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ABSOLUTELY CONVERGENT FOURIER SERIES AND GENERALIZED ZYGMUND CLASSES OF FUNCTIONS

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Abstract

We investigate the order of magnitude of the modulus of smoothness of a function f with absolutely convergent Fourier series. We give sufficient conditions in terms of the Fourier coefficients in order that f belongs to one of the generalized Zygmund classes $\mathrm{Zyg}(\alpha, L)$ and $\mathrm{Zyg}(\alpha, 1/L)$, where $0 \le \alpha \le 2$ and $L = L(x)$ is a positive, nondecreasing, slowly varying function and such that $L(x) \to \infty$ as $x \to \infty$. A continuous periodic function f with period 2π is said to belong to the class $\mathrm{Zyg}(\alpha, L)$ if

$$|f(x+h) - 2f(x) + f(x-h)| \le Ch^\alpha L\left(\frac{1}{h}\right) \qquad \text{for all } x \in \mathsf{T} \text{ and } h > 0,$$

where the constant C does not depend on x and h; and the class $\mathrm{Zyg}(\alpha, 1/L)$ is defined analogously. The above sufficient conditions are also necessary in case the Fourier coefficients of f are all nonnegative.

1. Introduction

Let $\{c_k : k \in \mathsf{Z}\}$ be a sequence of complex numbers, in symbols: $\{c_k\} \subset \mathsf{C}$, such that

$$\sum_{k\in\mathsf{Z}} |c_k| < \infty. \tag{1.1}$$

Then the trigonometric sereies

$$\sum_{k\in\mathsf{Z}} c_k e^{ikx} =: f(x) \tag{1.2}$$

converges uniformly; consequently, it is the Fourier series of its sum f.

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We recall (see, e.g., [1, p. 6]) that a positive measurable function L defined on some neighborhood $[a, \infty)$ of infinity is said to be *slowly varying* (in Karamata's sense) if

$$\frac{L(\lambda x)}{L(x)} \to 1 \quad \text{as} \quad x \to \infty \qquad \text{for every} \quad \lambda > 0. \tag{1.3}$$

The neighborhood $[a, \infty)$ is of little importance. One may suppose that L is defined on $(0, \infty)$, for instance, by setting $L(x) := L(a)$ on $(0, a)$. A typical slowly varying function is

$$L(x) := \begin{cases} 1 & \text{for } 0 < x < 2, \\ \log x & \text{for } x \geq 2. \end{cases}$$

In this paper, we consider positive, nondecreasing, slowly varying functions. In this case, it is enough to require the fulfillment of (1.3) only for a single value of λ, say $\lambda := 2$. To be more specific, the following condition $(*)$ will be required in our theorems and lemmas.

CONDITION $(*)$. L is a positive, nondecreasing function defined on $(0, \infty)$ and satisfies the limit relations

$$L(x) \to \infty \quad \text{and} \quad \frac{L(2x)}{L(x)} \to 1 \qquad \text{as} \quad x \to \infty.$$

Given $\alpha > 0$ and a function L satisfying condition $(*)$, a continuous periodic function f is said to belong to the *generalized Zygmund class* $\mathrm{Zyg}(\alpha, L)$ if for all $h > 0$,

$$\omega_2(f; h) := \sup_{x \in \mathsf{T}} |f(x+h) - 2f(x) + f(x-h)| \leq C h^\alpha L\left(\frac{1}{h}\right), \tag{1.4}$$

where the constant $C = C(f)$ does not depend on h, and $\omega_2(f; h)$ is the modulus of smoothness of the function f.

Furthermore, given $\alpha \geq 0$ and L with condition $(*)$, f is said to belong to the *generalized Zygmund class* $\mathrm{Zyg}(\alpha, 1/L)$ if for all $h > 0$,

$$\omega_2(f; h) \leq C h^\alpha \frac{1}{L\left(\frac{1}{h}\right)}. \tag{1.5}$$

It is worth observing that if $f \in \mathrm{Zyg}(\alpha, L)$ for some $\alpha > 2$, or if $f \in \mathrm{Zyg}(\alpha, 1/L)$ for some $\alpha \geq 2$, then $f \equiv$ constant (cf. [2, Ch. 2]). Clearly, we have

$\operatorname{Lip}(\alpha, L \subset \operatorname{Zyg}(\alpha, L)$ for $0 < \alpha \leq 1$;

$\operatorname{Lip}(\alpha, 1/L) \subset \operatorname{Zyg}(\alpha, 1/L)$ for $0 \leq \alpha \leq 1$.

