected or otherwise).

**Theorem 2.5.** Let $E \subset \mathbb{C}$ be an arbitrary compact set, $\text{cap}(E) > 0$. Then $M_E \geq 2$, where equality holds if and only if $\mathbb{C} \setminus \Omega_E$ is a closed disk.

In other words, $M_E = 2$ only for sets whose polynomial convex hull is a disk. This may also be described by saying that $M_E = 2$ if and only if $\partial U \subset E \subset U$, where $U$ is a closed disk.

Proving that the maximum of $M_E$ for arbitrary continua is attained for a segment is a more difficult problem. In fact, it is related to some old open
problems on the moments of the equilibrium measure (or circular means of conformal maps), see Pólya and Schiffer [27], and Pommerenke [28]. In particular, we use the results of [27] and [28] to show that

**Theorem 2.6.** Let $E \subset \mathbb{C}$ be a connected compact set, $\text{cap}(E) > 0$.

(i) If the center of mass $c := \int z \, d\mu_E(z)$ for $\mu_E$ belongs to $E$, then

\begin{equation}
M_E < 2 + 4.02/\pi \approx 3.279606.
\end{equation}

(ii) If $E$ is convex then

\begin{equation}
M_E < 2 + 4/\pi \approx 3.27324.
\end{equation}

This should be compared with $M_{[-2,2]} = M_{[-1,1]} \approx 3.2099123$.

After this paper had been written, a new related manuscript [3] appeared. That manuscript contains a proof of our conjecture $M_E \leq M_{[-2,2]}$ for centrally symmetric continua, as well as another quite general conjecture (if true) implying $M_E \leq M_{[-2,2]}$ holds for all continua.

### 3. Refined problem

The constant $M_E$ represents the base of rather crude exponential asymptotic for the constant in inequality (1.1). A more refined question is to find the sharp constant attained with equality. Such constants are known in the case of a segment, see (1.4) and [7]; and in the case of a disk, see (1.10) and [20]. Let $E$ be any compact set in the plane, and let $\prod_{k=1}^{m} p_k(z) = \prod_{j=1}^{n} (z - z_j)$, where $p_k(z)$ are arbitrary monic polynomials with complex coefficients. Define the constant

\begin{equation}
C_E(n) := \sup_{p_k} \frac{\prod_{k=1}^{m} \|p_k\|_E}{\prod_{k=1}^{m} \|p_k\|_E} = \sup_{c} \frac{\prod_{j=1}^{n} \|z - z_j\|_E}{\prod_{j=1}^{n} \|z - z_j\|_E}.
\end{equation}

If $\text{cap}(E) > 0$ then it follows from Theorem 1.1 that $1 \leq C_E(n) \leq M_E^n$. The refined version of our conjecture in (1.19) is as follows:

\begin{equation}
2^{n-1} = C_D(n) \leq C_E(n) \leq C_{[-2,2]}(n) = 2^{n-1} \prod_{k=1}^{[n/2]} \left(1 + \cos \frac{2k - 1}{2n} \pi \right)^2
\end{equation}

for any connected compact set $E$ of positive capacity.
4. Proofs

Proof of Theorem 2.1. Since $I \subset E$, we have that $\text{cap}(E) \geq \text{cap}(I) > 0$. Let $g_E(z, \infty)$ and $g_I(z, \infty)$ be the Green’s functions for $\Omega_E$ and $\Omega_I$, with poles in infinity. We follow the standard convention by setting $g_E(z, \infty) = 0$, $z \not\in \overline{\Omega}_E$ and $g_I(z, \infty) = 0$, $z \not\in \overline{\Omega}_I$. It follows from the maximum principle that $g_E(z, \infty) \leq g_I(z, \infty)$ for all $z \in \mathbb{C}$. Furthermore, this inequality is strict in $\Omega_E$, unless $\text{cap}(\Omega_I \setminus \Omega_E) = 0$.

