MATH. SCAND. 103 (2008), 11–39

# THE FIXED POINT FOR A TRANSFORMATION OF HAUSDORFF MOMENT SEQUENCES AND ITERATION OF A RATIONAL FUNCTION

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## Abstract

We study the fixed point for a non-linear transformation in the set of Hausdorff moment sequences, defined by the formula: $T((a_n))_n = 1/(a_0+\cdots+a_n)$. We determine the corresponding measure $\mu$, which has an increasing and convex density on $]0,1[$, and we study some analytic functions related to it. The Mellin transform $F$ of $\mu$ extends to a meromorphic function in the whole complex plane. It can be characterized in analogy with the Gamma function as the unique log-convex function on $]-1,\infty[$ satisfying $F(0)=1$ and the functional equation $1/F(s)=1/F(s+1)-F(s+1)$, $s>-1$.


## 1. Introduction and main results

Hausdorff moment sequences are sequences of the form $\int_0^1 t^n\,d\nu(t)$, $n\geq 0$, where $\nu$ is a positive measure on $[0,1]$. Hausdorff moment sequences were characterized as completely monotonic sequences in a fundamental paper by Hausdorff, see [17]. For a recent study of Hausdorff moment sequences see [14], [15]. Hausdorff moment sequences can also be characterized as bounded Stieltjes moment sequences, where Stieltjes moment sequences are of the form $\int_0^\infty t^n\,d\nu(t)$, $n\geq 0$ for a positive measure $\nu$ on $[0,\infty[$. For a treatment of these concepts and the more general Hamburger moment problem see the monograph by Akhiezer [1].

In [8] the authors introduced a non-linear multiplicative transformation from Hausdorff moment sequences to Stieltjes moment sequences. In [9] we introduced a non-linear transformation $T$ of the set of Hausdorff moment sequences into itself by the formula:

$$T((a_n))_n = 1/(a_0+a_1+\cdots+a_n),\qquad n\geq 0. \tag{1.1}$$

* This work was initiated during the visit of the second author to the University of Copenhagen partially supported by D.G.E.S, ref. BFM2003-6335-C03-01, FQM-262 (*Junta de Andalucía*), and by grant 21-03-0324 from the Danish Natural Science Research Council.

Received November 14, 2006.

The corresponding transformation of positive measures on $[0,1]$ is denoted $\widehat{T}$. We recall from [9] that if $\nu \neq 0$, then $\widehat{T}(\nu)(\{0\}) = 0$ and

$$\int_0^1 \frac{1-t^{z+1}}{1-t}\, d\nu(t) \int_0^1 t^z\, d\widehat{T}(\nu)(t) = 1 \qquad \text{for} \quad \operatorname{Re} z \geq 0. \tag{1.2}$$

Assuming $\operatorname{Re} z > 0$ we can consider $t^z = \exp(z \log t)$ as a continuous function on $[0,1]$ with value 0 for $t = 0$. Likewise $(1-t^z)/(1-t)$ is a continuous function for $t \in [0,1]$ with value $z$ for $t = 1$. If $\operatorname{Re} z = 0$, $z \neq 0$ the function $t^z$ is only considered for $t > 0$, so it is important that $\widehat{T}(\nu)$ has no mass at zero. Finally $t^0 \equiv 1$. It is clear that if $\nu$ is a probability measure, then so is $\widehat{T}(\nu)$, and in this way we get a transformation of the convex set of normalized Hausdorff moment sequences (i.e. $a_0 = 1$) as well as a transformation of the set of probability measures on $[0,1]$. By Kakutani's theorem the transformation has a fixed point, and by (1.1) it is clear that a fixed point $(m_n)_n$ is uniquely determined by the recursive equation

$$m_0 = 1, \qquad (1 + m_1 + \cdots + m_n)m_n = 1, \qquad n \geq 1. \tag{1.3}$$

Therefore

$$m_{n+1}^2 + \frac{m_{n+1}}{m_n} - 1 = 0, \tag{1.4}$$

giving

$$m_1 = \frac{-1+\sqrt{5}}{2}, \qquad m_2 = \frac{\sqrt{22+2\sqrt{5}} - \sqrt{5} - 1}{4}, \qquad \dots .$$

The purpose of this paper is to study the Hausdorff moment sequence $(m_n)_n$ and to determine its associated probability measure $\mu$, called the *fixed point measure*.

We already know that $\mu(\{0\}) = 0$ because $\mu = \widehat{T}(\mu)$, but it is also convenient to notice that $\mu(\{1\}) = 0$. It is clear that $(m_n)_n$ decreases to $c = \mu(\{1\}) \geq 0$, hence $m_0 + m_1 + \cdots + m_n \geq (n+1)m_n$. By (1.3) we get $1 \geq (n+1)m_n^2 \geq (n+1)c^2$, showing that $c = 0$.

In Section 4 we prove much more, namely

$$m_n \sim 1/\sqrt{2n} \qquad \text{for} \quad n \to \infty. \tag{1.5}$$

We will study $\mu$ by determining what we call the *Bernstein transform*

$$f(z) = \mathscr{B}(\mu)(z) = \int_0^1 \frac{1-t^z}{1-t}\, d\mu(t), \qquad \operatorname{Re} z > 0 \tag{1.6}$$

as well as the *Mellin transform*

$$F(z) = \mathcal{M}(\mu)(z) = \int_0^1 t^z \, d\mu(t), \qquad \operatorname{Re} z > 0. \tag{1.7}$$

These functions are clearly holomorphic in the half-plane $\operatorname{Re} z > 0$ and continuous in $\operatorname{Re} z \geq 0$, the latter because $\mu(\{0\}) = 0$.

As a first result we prove:

THEOREM 1.1. *The functions $f$, $F$ can be extended to meromorphic functions in* $\mathbb{C}$ *and they satisfy*

$$f(z+1)F(z) = 1, \qquad z \in \mathbb{C} \tag{1.8}$$

$$f(z) = f(z+1) - \frac{1}{f(z+1)}, \qquad z \in \mathbb{C}. \tag{1.9}$$

*They are holomorphic in* $\operatorname{Re} z > -1$. *Furthermore $z = -1$ is a pole of $f$ and $F$.*

*The fixed point measure $\mu$ has the properties*

$$\int_0^1 t^x \, d\mu(t) < \infty, \quad x > -1; \qquad \int_0^1 \frac{d\mu(t)}{t} = \infty. \tag{1.10}$$

PROOF. By (1.2) with $\nu$ replaced by the fixed point measure $\mu$ we get $f(z+1)F(z) = 1$ for $\operatorname{Re} z \geq 0$, showing in particular that $f(z+1)$ and $F(z)$ are different from zero for $\operatorname{Re} z \geq 0$. For $\operatorname{Re} z \geq 0$ we get by (1.6)

$$f(z+1) - f(z) = \int_0^1 \frac{t^z - t^{z+1}}{1-t} \, d\mu(t) = \int_0^1 t^z \, d\mu(t) = F(z) = \frac{1}{f(z+1)},$$

which shows (1.9) for these values of $z$.

We remark that $\operatorname{Re} f(z) > 0$ and in particular $f(z) \neq 0$ for $\operatorname{Re} z > 0$. This follows by (1.6) because $\operatorname{Re}(t^z) \leq |t^z| < 1$ for $0 < t < 1$ and $\operatorname{Re} z > 0$.