We note that the generalized Lipschitz classes $\operatorname{Lip}(\alpha, L)$ and $\operatorname{Lip}(\alpha, 1/L)$ were defined analogously in [6] where $\omega_2(f; h)$ is replaced in (1.4) and (1.5) by

$$\omega(f; h) = \omega_1(f; h) := \sup_{x \in \mathsf{T}} |f(x + h) - f(x)|, \qquad h > 0,$$

the ordinary modulus of continuity of the function f.

Various kinds of generalized Lipschitz and/or Zygmund classes of periodic functions were introduced in [3], [4], [7], [8], in which necessary and/or sufficient conditions were given in order that the sum of an absolutely convergent sine or cosine series with nonnegative coefficients belong to one of those generalized classes of order α for some $0 < \alpha \leq 1$. However, the case $1 < \alpha \leq 2$ was not considered at all in the papers indicated above.

2. New results

THEOREM 1. *Suppose* $\{c_k\} \subset \mathsf{C}$ *with* (1.1)*,* f *is defined in* (1.2)*, and L satisfies condition* $(*)$.

(i) *If for some* $0 < \alpha \leq 2$,

$$\sum_{|k| \leq n} k^2 |c_k| = O(n^{2-\alpha} L(n)), \qquad n \in \mathsf{N}, \tag{2.1}$$

then $f \in \operatorname{Zyg}(\alpha, L)$.

(ii) *Conversely, if* $\{c_k\}$ *is a sequence of nonnegative real numbers, in symbols:* $\{c_k\} \subset \mathsf{R}_+$*, and* $f \in \operatorname{Zyg}(\alpha, L)$ *for some* $0 < \alpha \leq 2$*, then* (2.1) *holds.*

We note that in case $0 < \alpha < 2$ condition (2.1) is equivalent to the following condition:

$$\sum_{|k| \geq n} |c_k| = O(n^{-\alpha} L(n)), \qquad n \in \mathsf{N}. \tag{2.2}$$

This claim is a straightforward consequence of Lemma 1 in Section 3.

We also note that, in case $\alpha = 1$ and $L \equiv 1$, Theorem 1 was proved in [5, Theorem 3].

The next Theorem 2 is a natural counterpart of Theorem 1.

THEOREM 2. *Suppose* $\{c_k\} \subset \mathsf{C}$ *with* (1.1)*,* f *is defined in* (1.2)*, and L satisfies condition* $(*)$.

(i) *If for some* $0 \le \alpha < 2$*,*

$$\sum_{|k|\ge n} |c_k| = O\left(\frac{n^{-\alpha}}{L(n)}\right), \qquad n \in \mathsf{N}, \tag{2.3}$$

then $f \in \mathrm{Zyg}(\alpha, 1/L)$*.*

(ii) *Conversely, if* $\{c_k\} \subset \mathsf{R}_+$ *and* $f \in \mathrm{Zyg}(\alpha, 1/L)$ *for some* $0 \le \alpha < 2$*, then* (2.3) *holds.*

We note that in case $0 < \alpha < 2$ condition (2.3) is equivalent to the following condition:

$$\sum_{|k|\le n} k^2 |c_k| = O\left(\frac{n^{2-\alpha}}{L(n)}\right), \qquad n \in \mathsf{N}. \tag{2.4}$$

This claim is a straightforward consequence of Lemma 2 in Section 3.

3. Auxiliary results

We recall three lemmas from [6, Lemmas 3, 4 and 6].

LEMMA 1. *Suppose* $\{a_k : k \in \mathsf{N}\} \subset \mathsf{R}_+$ *with* $\sum a_k < \infty$ *and* L *satisfies condition* $(*)$*.*

(i) *If for some* $\delta > \gamma \ge 0$*,*

$$\sum_{k=1}^{n} k^\delta a_k = O(n^\gamma L(n)), \tag{3.1}$$

then

$$\sum_{k=n}^{\infty} a_k = O(n^{\gamma-\delta} L(n)), \qquad n \in \mathsf{N}. \tag{3.2}$$

(ii) *Conversely, if* (3.2) *holds for some* $\delta \ge \gamma > 0$*, then* (3.1) *also holds.*

Consequently, in case $\delta > \gamma > 0$ conditions (3.1) and (3.2) are equivalent.