Using the integral representation for $d_E(z)$ from Lemma 5.1 of [31] (see also [22] and [14]) and the Fubini theorem, we obtain that

$$\log M_E = \int \log d_E(z) \, d\mu_E(z) - \log \text{cap} \text{cap}(E)$$

$$= \int \left( \int \log |z - t| \, d\mu_E(z) - \log \text{cap} \text{cap}(E) \right) \, d\sigma_E(t)$$

$$= \int g_E(t, \infty) \, d\sigma_E(t),$$

where the last equality follows from the well known identity $g_E(t, \infty) = \int \log |z - t| \, d\mu_E(z) - \log \text{cap} \text{cap}(E)$ [35]. It is clear that

$$\int g_E(t, \infty) \, d\sigma_E(t) \leq \int g_I(t, \infty) \, d\sigma_E(t),$$

with equality possible if and only if $\text{cap}(\Omega_I \setminus \Omega_E) = 0$. Indeed, if we have equality in the above inequality, then $g_E(z, \infty) = g_I(z, \infty)$ for all $z \in \text{supp} \sigma_E$. But $\text{supp} \sigma_E$ is unbounded, so that $g_E(z, \infty) = g_I(z, \infty)$ in $\Omega_E$ by the maximum principle. Hence we obtain that

$$\log M_E \leq \int g_I(t, \infty) \, d\sigma_E(t)$$

$$= \int \left( \int \log |z - t| \, d\mu_I(z) - \log \text{cap} \text{cap}(I) \right) \, d\sigma_E(t)$$

$$= \int \log d_E(z) \, d\mu_I(z) - \log \text{cap} \text{cap}(I)$$

$$= \int \log d_I(z) \, d\mu_I(z) - \log \text{cap} \text{cap}(I) = \log M_I,$$

with equality if and only if $\text{cap}(\Omega_I \setminus \Omega_E) = 0$. Note that we used $\text{supp} \mu_I \subset \partial \Omega_I$, so that $d_E(z) = d_I(z)$ for $z \in \text{supp} \mu_I$. 

154

I. E. PRITSKER AND S. RUSCHEWEYH
Proof of Corollary 2.2. Let $I = [z, w]$ be the segment connecting the points $z$ and $w$, i.e., the common diameter of $E$ and the disk containing it. Observe that we have $d_E(t) = d_I(t)$ for all $t \in \partial \Omega_1 = I$ under the stated geometric conditions. Since all assumptions of Theorem 2.1 are satisfied, we obtain that $M_E \leq M_{[z,w]} = M_{[-2,2]}$, where the last equality follows from the invariance with respect to the similarity transformations of the plane.

Proof of Corollary 2.3. Observe that $E \subset D(z, d_E(z))$ for any $z \in \mathbb{C}$. Hence $E \subset E^*$. Since $E \subset H \subset E^*$, we immediately obtain that $d_E(z) \leq d_H(z) \leq d_E^*(z)$, $z \in \mathbb{C}$. On the other hand, the definition of $E^*$ gives that $d_E(z) = d_E^*(z)$ for all $z \in \partial \Omega_E$. Therefore $d_E(z) = d_H(z)$ for all $z \in \partial \Omega_E$, and the result follows from Theorem 2.1.

Proof of Corollary 2.4. We apply Theorem 2.1 again, with $I = H$ and $E = \text{conv}(H)$. It was shown in [22] that $d_H(z) = d_{\text{conv}(H)}(z)$ for all $z \in \mathbb{C}$, where $H$ is an arbitrary compact set. Since $H$ is not convex in our case, we obtain that $\text{cap}(\Omega_I \setminus \Omega_E) > 0$ and $M_E < M_I$.

For the proof of Theorem 2.5 we need a special case of the following lemma, which may be of some independent interest. Let $\Delta := \{w : |w| > 1\}$, and $D := \{z : |z| < 1\}$ the unit disk.

Lemma 4.1. Let $\Gamma$ be a Jordan domain and let $\Psi(z) := cw + \sum_{k=0}^{\infty} a_kw^{-k}$ be a conformal map of $\Delta$ onto $\Omega_\Gamma$. Furthermore assume that

$$(4.1) \quad \forall x, z \in \partial \Delta : \quad |\Psi(z) - \Psi(x)| \leq |\Psi(z) - \Psi(-z)|.$$ 

Then $\Gamma$ is a disk.

Proof. First note that by Carathéodory’s theorem [30, p. 18] $\Psi$ extends to a homeomorphism of $\overline{\Delta}$, so that (4.1) makes sense. Also there is no loss of generality in assuming $0 \in \Gamma$, so that $\Psi(z) \neq 0$ in $\overline{\Delta}$. Let

$$g(z) := \frac{1}{\Psi(1/z)}, \quad z \in \overline{D}.$$ 

Then $g(z) = z/c + \sum_{k=2}^{\infty} b_kz^k$ is a homeomorphism of $\overline{D}$ onto the closure of the Jordan domain $\Gamma^*$, the interior domain of the Jordan curve $1/\partial \Gamma$. Note that $g(0) = 0$, $g'(0) = 1/c \neq 0$.