We next use equation (1.9) to define $f(z)$ for $\operatorname{Re} z \geq -1$, yielding a holomorphic continuation of $f$ to the open half-plane $\operatorname{Re} z > -1$ because $f(z+1) \neq 0$.

Using equation (1.9) once more we obtain a meromorphic extension of $f$ to the half-plane $\operatorname{Re} z > -2$. There will be poles at points $z$ for which $f(z+1) = 0$, in particular for $z = -1$ because $f(0) = 0$.

Repeated use of equation (1.9) makes it possible to obtain a meromorphic extension to $\mathbb{C}$. At each step, $z$ will be a pole if $z+1$ is a zero or a pole.

At this stage we cannot give a complete picture of the poles of $f$, but we return to that in Theorem 1.4.

Having extended $f$ to a meromorphic function in $\mathsf{C}$ such that (1.9) holds, we extend $F$ to a meromorphic function in $\mathsf{C}$ such that equation (1.8) holds.

Let us notice that also $F$ has no poles in $\operatorname{Re} z > -1$ because $f(z+1) \neq 0$. Moreover $z = -1$ is a pole of $F$ because $f(0) = 0$.

By a classical result (going back to Landau for Dirichlet series), see [23, p. 58], we then get equation (1.10).

The function $f$ can be characterized in analogy with the Bohr-Mollerup theorem about the Gamma function, cf. [2]. More precisely we have:

THEOREM 1.2. *The Bernstein transform (1.6) of the fixed point measure is a function* $f : ]0,\infty[ \to ]0,\infty[$ *with the following properties*

(i) $f(1) = 1$,

(ii) $\log(1/f)$ *is convex,*

(iii) $f(s) = f(s+1) - 1/f(s+1)$, $s > 0$.

*Conversely, if* $\tilde{f} : ]0,\infty[ \to ]0,\infty[$ *satisfies* (i)–(iii)*, then it is equal to* $f$ *and for* $0 < s \leq 1$ *we have*

$$\tilde{f}(s) = \lim_{n\to\infty} \psi^{\circ n}\left(\frac{1}{m_{n-1}}\left(\frac{m_{n-1}}{m_n}\right)^s\right), \tag{1.11}$$

*where* $\psi$ *is the rational function* $\psi(z) = z - 1/z$*. In particular* (1.11) *holds for* $f$*.*

Here and elsewhere we use the notation for composition of mappings $\psi^{\circ 1}(z) = \psi(z)$, $\psi^{\circ n}(z) = \psi(\psi^{\circ(n-1)}(z))$, $n \geq 2$. Theorem 1.2 will be proved in Section 3. Using the relation $f(s+1)F(s) = 1$ it is clear that Theorem 1.2 can be reformulated to a characterization of $F$:

THEOREM 1.3. *There exists one and only one function* $F : ]-1,\infty[ \to ]0,\infty[$ *with the following properties*

(i) $F(0) = 1$,

(ii) $F$ *is log-convex,*

(iii) $1/F(s) = 1/F(s+1) - F(s+1)$, $s > -1$,

*namely* $F$ *is the Mellin transform*

$$F(s) = \int_0^1 t^s \, d\mu(t), \qquad s > -1$$

*of the fixed point measure.*

Let $\mathscr{H}$ denote the set of normalized Hausdorff moment sequences considered as a subset of $[0, 1]^{\mathsf{N}_0}$ with the product topology, $\mathsf{N}_0 = \{0, 1, \ldots\}$. In Section 2 we prove that the fixed point $\mathbf{m} = (m_n)_n$ is attractive in the sense that for each $\mathbf{a} = (a_n)_n \in \mathscr{H}$ the sequence of iterates $T^{\circ n}(\mathbf{a})$ converges to $\mathbf{m}$ in $\mathscr{H}$. Focusing on probability measures we see that every probability measure $\tau$ on $[0, 1]$ belongs to the domain of attraction of the fixed point measure $\mu$ in the sense that $\lim_{n\to\infty} \widehat{T}^{\circ n}(\tau) = \mu$ weakly. For $q \in \mathsf{R}$ we denote by $\delta_q$ the probability measure with mass 1 concentrated at the point $q$. By specializing the iteration using $\tau = \delta_0$ we prove the following result:

THEOREM 1.4. *Let $f$ and $F$ be the meromorphic functions in $\mathsf{C}$ extending* (1.6) *and* (1.7) *respectively. The zeros and poles of $f$ are all simple and are contained in $]-\infty, 0]$. The zeros of $f$ are denoted $\xi_0 = 0$ and $\xi_{p,k}$, $p = 1, 2, \ldots, k = 1, \ldots, 2^{p-1}$ with $-p-1 < \xi_{p,1} < \xi_{p,2} < \cdots < \xi_{p,2^{p-1}} < -p$.*

*The poles of $f$ are $-l, \xi_{p,k} - l$, $l = 1, 2, \ldots$ with $p, k$ as above.*

*Defining*

$$\rho_0 = \frac{1}{f'(0)}; \qquad \rho_{p,k} = \frac{1}{f'(\xi_{p,k})}, \tag{1.12}$$

*then $\rho_0, \rho_{p,k} > 0$.*

*The following representations hold*

$$F(z) = \frac{\rho_0}{z+1} + \sum_{p=1}^{\infty}\sum_{k=1}^{2^{p-1}} \frac{\rho_{p,k}}{z+1-\xi_{p,k}}, \tag{1.13}$$

*and*

$$f(z) = z\sum_{l=1}^{\infty}\left[\frac{\rho_0}{l(z+l)} + \sum_{p=1}^{\infty}\sum_{k=1}^{2^{p-1}} \frac{\rho_{p,k}}{(l-\xi_{p,k})(z+l-\xi_{p,k})}\right]. \tag{1.14}$$

*The fixed point measure $\mu$ has an increasing and convex density $\mathscr{D}$ with respect to Lebesgue measure on $]0, 1[$ and it is given by*

$$\mathscr{D}(t) = \rho_0 + \sum_{p=1}^{\infty}\sum_{k=1}^{2^{p-1}} \rho_{p,k} t^{-\xi_{p,k}}. \tag{1.15}$$

While clearly $\mathscr{D}(0) = \rho_0$, we prove in Theorem 3.9 that

$$\mathscr{D}(t) \sim 1/\sqrt{2\pi(1-t)}, \qquad t \to 1.$$

It is possible to obtain expressions for $\xi_{p,k}$ and $\rho_{p,k}$ in terms of the moments $(m_n)$. This is quite technical and is given in Theorem 3.8.

We recall that a function $\varphi$ is called a *Stieltjes transform* if it can be written in the form

$$\varphi(z) = a + \int_0^\infty \frac{d\sigma(x)}{x+z}, \qquad z \in \mathsf{C} \setminus\, ]-\infty, 0], \tag{1.16}$$

where $a \geq 0$ and $\sigma$ is a positive measure on $[0, \infty[$ such that (1.16) makes sense, i.e. $\int 1/(x+1)\, d\sigma(x) < \infty$.