LEMMA 2. *Suppose* $\{a_k\} \subset \mathsf{R}_+$ *with* $\sum a_k < \infty$ *and* L *satisfies condition* $(*)$.

(i) *If for some* $\delta > \gamma > 0$*,*

$$\sum_{k=1}^{n} k^\delta a_k = O\left(\frac{n^\gamma}{L(n)}\right), \tag{3.3}$$

then

$$\sum_{k=n}^{\infty} a_k = O\left(\frac{n^{\gamma-\delta}}{L(n)}\right), \qquad n \in \mathsf{N}. \tag{3.4}$$

(ii) *Conversely, if* (3.4) *holds for some* $\delta \geq \gamma > 0$, *then* (3.3) *also holds.*

Consequently, in case $\delta > \gamma > 0$ conditions (3.3) and (3.4) are equivalent.

LEMMA 3. *If L satisfies condition* $(*)$ *and* $\eta > -1$, *then*

$$\int_0^h \frac{x^\eta}{L\left(\frac{1}{x}\right)} dx = O\left(\frac{h^{\eta+1}}{L\left(\frac{1}{h}\right)}\right), \qquad h > 0.$$

4. Proofs of Theorems 1 and 2

PROOF OF THEOREM 1. (i) Suppose (2.1) is satisfied for some $0 < \alpha \leq 2$. By (1.1) and (1.2), we may write that

$$\begin{aligned} |f(x+h) - 2f(x) + f(x-h)| &= \left|\sum_{k\in\mathsf{Z}} c_k e^{ikx}(e^{ikh} - 2 + e^{-ikh})\right| \\ &\leq \left\{\sum_{|k|\leq n} + \sum_{|k|>n}\right\}|c_k||e^{ikh} - 2 + e^{-ikh}| \\ &=: A_n + B_n, \end{aligned} \tag{4.1}$$

say, where n is defined by

$$n := [1/h], \qquad h > 0, \tag{4.2}$$

and $[\cdot]$ means the integer part.

We will use the inequality

$$\begin{aligned} |e^{ikh} - 2 + e^{-ikh}| &= |2\cos kh - 2| \\ &= 4\sin^2\frac{kh}{2} \leq \min\{4, k^2h^2\}, \qquad k \in \mathsf{Z}. \end{aligned} \tag{4.3}$$

By (2.1) and (4.2), we obtain

$$\begin{aligned} |A_n| \leq h^2 \sum_{|k|\leq n} k^2|c_k| &= h^2 O(n^{2-\alpha}L(n)) \\ &= h^2 O\left(h^{\alpha-2}L\left(\frac{1}{h}\right)\right) = O\left(h^\alpha L\left(\frac{1}{h}\right)\right). \end{aligned} \tag{4.4}$$

Due to Lemma 1, Part (i) (applied with $\gamma := 2 - \alpha$ and $\delta := 2$), condition (2.1) implies (2.2). Now, by (2.2) and (4.2), we find that

$$|B_n| \le 4 \sum_{|k|>n} |c_k| = 4O(n^{-\alpha} L(n)) = O\left(h^\alpha L\left(\frac{1}{h}\right)\right). \tag{4.5}$$

Combining (4.1), (4.4) and (4.5) yields $f \in \mathrm{Zyg}(\alpha, L)$.