Let $1/z \in \partial D$, and in (4.1) we replace $1/x \in \partial D$ by $-1/xz$ which is also in $\partial D$. Condition (4.1) then becomes

$$1 \geq \left| \frac{1}{g(z)} - \frac{1}{g(-xz)} \right| = \left| \frac{xg(-z)g(-xz) - g(z)}{g(-xz)g(-z) - g(z)} \right|, \quad x, z \in \partial D.$$
Note that the function
\[ F(x, z) := \frac{xg(-z) g(-xz) - g(z)}{g(-xz) g(-z) - g(z)} \]
is analytic in \((x, z) \in \mathbb{D}^2\), and by the maximum principle, applied to both variables separately, we find that
\[ |F(x, z)| \leq 1, \quad x, z \in \mathbb{D}. \]
Now fix \(z_0\) with \(0 < |z_0| < 1\). Then \(x \mapsto F(x, z_0)\) is analytic in \(\mathbb{D}\), satisfies \(|F(x, z_0)| \leq 1\) for \(x \in \mathbb{D}\), and, in addition, \(F(1, z_0) = 1\). The Julia-Wolf Lemma [30, p. 82] then says that \(F'(1, z_0) > 0\), or
\[ 1 + \frac{-z_0g'(-z_0)}{g(-z_0)} \cdot \frac{g(z_0)}{g(-z_0) - g(z_0)} > 0. \]
Obviously this must be true for any \(z_0\), and so, by the identity principle, we are left with the relation
\[ \frac{-zg'(-z)}{g(-z)} \cdot \frac{g(z)}{g(-z) - g(z)} \equiv \alpha, \quad z \in \mathbb{D}, \]
where \(\alpha > -1\) is some real constant. Letting \(z \to 0\), we find \(\alpha = -\frac{1}{2}\). Hence we are left with the difference-differential equation
\[ z \frac{g'(z)}{g(z)} \frac{g(-z)}{g(-z) - g(z)} = \frac{1}{2}, \quad z \in \mathbb{D}. \]
In terms of \(\Psi\) this reads
\[ 2w\Psi'(w) = \Psi(w) - \Psi(-w), \quad w \in \Omega_\Gamma. \]
From this we conclude that \(w\Psi'(w)\) is an odd function, which, in turn, implies that \(\Phi(w) := \Psi(w) - a_0\) is odd as well. For \(\Phi\) we then get the equation \(w\Phi'(w) = \Phi(w)\), or \(\Phi(w) = cw.\) This implies \(\Psi(w) = cw + a_0\) and therefore that \(\Gamma\) is a disk.

**Proof of Theorem 2.5.** Note that for any compact set \(E\), we have \(M_E = M_W\), where \(W := \overline{\mathbb{C}} \setminus \Omega_E\). This follows because \(\mu_E = \mu_W [35]\) and \(d_E(z) = d_W(z), z \in \mathbb{C}\). Corollary 2.4 now implies that
\[ \inf\{M_E : E \text{ is compact}\} = \inf\{M_H : H \text{ is convex and compact}\}. \]
Hence we can assume that \(E\) is convex from the start. We also set \(\text{cap}(E) = 1\), because \(M_E\) is invariant under similarity transforms. Thus \(\partial E\) is a rectifiable
Jordan curve (or a segment when \( E = \partial E \)). The following argument that shows \( M_E \geq 2 \) for all connected sets is due to A. Solynin. Let \( \Psi : \Delta \rightarrow \Omega_E \) be the standard conformal map:

\[
\Psi(w) = w + a_0 + \sum_{k=1}^{\infty} \frac{a_k}{w^k}, \quad w \in \Delta.
\]

Recall that \( \Psi \) can be extended as a homeomorphism of \( \overline{\Delta} \) onto \( \overline{\Omega}_E \), with \( \Psi(T) = \partial E, T := \partial \Delta \). It is clear that

\[
d_E(\Psi(e^{it})) \geq |\Psi(e^{it}) - \Psi(-e^{it})|, \quad t \in [0, 2\pi).
\]