It is clear that if $\sigma \neq 0$ then $\varphi$ is strictly decreasing on $]0, \infty[$ with $a = \lim_{s\to\infty} \varphi(s)$. Furthermore, $\varphi$ is holomorphic in $\mathsf{C} \setminus\, ]-\infty, 0]$ with

$$\frac{\operatorname{Im} \varphi(z)}{\operatorname{Im} z} < 0 \qquad \text{for} \quad z \in \mathsf{C} \setminus \mathsf{R},$$

so in particular $\varphi$ is never zero in $\mathsf{C} \setminus\, ]-\infty, 0]$. The Stieltjes transforms we are going to consider will be meromorphic in $\mathsf{C}$. The function (1.16) is meromorphic precisely when the measure $\sigma$ is discrete and the set of mass-points have no finite accumulation points, i.e. if and only if

$$\varphi(z) = a + \sum_{p=0}^{\infty} \frac{\sigma_p}{z + \eta_p}$$

with $\sigma_p > 0,\ 0 \leq \eta_0 < \eta_1 < \eta_2 < \cdots \to \infty$.

For results about Stieltjes transforms see [10]. Stieltjes transforms are closely related to Pick functions, cf. [1], [16]. We recall that a Pick function is a holomorphic function $\varphi : \mathsf{C} \setminus \mathsf{R} \to \mathsf{C}$ satisfying

$$\frac{\operatorname{Im} \varphi(z)}{\operatorname{Im} z} \geq 0 \qquad \text{for} \quad z \in \mathsf{C} \setminus \mathsf{R},$$

so if $\varphi \neq 0$ is a Stieltjes transform, then $1/\varphi$ is a Pick function. Notice that $z/(z+a)$ is a Pick function for any $a > 0$.

COROLLARY 1.5. *In the notation of Theorem 1.4 $f(z)/z$ and $F(z)$ are Stieltjes transforms and $f$ is a Pick function.*

We have used the name Bernstein transform for (1.6). In general, if $\nu$ is a positive finite measure on $]0, 1]$, we call

$$\mathscr{B}(\nu)(z) = \int_0^1 \frac{1 - t^z}{1 - t}\, d\nu(t) \tag{1.17}$$

the Bernstein transform of $\nu$, because it is a Bernstein function in the terminology of [10]. In fact we can write

$$\mathscr{B}(\nu)(z) = \nu(\{1\})z + \int_0^\infty \left(1 - e^{-xz}\right) d\lambda(x), \qquad \operatorname{Re} z \geq 0,$$

where $\lambda$ is defined as the image measure of $(1-t)^{-1}(\nu|]0,1[)$ under $\log(1/x)$ mapping $]0,1[$ onto $]0,\infty[$. We recall that $\lambda$ is called the Lévy measure of the Bernstein function. It follows that $\mathscr{B}(\nu)'$ is a completely monotonic function. Bernstein functions are very important in the theory of Lévy processes, see [11].

In Section 4 we prove that $(m_n)_n$ is infinitely divisible in the sense that $(m_n^\alpha)_n$ is a Hausdorff moment sequence for all $\alpha > 0$.

## 2. An iteration leading to the fixed point measure

For $n = 0, 1, \ldots$ we denote the moments of $\mu_n = \widehat{T}^{\circ n}(\delta_0)$ by $(m_{n,k})_k$, i.e.

$$\int_0^1 t^k \, d\widehat{T}^{\circ n}(\delta_0)(t) = m_{n,k},$$

hence for $n \geq 1$

$$m_{n,k} = \left(m_{n-1,0} + m_{n-1,1} + \cdots + m_{n-1,k}\right)^{-1}. \tag{2.1}$$

Notice that $m_{n,0} = 1$ for all $n$ and $m_{0,k} = \delta_{0k}$, $m_{1,k} = 1$, $m_{2,k} = 1/(k+1)$ for all $k$.

LEMMA 2.1. *For fixed $k = 0, 1, \ldots$ we have*

$$m_{0,k} \leq m_{2,k} \leq m_{4,k} \leq \cdots$$
$$m_{1,k} \geq m_{3,k} \geq m_{5,k} \geq \cdots$$

*and these sequences have the same limit*

$$\lim_{n\to\infty} m_{2n,k} = \lim_{n\to\infty} m_{2n+1,k} = m_k,$$

*where $(m_k)_k$ is the fixed point given by* (1.3).

*Furthermore,* $\lim_{k\to\infty} m_{n,k} = 0$ *for $n \geq 2$, implying that $\mu_n = \widehat{T}^{\circ n}(\delta_0)$ has no mass at $t = 1$ for $n \geq 2$.*

PROOF. Since the result is trivial for $k = 0$, we assume that $k \geq 1$ and have

$$0 = m_{0,k} < m_{2,k} = \frac{1}{k+1}; \qquad 1 = m_{1,k} > m_{3,k} = \frac{1}{\mathscr{H}_{k+1}},$$

where $\mathscr{H}_p = 1 + \frac{1}{2} + \cdots + \frac{1}{p}$ is the $p$'th harmonic number. We now get

$$\frac{1}{m_{4,k}} = \sum_{j=0}^{k} m_{3,j} < k+1$$

hence $m_{4,k} > m_{2,k}$. We next use this to conclude

$$\frac{1}{m_{5,k}} = \sum_{j=0}^{k} m_{4,j} > \sum_{j=0}^{k} m_{2,j} = \frac{1}{m_{3,k}},$$

hence $m_{5,k} < m_{3,k}$. It is clear that this procedure can be continued and reformulated to a proof by induction.

Defining

$$m'_k = \lim_{n\to\infty} m_{2n,k}, \qquad m''_k = \lim_{n\to\infty} m_{2n+1,k},$$

we get the following relations from (2.1)

(2.2)

$$m'_k = \left(1 + m''_1 + \cdots + m''_k\right)^{-1}, \quad m''_k = \left(1 + m'_1 + \cdots + m'_k\right)^{-1}, \quad k \geq 1,$$

because clearly $m'_0 = m''_0 = m_0 = 1$. It follows easily by induction using (2.2) that $m'_k = m''_k = m_k$ for all $k$.

Since $m_{2n,k} \leq m_k$ we get $\lim_{k\to\infty} m_{2n,k} = 0$. Furthermore, for $n \geq 1$

$$\frac{1}{m_{2n+1,k}} = \sum_{j=0}^{k} m_{2n,j} \geq \sum_{j=0}^{k} m_{2,j} = \mathscr{H}_{k+1}$$

and hence $\lim_{k\to\infty} m_{2n+1,k} = 0$.

We recall that $\mathscr{H}$ denotes the set of normalized Hausdorff moment sequences $\mathbf{a} = (a_n)_n$. The mapping $\nu \to (\int x^n \, d\nu(x))_n$ from the set of probability measures $\nu$ on $[0, 1]$ to $\mathscr{H}$ is a homeomorphism between compact sets, when the set of probability measures carries the weak topology and $\mathscr{H}$ carries the topology inherited from $[0, 1]^{\mathbb{N}_0}$ equipped with the product topology.