(ii) Conversely, suppose $c_k \ge 0$ for all k and that $f \in \mathrm{Zyg}(\alpha, L)$ for some $0 < \alpha \le 2$. Then there exists a constant C such that

$$\begin{aligned} |f(h) - 2f(0) + f(-h)| &= \left|\sum_{k\in\mathbf{Z}} c_k(e^{ikh} - 2 + e^{-ikh})\right| \\ &= \left|\sum_{k\in\mathbf{Z}} c_k(2\cos kh - 2)\right| = \sum_{k\in\mathbf{Z}} c_k(2 - 2\cos kh) \\ &= 4\sum_{k\in\mathbf{Z}} c_k \sin^2\frac{kh}{2} \le Ch^\alpha L\left(\frac{1}{h}\right), \qquad h > 0 \end{aligned} \tag{4.6}$$

(cf. (4.3)). Making use of the well-known inequality

$$\sin t \ge \frac{2}{\pi} t \qquad \text{for} \quad 0 \le t \le \frac{\pi}{2},$$

from (4.6) we conclude that

$$4\sum_{|k|\le n} k^2 c_k \frac{h^2}{\pi^2} \le 4\sum_{k\in\mathbf{Z}} c_k \sin^2\frac{kh}{2} \le Ch^\alpha L\left(\frac{1}{h}\right), \qquad h > 0,$$

where n is defined in (4.2). Now, hence it follows that

$$\sum_{|k|\le n} k^2 c_k \le \frac{C\pi^2}{4} h^{\alpha-2} L\left(\frac{1}{h}\right) = O(n^{2-\alpha} L(n)),$$

which is (2.1) to be proved.

Proof of Theorem 2. (i) Suppose (2.3) is satisfied for some $0 \le \alpha < 2$. We start with (4.1), where n is defined in (4.2). Making use of the first inequality in (4.4) and applying Lemma 2, Part (ii) (with $\gamma := 2 - \alpha$ and $\delta := 2$) yield

$$|A_n| \le h^2 \sum_{|k|\le n} k^2 |c_k| = h^2 O\left(\frac{n^{2-\alpha}}{L(n)}\right) = O\left(\frac{h^\alpha}{L\left(\frac{1}{h}\right)}\right). \tag{4.7}$$

On the other hand, by (2.3), (4.2) and (4.3), we find that

$$|B_n| \leq 4 \sum_{|k|>n} |c_k| = O\bigg(\frac{n^{-\alpha}}{L(n)}\bigg) = O\bigg(\frac{h^\alpha}{L\left(\frac{1}{h}\right)}\bigg). \tag{4.8}$$

Combining (4.1), (4.7) and (4.8) gives $f \in \mathrm{Zyg}(\alpha, 1/L)$.

(ii) Conversely, suppose that $c_k \geq 0$ for all k and that $f \in \mathrm{Zyg}(\alpha, 1/L)$ for some $0 \leq \alpha < 2$. Similarly to (4.6), this time we have

(4.9)

$$\begin{aligned} |f(x) - 2f(0) + f(-x)| &= \Big| \sum_{k\in\mathbb{Z}} c_k (2\cos kx - 2\Big| = 2 \sum_{k\in\mathbb{Z}} c_k (1 - \cos kx) \\ &= O\bigg(\frac{x^\alpha}{L\left(\frac{1}{x}\right)}\bigg), \qquad x > 0. \end{aligned}$$

By uniform convergence, due to (1.1), the series $\sum c_k (1 - \cos kx)$ may be integrated term by term on any interval $(0, h)$, $h > 0$. By Lemma 3, we conclude from (4.9) that

$$\sum_{|k|\geq 1} c_k \bigg(h - \frac{\sin kh}{k}\bigg) \leq \frac{Ch^{\alpha+1}}{L\left(\frac{1}{h}\right)}, \qquad h > 0,$$

where C is a constant. Setting $h := 1/n$ and perhaps neglecting a finite number of nonnegative terms, we even have

$$\sum_{|k|\geq 2n} c_k \bigg(\frac{1}{n} - \frac{\sin \frac{k}{n}}{k}\bigg) \leq \frac{Cn^{-\alpha-1}}{L(n)}, \qquad n \in \mathbb{N}. \tag{4.10}$$

Since

$$\frac{1}{n} - \frac{\sin \frac{k}{n}}{k} \geq \frac{1}{2n} \qquad \text{for all} \quad |k| \geq 2n,$$

it follows from (4.10) that

$$\frac{1}{2} n^{-1} \sum_{|k|\geq 2n} c_k \leq \frac{Cn^{-\alpha-1}}{L(n)}, \qquad n \in \mathbb{N}.$$

Due to (1.3), this inequality is equivalent to (2.3) to be proved.

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