Since \( \Psi(w) \) is univalent in \( \Delta \), the function

\[
H(w) := \frac{\Psi(w) - \Psi(-w)}{w}
\]

is analytic and non-vanishing in \( \Delta \), including \( w = \infty \). Furthermore, \( H(\infty) := \lim_{w \to \infty} H(w) = 2 \). It follows that \( h(w) := \log |H(w)| \) is harmonic in \( \Delta \). Recall that the equilibrium measure \( \mu_E \) is the harmonic measure of \( \Omega_E \) at \( \infty \), which is invariant under the conformal transformation \( \Psi \), see [35]. Hence

\[
\log M_E = \int \log d_E(z) \, d\mu_E(z) = \frac{1}{2\pi} \int_0^{2\pi} \log d_E(\Psi(e^{it})) \, dt \\
\geq \frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{\Psi(e^{it}) - \Psi(-e^{it})}{e^{it}} \right| \, dt = \log 2,
\]

where we used the Mean Value Theorem for \( h(w) \) on the last step. Thus we conclude that \( M_E \geq 2 = M_D \) holds for all compact sets \( E \).

Recall that \( M_E = M_W \), where \( W = \overline{C} \setminus \Omega_E \). If \( M_E = 2 \) then \( M_W = 2 \), so that \( W \) must be convex by Corollary 2.4. Since \( M_W > 3.2 \) for any segment, we have that \( W \) is the closure of a convex domain. We can assume that \( \text{cap}(W) = 1 \) after a dilation. Repeating the above argument for \( W \) instead of \( E \), we obtain that

\[
\log 2 = \log M_W = \frac{1}{2\pi} \int_0^{2\pi} \log d_W(\Psi(e^{it})) \, dt \\
\geq \frac{1}{2\pi} \int_0^{2\pi} \log \left| \Psi(e^{it}) - \Psi(-e^{it}) \right| \, dt = \log 2.
\]

It follows that

\[
\int_0^{2\pi} \left( \log d_W(\Psi(e^{it})) - \log \left| \Psi(e^{it}) - \Psi(-e^{it}) \right| \right) \, dt = 0,
\]
and that \( d_W(\Psi(e^{it})) = |\Psi(e^{it}) - \Psi(-e^{it})| \) a.e. on \([0, 2\pi)\). But these functions are clearly continuous, so that
\[
d_W(\Psi(e^{it})) = |\Psi(e^{it}) - \Psi(-e^{it})| \quad \forall t \in \mathbb{R}.
\]

An application of Lemma 4.1 with \( \Gamma \) the interior domain of \( W \) shows that \( W \) must be a disk. We would also like to mention that A. Solynin obtained a different proof of the fact that \( M_E = 2 \) for a connected set \( E \) implies \( W \) is a disk.

**Proof of Theorem 2.6.** Recall that \( M_E \) is invariant under similarity transformations. Hence we can assume again that \( \text{cap}(E) = 1 \) and \( \int z \, d\mu_E(z) = 0 \). The latter condition means that the center of mass for the equilibrium measure is at the origin. If we introduce the conformal map \( \Psi : \Delta \to \Omega_E \), as in the previous proof, then this condition translates into \( a_0 = 0 \), i.e.,
\[
\Psi(w) = w + \sum_{k=1}^{\infty} \frac{a_k}{w^k}, \quad w \in \Delta.
\]

Theorem 1.4 of [29, p. 19] gives that \( E \subset D(0, 2) \), so that \( d_E(z) \leq 2 + |z|, z \in E \), by the triangle inequality. Note that this is sharp for \( E = [-2, 2] \). Applying Jensen’s inequality, we have
\[
\log M_E = \int \log d_E(z) \, d\mu_E(z) \leq \int \log(2 + |z|) \, d\mu_E(z)
< \log \left( 2 + \int |z| \, d\mu_E(z) \right).
\]

Estimates (2.1) and (2.2) now follow from the results of Pommerenke [28], and of Pólya and Schiffer [27], who estimated the integral
\[
\int |z| \, d\mu_E(z) = \frac{1}{2\pi} \int_0^{2\pi} |\Psi(e^{it})| \, dt < 4.02/\pi \quad \text{(or } \leq 4/\pi),
\]
under the corresponding assumptions.

**REFERENCES**