Defining an order relation $\leq$ on $\mathscr{H}$ by writing $\mathbf{a} \leq \mathbf{b}$ if $a_k \leq b_k$ for $k = 0, 1, \ldots$, we easily get the following Lemma:

LEMMA 2.2. *The transformation $T : \mathscr{H} \to \mathscr{H}$ is decreasing, i.e.*

$$\mathbf{a} \leq \mathbf{b} \Rightarrow T(\mathbf{a}) \geq T(\mathbf{b}).$$

THEOREM 2.3. *For every $\mathbf{a} \in \mathscr{H}$ we have*

$$\lim_{n\to\infty} T^{\circ n}(\mathbf{a}) = \mathbf{m},$$

*where* $\mathbf{m} = (m_n)_n$ *is the fixed point.*

Proof. For $0 \le q \le 1$ we write $\underline{\boldsymbol{q}} = (q^n)_n$, hence $\underline{\mathbf{0}} \le \mathbf{a} \le \underline{\mathbf{1}}$ for every $\mathbf{a} \in \mathscr{H}$. By Lemma 2.2 we get

$$T^{\circ(2n)}(\underline{\mathbf{0}}) \le T^{\circ(2n)}(\mathbf{a}) \le T^{\circ(2n)}(\underline{\mathbf{1}}) = T^{\circ(2n+1)}(\underline{\mathbf{0}})$$

$$T^{\circ(2n+1)}(\underline{\mathbf{0}}) \ge T^{\circ(2n+1)}(\mathbf{a}) \ge T^{\circ(2n+1)}(\underline{\mathbf{1}}) = T^{\circ(2n+2)}(\underline{\mathbf{0}}),$$

and since $\lim_{n\to\infty} T^{\circ n}(\underline{\mathbf{0}}) = \mathbf{m}$ by Lemma 2.1, we get

$$\lim_{n\to\infty} T^{\circ(2n)}(\mathbf{a}) = \lim_{n\to\infty} T^{\circ(2n+1)}(\mathbf{a}) = \mathbf{m}.$$

Theorem 2.3 can also be expressed that $\widehat{T}^{\circ n}(\tau) \to \mu$ weakly for any probability measure $\tau$ on [0, 1]. Specializing this to $\tau = \delta_0$ and using formula (1.2), we obtain:

Corollary 2.4. *The iterated sequence* $\mu_n = \widehat{T}^{\circ n}(\delta_0)$ *of measures converges weakly to the fixed point measure* $\mu$ *and*

(2.3)
$$\int_0^1 \frac{1-t^{z+1}}{1-t}\, d\mu_n(t) \int_0^1 t^z\, d\mu_{n+1}(t) = 1, \qquad \operatorname{Re} z \ge 0, \;\; n = 0, 1, \dots$$

We have $\mu_0 = \delta_0, \mu_1 = \delta_1, \mu_2 = \chi_{]0,1[}(t)dt$, where $\chi_{]0,1[}(t)$ denotes the indicator function for the interval ]0, 1[. The Bernstein transform of the measure $\mu_2$ is

$$\text{(2.4)} \quad \mathscr{B}(\mu_2)(z) = \int_0^1 \frac{1-t^z}{1-t}\, dt = \sum_{l=1}^{\infty} \left(\frac{1}{l} - \frac{1}{z+l}\right) = \Psi(z+1) + \gamma,$$

where $\gamma$ is Euler's constant and $\Psi(x) = \Gamma'(x)/\Gamma(x)$ is the Digamma function.

The measure $\mu_3$ has been calculated in [9] and the result is

$$\mu_3 = \left(\sum_{p=0}^{\infty} \alpha_p t^{-\xi_p}\right) \chi_{]0,1[}(t)\, dt,$$

where $0 = \xi_0 > \xi_1 > \xi_2 > \cdots$ satisfy $-p-1 < \xi_p < -p$ for $p = 1, 2, \dots$ and $\alpha_p > 0$, $p = 0, 1, \dots$. More precisely, it was proved that $\xi_p$ is the unique solution $x \in \,]-p-1, -p[$ of the equation $\Psi(1+x) = -\gamma$. Writing $\xi_p = -p - 1 + \delta_p$, we have $0 < \delta_{p+1} < \delta_p < \frac{1}{2}$, $\delta_p \sim 1/\log p$, $p \to \infty$. Furthermore, $\alpha_p = 1/\Psi'(1+\xi_p) \sim 1/\log^2 p$. Since $\sum \alpha_p/(1-\xi_p) = 1$, we have the crude estimate $\alpha_p < p + 2$.

We shall now prove that all the measures $\mu_n$, $n \geq 4$ have a form similar to that of $\mu_3$.

LEMMA 2.5. *For $n \geq 3$ the measure $\mu_n$ has the form*

$$\mu_n = \left(\rho_0^{(n)} + \sum_{p=1}^{\infty} \sum_{k=1}^{N(n,p)} \rho_{p,k}^{(n)} t^{-\xi_{p,k}^{(n)}}\right) \chi_{]0,1[}(t)\, dt, \tag{2.5}$$

*where for each $p \geq 1$*

(i) $1 \leq N(n, p) \leq 2^{p-1}$,

(ii) $-p-1 < \xi_{p,1}^{(n)} < \xi_{p,2}^{(n)} < \cdots < \xi_{p,N(n,p)}^{(n)} < -p$,

(iii) $0 < \rho_0^{(n)} < 1$, $0 < \rho_{p,k}^{(n)} < p+2$, $k = 1, \ldots, N(n, p)$.

PROOF. The result for $n = 3$ follows from the description above from [9] with $\rho_0^{(3)} = \alpha_0$, $N(3, p) = 1$, $\rho_{p,1}^{(3)} = \alpha_p$, $\xi_{p,1}^{(3)} = \xi_p$.

Assume now that the result holds for $\mu_n$ and let us prove it for $\mu_{n+1}$. For $\operatorname{Re} z > 0$ we then have

$$\begin{aligned}
f_n(z) &:= \mathscr{B}(\mu_n)(z)\\
&= \int_0^1 \frac{1-t^z}{1-t}\, d\mu_n(t) = \sum_{l=0}^{\infty} \int_0^1 \left(t^l - t^{z+l}\right) d\mu_n(t)\\
&= \sum_{l=0}^{\infty} \left[\rho_0^{(n)} \int_0^1 \left(t^l - t^{z+l}\right) dt + \sum_{p=1}^{\infty} \sum_{k=1}^{N(n,p)} \rho_{p,k}^{(n)} \int_0^1 \left(t^{l-\xi_{p,k}^{(n)}} - t^{z+l-\xi_{p,k}^{(n)}}\right) dt\right]\\
&= z \sum_{l=1}^{\infty} \left[\frac{\rho_0^{(n)}}{l(z+l)} + \sum_{p=1}^{\infty} \sum_{k=1}^{N(n,p)} \frac{\rho_{p,k}^{(n)}}{(l-\xi_{p,k}^{(n)})(z+l-\xi_{p,k}^{(n)})}\right].
\end{aligned}$$

This shows that $f_n(z)/z$ is a Stieltjes transform and a meromorphic function in $\mathbb{C}$ with poles at the points

$$-l,\ \xi_{p,k}^{(n)} - l, \qquad l = 1, 2, \ldots, \quad p = 1, 2, \ldots, \quad k = 1, \ldots, N(n, p),$$

so in the interval $]-p-1, -p]$ we have the poles

$$-p,\ \xi_{p-l,k}^{(n)} - l, \qquad k = 1, \ldots, N(n, p-l),\ \ l = 1, \ldots, p-1. \tag{2.6}$$

Since $f_n(x)/x$ is strictly decreasing between the poles, we conclude that there is precisely one simple zero between two consecutive poles. Let $N(n+1, p)$ denote the number of zeros of $f_n$ in $]-p-1, -p[$ and let $\xi_{p,k}^{(n+1)}$ denote the zeros numbered such that

$$-p-1 < \xi_{p,1}^{(n+1)} < \xi_{p,2}^{(n+1)} < \cdots < \xi_{p,N(n+1,p)}^{(n+1)} < -p.$$

In addition also $z = 0$ is a zero of $f_n$. There are no zeros or poles in $\mathsf{C}\setminus\,]-\infty, 0]$ because $f_n(z)/z$ is a Stieltjes transform.

We are now ready to prove equation (2.5) and (i)–(iii) with $n$ replaced by $n+1$.

(i). By (2.6) we get

$$N(n+1, p) \leq 1 + \sum_{l=1}^{p-1} N(n, p-l) \leq 1 + \sum_{l=1}^{p-1} 2^{p-l-1} = 2^{p-1}.$$

(ii) is clear by definition, when we have proved that the measure $\mu_{n+1}$ has the form (2.5) using the numbers $\xi_{p,k}^{(n+1)}$.

(iii). By a classical result, see [19], [18], [4], $1/f_n(z)$ is a Stieltjes transform because $f_n(z)/z$ is so, i.e.

$$\frac{1}{f_n(z)} = \frac{\rho_0^{(n+1)}}{z} + \sum_{p=1}^{\infty} \sum_{k=1}^{N(n+1,p)} \frac{\rho_{p,k}^{(n+1)}}{z - \xi_{p,k}^{(n+1)}},$$

with $\rho_0^{(n+1)}, \rho_{p,k}^{(n+1)} > 0$. There is no constant term in the Stieltjes representation because $f_n(x) \to \infty$ for $x \to \infty$. In fact, by Lemma 2.1 we get

$$\lim_{x\to\infty} f_n(x) = \int_0^1 \frac{d\mu_n(t)}{1-t} = \sum_{k=0}^{\infty} m_{n,k} = \lim_{k\to\infty} \frac{1}{m_{n+1,k}} = \infty.$$

Note that

$$\rho_0^{(n+1)} = \frac{1}{f_n'(0)}, \qquad \rho_{p,k}^{(n+1)} = \frac{1}{f_n'(\xi_{p,k}^{(n+1)})}. \tag{2.7}$$

By (2.3) we get

$$\int_0^1 t^z \, d\mu_{n+1}(t) = \frac{1}{f_n(z+1)} = \frac{\rho_0^{(n+1)}}{z+1} + \sum_{p=1}^{\infty} \sum_{k=1}^{N(n+1,p)} \frac{\rho_{p,k}^{(n+1)}}{z+1-\xi_{p,k}^{(n+1)}},$$

which shows that

$$\mu_{n+1} = \left( \rho_0^{(n+1)} + \sum_{p=1}^{\infty} \sum_{k=1}^{N(n+1,p)} \rho_{p,k}^{(n+1)} t^{-\xi_{p,k}^{(n+1)}} \right) \chi_{]0,1[}(t)\, dt,$$

which is (2.5) with $n$ replaced by $n+1$.

Since $\mu_{n+1}$ is a probability measure we get

$$\rho_0^{(n+1)} < 1, \qquad \rho_{p,k}^{(n+1)} \int_0^1 t^{-\xi_{p,k}^{(n+1)}}\,dt < 1,$$

hence

$$\rho_{p,k}^{(n+1)} < 1 - \xi_{p,k}^{(n+1)} < p+2.$$

COROLLARY 2.6. *For $n \geq 0$ let $\mu_n = \widehat{T}^{\circ n}(\delta_0)$. The functions $f_n = \mathscr{B}(\mu_n)$ are meromorphic Pick functions and the functions $F_n = \mathscr{M}(\mu_n)$ are meromorphic Stieltjes transforms satisfying*

$$f_n(z+1)F_{n+1}(z) = 1, \qquad z \in \mathbb{C}. \tag{2.8}$$

*All zeros and poles of $f_n$ are contained in $]-\infty, 0]$.*

PROOF. We have $f_0(z) = 1$, $f_1(z) = z$, $F_0(z) = 0$, $F_1(z) = 1$, $F_2(z) = 1/(z+1)$ and for $n \geq 2$ the result follows from Lemma 2.5 and its proof.

In order to obtain a limit result for $n \to \infty$ in Corollary 2.6 we need the following:

LEMMA 2.7. *Let $(\varphi_n)_n$ be a sequence of Stieltjes transforms of the form*

$$\varphi_n(z) = \int_0^\infty \frac{d\sigma_n(x)}{x+z}, \qquad n = 1, 2, \ldots$$

*and assume that $\varphi_n(z) \to \varphi(z)$ uniformly on compact subsets of $\operatorname{Re} z > 0$ for some holomorphic function $\varphi$ on the right half-plane.*

*Then $\varphi$ is a Stieltjes transform*

$$\varphi(z) = a + \int_0^\infty \frac{d\sigma(x)}{x+z}$$

*and $\lim_{n\to\infty} \sigma_n = \sigma$ vaguely. Furthermore, $\varphi_n(z) \to \varphi(z)$ uniformly on compact subsets of $\mathbb{C} \setminus ]-\infty, 0]$.*

PROOF. Since

$$\int_0^\infty \frac{d\sigma_n(x)}{x+1} = \varphi_n(1) \to \varphi(1),$$

there exists a constant $K > 0$ such that $\int 1/(x+1)\,d\sigma_n(x) \leq K$ for all $n$. Let $\sigma$ be a vague accumulation point for $(\sigma_n)_n$. Replacing $(\sigma_n)_n$ by a subsequence we can assume without loss of generality that $\sigma_n \to \sigma$ vaguely. By standard results in measure theory, cf. [7, Prop. 4.4], we have

$$\int_0^\infty \frac{d\sigma(x)}{x+1} \leq K, \qquad \lim_{n\to\infty} \int f\,d\sigma_n = \int f\,d\sigma$$

for any continuous function $f : [0, \infty[ \to \mathbb{C}$ which is $o(1/(x+1))$ for $x \to \infty$. In particular

$$\varphi_n'(z) = -\int_0^\infty \frac{d\sigma_n(x)}{(x+z)^2} \to -\int_0^\infty \frac{d\sigma(x)}{(x+z)^2}, \qquad z \in \mathbb{C} \setminus ]-\infty, 0],$$

showing that

$$\varphi'(z) = -\int_0^\infty \frac{d\sigma(x)}{(x+z)^2}, \qquad \operatorname{Re} z > 0,$$

hence

$$\varphi(z) = a + \int_0^\infty \frac{d\sigma(x)}{x+z}, \qquad \operatorname{Re} z > 0$$

for some constant $a$. Using $\varphi(x) = \lim_{n\to\infty} \varphi_n(x) \geq 0$ for $x > 0$, we get $a \geq 0$, showing that $\varphi$ is a Stieltjes transform. By uniqueness of $a$ and $\sigma$ in the representation of $\varphi$ as a Stieltjes transform, we conclude that the accumulation point $\sigma$ is unique, hence $\lim_{n\to\infty} \sigma_n = \sigma$ vaguely.

It is now easy to see that $(\varphi_n(z))_n$ is uniformly bounded on compact subsets of $\mathbb{C} \setminus ]-\infty, 0]$, and the last assertion of Lemma 2.7 is a consequence of the Stieltjes-Vitali theorem.

Proof of Theorem 1.4. From Lemma 2.5 follows that the Mellin transform $\mathcal{M}(\mu_n)(z)$ coincides on $\operatorname{Re} z \geq 0$ with the meromorphic function

$$\frac{\rho_0^{(n)}}{z+1} + \sum_{p=1}^{\infty} \sum_{k=1}^{N(n,p)} \frac{\rho_{p,k}^{(n)}}{z+1-\xi_{p,k}^{(n)}} = \int_0^\infty \frac{d\sigma_n(x)}{x+z},$$

where $\sigma_n$ is the discrete measure

$$\sigma_n = \rho_0^{(n)} \delta_1 + \sum_{p=1}^{\infty} \sum_{k=1}^{N(n,p)} \rho_{p,k}^{(n)} \delta_{1-\xi_{p,k}^{(n)}}.$$

Since $\mathcal{M}(\mu_n)(z) \to \mathcal{M}(\mu)(z)$ uniformly on compact subsets of $\operatorname{Re} z > 0$ by Corollary 2.4, it follows by Lemma 2.7 that $\mathcal{M}(\mu)$ is a Stieltjes transform

$$\mathcal{M}(\mu)(z) = a + \int_0^\infty \frac{d\sigma(x)}{x+z},$$

and $\sigma_n \to \sigma$ vaguely. Since $\mathcal{M}(\mu)(k) = m_k \to 0$ as $k \to \infty$, we get $a = 0$. Using that $\sigma_n$ has at most $2^{p-1}$ mass points in $[p+1, p+2]$, $p = 1, 2, \ldots$ and that $\rho_{p,k}^{(n)} < p+2$ by Lemma 2.5, we can write

$$\sigma = \rho_0 \delta_1 + \sum_{p=1}^{\infty} \sum_{k=1}^{N_p} \rho_{p,k} \delta_{1-\xi_{p,k}},$$

with $\rho_0 \geq 0$, $0 < \rho_{p,k} \leq p+2$ and $-p-1 \leq \xi_{p,1} < \xi_{p,2} < \cdots < \xi_{p,N_p} < -p$, where $N_p \leq 2^{p-1}$. At this stage we cannot confirm that $\rho_0 > 0$, $-p-1 < \xi_{p,1}$, $N_p = 2^{p-1}$ and that $\xi_{p,k}$ are the zeros of $f$. The function

$$\frac{\rho_0}{z+1} + \sum_{p=1}^{\infty} \sum_{k=1}^{N_p} \frac{\rho_{p,k}}{z+1-\xi_{p,k}} \tag{2.9}$$

is a meromorphic extension of $\mathscr{M}(\mu)$ and therefore equal to the meromorphic function $F$ of Theorem 1.1. This shows that $\mu$ has the density

$$\mathscr{D}(t) = \rho_0 + \sum_{p=1}^{\infty} \sum_{k=1}^{N_p} \rho_{p,k} t^{-\xi_{p,k}}, \tag{2.10}$$

which is clearly increasing and convex since $-\xi_{p,k} \geq 1$. Finally, by (2.10) the Bernstein transform $\mathscr{B}(\mu)$ has the meromorphic extension

$$z \sum_{l=1}^{\infty} \left[ \frac{\rho_0}{l(z+l)} + \sum_{p=1}^{\infty} \sum_{k=1}^{N_p} \frac{\rho_{p,k}}{(l-\xi_{p,k})(z+l-\xi_{p,k})} \right], \tag{2.11}$$

which is a Pick function. The function given by (2.11) equals the meromorphic function $f$ of Theorem 1.1. By Lemma 2.7 applied to the Stieltjes transforms $f_n(z)/z$, we conclude that $f_n(z) \to f(z)$ uniformly on compact subsets of $\mathbb{C} \setminus ]-\infty, 0]$.

We already know from Theorem 1.1 that $F$ has a pole at $z = -1$ and hence $\rho_0 > 0$. The remaining poles of $F$ are $\xi_{p,k} - 1$, so by formula (1.8) the zeros of $f$ are $z = 0$ and $z = \xi_{p,k}$. By the expression (2.11) for $f$ the poles of $f$ are $-l$, $\xi_{p,k} - l$ and therefore $-p-1 < \xi_{p,1}$, $p = 1, 2, \ldots$.

We have now proved that the zeros and poles of $f$ are all simple and are contained in $]-\infty, 0]$. Since $f(z+1)F(z) = 1$ we get by (2.9) that

$$\frac{1}{f(z)} = \frac{\rho_0}{z} + \sum_{p=1}^{\infty} \sum_{k=1}^{N_p} \frac{\rho_{p,k}}{z-\xi_{p,k}},$$

which shows equation (1.12).

To finish the proof we shall establish that $N_p = 2^{p-1}$.

From the functional equation (1.9) and the fact that $f$ is strictly increasing between the poles, we see the following about the generation of zeros and poles of $f$:

(1) If $z+1$ is regular point, then $f(z+1) = \pm 1$ if and only if $f(z) = 0$.

(2) If $z+1$ is regular point, then $f(z+1)=0$ if and only if $z$ is a pole. In the affirmative case $\mathrm{Res}(f,z)=-1/f'(z+1)$.

(3) If $z+1$ is a pole then $z$ is a pole with the same residue as in $z+1$.

(4) For a pole $\beta$ let $\alpha_\beta$ be the smallest zero in $]\beta,\infty[$. Then $f(]\beta,\alpha_\beta[)=\,]-\infty,0[$ and there exists a unique point $x_*$ in $]\beta,\alpha_\beta[$ such that $f(x_*)=-1$.

(5) For a pole $\beta$ let $\gamma_\beta$ be the biggest zero in $]-\infty,\beta[$. Then $f(]\gamma_\beta,\beta[)=\,]0,\infty[$ and there exists a unique point $x^*$ in $]\gamma_\beta,\beta[$ such that $f(x^*)=1$.

From (1)–(5) we deduce that $f$ has the following properties. Since $f(0)=0$ we see that $f$ has poles at $z=-1,-2,\dots$ in accordance with (2.11). There are no poles in $]-2,-1[$ since $f$ is regular in $]-1,0[$ and non-zero. Notice that $f$ is strictly increasing on $]-1,\infty[$ mapping this interval onto the whole real line by (2.11). There is a unique point $x_*\in\,]-1,0[$ such that $f(x_*)=-1$, hence $x_*-1$ is a zero and $x_*-2, x_*-3,\dots$ are poles. In $]-3,-2]$ there are two poles namely $x_*-2$ and $-2$ and since $f$ is strictly increasing between consecutive poles we have two zeros in $]-3,-2[$. By induction it is easy to see that there are exactly $2^{p-1}$ poles in each interval $]-p-1,-p]$ and $2^{p-1}$ zeros in the open interval $]-p-1,-p[$, $p\ge 1$. This shows that $N_p=2^{p-1}$. Note that $\xi_{1,1}=x_*-1$.

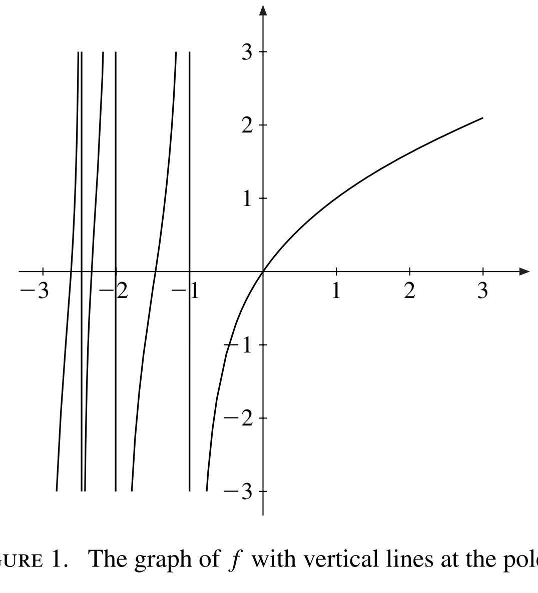


FIGURE 1. The graph of $f$ with vertical lines at the poles.

We give some further information about the poles of $f$.

We call the negative integers poles of the *first generation* of $f$ and say that a pole of $f$ is of the *l-th generation*, $l\ge 2$, if it is generated by a zero $\xi_{l-1,k}$, i.e. the pole is of the form $\xi_{l-1,k} - m$, for some integer $m \geq 1$. Then it can easily be proved by induction on $p$ that:

(1) In $]-p-1, -p]$ there is one pole of the first generation (namely, $-p$), one pole of the second generation (namely $\xi_{1,1} - p + 1$), and for $l = 3, \dots, p$, $2^{l-2}$ poles of the $l$-th generation (so that the total number of poles is $1 + \sum_{l=2}^{p} 2^{l-2} = 2^{p-1}$).

(2) For each interval $[-p-1, -p]$, the poles of one generation separate the set of poles of lower generations, and the zeros $\xi_{p,k}$, $k = 1, \dots, 2^{p-1}$, separate the set of all poles. That means that the set of poles of generation less than or equal to $l$ separate the zeros $\xi_{p,k}$, $k = 1, \dots, 2^{p-1}$, in groups of $2^{p-l}$ consecutive elements.

(3) For $l \geq 2$ the poles in $]-p-1, -p[$ of the $l$-th generation are zeros of $f(z + p - l + 1)$ but they are still poles of $f(z + j)$ if $0 \leq j \leq p - l$.

## 3. Iteration of the rational function $\psi$

In this section we will prove Theorem 1.2 and discuss the relationship with the classical study of iteration of rational functions of degree $\geq 2$, cf. e.g. [3].

We have already introduced the rational function $\psi$ by

$$\psi(z) = z - \frac{1}{z}. \tag{3.1}$$

It is a mapping of $\mathbb{C} \setminus \{0\}$ onto $\mathbb{C}$ with a simple pole at $z = 0$. Moreover, $\psi(0) = \psi(\infty) = \infty$. It is two-to-one with the exception that $\psi(z) = \pm 2i$ has only one solution $z = \pm i$. It is strictly increasing on the half-lines $]-\infty, 0[$ and $]0, \infty[$, mapping each of them onto $\mathbb{R}$. The functional equation (1.9) can be written

$$f(z) = \psi(f(z+1)). \tag{3.2}$$

We notice that $\psi$ and hence all iterates $\psi^{\circ n}$ are Pick functions. It is convenient to define $\psi^{\circ 0}(z) = z$. We claim that the Julia set is $J(\psi) = \mathbb{R}^*$, and the Fatou set is $F(\psi) = \mathbb{C} \setminus \mathbb{R}$. This is because $\psi$ is conjugate to the rational function

$$R(z) = \frac{3z^2 + 1}{z^2 + 3}$$

i.e. $g \circ R = \psi \circ g$, where $g$ is the Möbius transformation $g(z) = i(1+z)/(1-z)$. Note that $g$ is the Cayley transformation mapping the unit circle $\mathbb{T}$ onto $\mathbb{R}^*$. In [3, p. 200] the Julia set of $R$ is determined as $J(R) = \mathbb{T}$, and the assertion follows.

The sequence $(\lambda_n)_n$ is defined in terms of $(m_n)_n$ from (1.3) by

$$\lambda_0 = 0, \quad \lambda_{n+1} = 1/m_n, \qquad n \geq 0. \tag{3.3}$$

By (1.7) and (1.8) we clearly have

$$m_n = F(n), \quad \lambda_n = f(n), \qquad n \geq 0, \tag{3.4}$$

hence by (3.2)

$$\lambda_n = \psi(\lambda_{n+1}), \qquad n \geq 0, \tag{3.5}$$

which can be reformulated to

$$\lambda_{n+1} = \frac{1}{2}\left(\lambda_n + \sqrt{\lambda_n^2 + 4}\right), \qquad n \geq 0. \tag{3.6}$$

The following result is easy and the proof is left to the reader.

LEMMA 3.1. *Defining*

$$Y_n = (\psi^{\circ n})^{-1}(\{0\}) = \{z \in \mathsf{C} \mid \psi^{\circ n}(z) = 0\}, \tag{3.7}$$

*i.e.*

$$Y_0 = \{0\}, \qquad Y_1 = \{-1, 1\}, \qquad Y_2 = \{(\pm 1 \pm \sqrt{5})/2\}, \ldots$$

*we have for* $n \geq 1$

(i) $\psi(Y_n) = Y_{n-1}$, $Y_n = \psi^{\circ -1}(Y_{n-1})$,

(ii) *The set of poles of* $\psi^{\circ n}$ *is* $\cup_{j=0}^{n-1} Y_j$,

(iii) $Y_n$ *consists of* $2^n$ *real numbers and is symmetric with respect to zero.*

(iv) *The function* $\psi^{\circ n}$ *is strictly increasing from* $-\infty$ *to* $\infty$ *in each of the* $2^n$ *intervals in which* $\cup_{j=0}^{n-1} Y_j$ *divides* $\mathsf{R}$. *There is exactly one zero of* $\psi^{\circ n}$ *in each of these intervals, and these zeros form the set* $Y_n$.

We write $Y_n = \{\alpha_{n,k} : k = 1, \ldots, 2^n\}$ arranged in increasing order ($n \geq 1$):

$$\alpha_{n,1} < \alpha_{n,2} < \cdots < \alpha_{n,2^{n-1}} < 0 < \alpha_{n,2^{n-1}+1} < \cdots < \alpha_{n,2^n}.$$

It is easy to see that $-\alpha_{n,1} = \alpha_{n,2^n} = \lambda_n$ for $n \geq 0$.

PROPOSITION 3.2. *The set*

$$\cup_{p=0}^{\infty} Y_p = \left\{\alpha_{p,k} \mid p \geq 0, k = 1, \ldots, 2^p\right\}$$

*is dense in* $\mathsf{R}$.

PROOF. The set in question is the so-called backward orbit of 0 for $\psi$, and since $0 \in J(\psi)$ the result follows by [3, Theorem 4.2.7].

We next give some asymptotic properties of the sequence $(\lambda_n)_n$ and the function $f$:

LEMMA 3.3. (1) $\sqrt{n} \le \lambda_n \le \sqrt{2n}$, *$n \ge 0$.*

(2) $(\lambda_n)_n$ *is an increasing divergent sequence and* $\lambda_{n+1}/\lambda_n$ *is decreasing with* $\lim_{n\to\infty} \dfrac{\lambda_{n+1}}{\lambda_n} = 1$.

(3) $\lim_{n\to\infty} (\lambda_{n+1}^2 - \lambda_n^2) = 2$.

(4) $\lim_{n\to\infty} \dfrac{\lambda_n^2}{n} = 2$.

(5) $\lim_{n\to\infty} \dfrac{\lambda_n^2 - 2n}{\log n} = -\dfrac{1}{2}$.

(6) $\lim_{s\to\infty} f(s)/\sqrt{2s} = 1$.

(7) $\lim_{s\to\infty} f'(s)\sqrt{2s} = 1$.

PROOF.

(1) These inequalities follow easily from (3.6) using induction on $n$.

(2) The sequence $(\lambda_n)_n$ increases to infinity since it is the reciprocal of the Hausdorff moment sequence $(m_n)_n$. By the Cauchy-Schwarz inequality $m_n^2 \le m_{n-1}m_{n+1}$, which proves that $(\lambda_{n+1}/\lambda_n)_n$ is decreasing. The limit follows now easily from (3.6).

(3) Using (3.5) we can write

$$\lambda_{n+1}^2 - \lambda_n^2 = \frac{\lambda_{n+1} + \lambda_n}{\lambda_{n+1}} = 1 + \frac{\lambda_n}{\lambda_{n+1}},$$

and it suffices to apply part 2.

(4) is a consequence of part 3 and the following version of the Stolz criterion going back to [21]:

LEMMA 3.4. *Let* $(a_n)_n, (b_n)_n$ *be real sequences, where* $(b_n)_n$ *is strictly increasing tending to infinity. Then*

$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = L \Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = L.$$

(5) follows by using again the Stolz criterion and taking into account that

$$\frac{\lambda_{n+1}^2-\lambda_n^2-2}{\log\frac{n+1}{n}}=\frac{\lambda_{n+1}^2-\lambda_n^2-2\lambda_{n+1}^2+2\lambda_{n+1}\lambda_n}{\log\frac{n+1}{n}}=-\frac{(\lambda_{n+1}-\lambda_n)^2}{\log\frac{n+1}{n}}$$
$$=-\frac{1}{n\log\frac{n+1}{n}}\frac{n}{\lambda_{n+1}^2}\to-\frac{1}{2}.$$

(6) Since $f$ is increasing and $f(n)=\lambda_n$, the assertion follows from part 4.

(7) We write $f(n+1)-f(n)=f'(t_n)$, for a certain $t_n\in(n,n+1)$. Since $f'$ is decreasing ($f'(s)$ is completely monotonic), part 7 follows if we prove that $f'(t_n)\sqrt{2t_n}$ tends to 1 as $n$ tends to $\infty$. However, using the recursion formula for $(\lambda_n)_n$, we get

$$f'(t_n)\sqrt{2t_n}=(\lambda_{n+1}-\lambda_n)\sqrt{2t_n}=\frac{\sqrt{2(n+1)}}{\lambda_{n+1}}\frac{\sqrt{2t_n}}{\sqrt{2(n+1)}},$$

and it suffices to apply part 4.

PROOF OF THEOREM 1.2. We have already proved the properties (i) and (iii). To see (ii) we notice that $f=\mathscr{B}(\mu)$ is a Bernstein function, and therefore $1/f$ is completely monotonic. Every completely monotonic function is logarithmically convex. For these statements see e.g. [10, § 14].

Suppose next that $\tilde f$ is a function satisfying (i)-(iii). Since $\tilde f(1)=1=\lambda_1$, we see by (iii) and (3.5) that $\tilde f(n)=\lambda_n$ for $n\ge1$. Equation (1.11) is equivalent with

$$\tilde f(s)=\lim_{n\to\infty}\psi^{\circ n}\left(\lambda_n\left(\frac{\lambda_{n+1}}{\lambda_n}\right)^s\right),\tag{3.8}$$

and if we prove this equation for $0<s\le1$, then $\tilde f$ is uniquely determined on $]0,1]$ and hence by (iii) for all $s>0$.

We prove that the limit in (3.8) exists and coincides with $\tilde f(s)$ for $0<s\le1$. This is clear for $s=1$ since $\psi^{\circ n}(\lambda_{n+1})=1$ for $n\ge0$.

For any convex function $\phi$ on $]0,\infty[$ we have for $0<s\le1$ and $n\ge2$

$$\phi(n)-\phi(n-1)\le\frac{\phi(n+s)-\phi(n)}{s}\le\phi(n+1)-\phi(n).$$

By taking $\phi=\log(1/\tilde f)$, which is convex by assumption, we get

$$\log\frac{\lambda_{n-1}}{\lambda_n}\le\frac{1}{s}\log\frac{\tilde f(n)}{\tilde f(n+s)}\le\log\frac{\lambda_n}{\lambda_{n+1}};$$

that is

$$\left(\frac{\lambda_{n-1}}{\lambda_n}\right)^s \le \frac{\lambda_n}{\tilde{f}(n+s)} \le \left(\frac{\lambda_n}{\lambda_{n+1}}\right)^s,$$

which finally gives:

$$\lambda_n \left(\frac{\lambda_{n+1}}{\lambda_n}\right)^s \le \tilde{f}(n+s) \le \lambda_n \left(\frac{\lambda_n}{\lambda_{n-1}}\right)^s, \qquad 0 < s < 1.$$

Using that $\psi$ is increasing on $]0,\infty[$, we get by applying $\psi^{\circ n}$ to the previous inequality

$$\psi^{\circ n}(b_n(s)) \le \tilde{f}(s) = \psi^{\circ n}(\tilde{f}(n+s)) \le \psi^{\circ n}(a_n(s)),$$

where we have introduced

$$a_n(s) = \lambda_n \left(\frac{\lambda_n}{\lambda_{n-1}}\right)^s, \qquad b_n(s) = \lambda_n \left(\frac{\lambda_{n+1}}{\lambda_n}\right)^s.$$

It is now enough to prove that

$$\lim_{n\to\infty} (\psi^{\circ n}(a_n(s)) - \psi^{\circ n}(b_n(s)) = 0.$$

By applying the mean value theorem, we get for a certain $w \in ]b_n(s), a_n(s)[$ that

$$\begin{aligned}
&\psi^{\circ n}(a_n(s)) - \psi^{\circ n}(b_n(s)) \\
&\quad = (a_n(s) - b_n(s))(\psi^{\circ n})'(w) \\
&\quad = (a_n(s) - b_n(s))\psi'(\psi^{\circ n-1}(w))\psi'(\psi^{\circ n-2}(w)) \cdots \psi'(w).
\end{aligned}$$

Since $\lambda_n < b_n(s) < w < a_n(s)$, we get $\lambda_{n-k} < \psi^{\circ k}(b_n(s)) < \psi^{\circ k}(w)$, $k = 0, 1, \dots, n$, hence

$$\begin{aligned}
\left|\psi^{\circ n}(a_n(s)) - \psi^{\circ n}(b_n(s))\right| &\le \left|a_n(s) - b_n(s)\right| \prod_{k=0}^{n-1} \left|\psi'(\psi^{\circ k}(w))\right| \\
&\le \left|a_n(s) - b_n(s)\right| \prod_{k=0}^{n-1} \left(1 + \frac{1}{\lambda_{n-k}^2}\right) \\
&= \lambda_n \left(\left(\frac{\lambda_n}{\lambda_{n-1}}\right)^s - \left(\frac{\lambda_{n+1}}{\lambda_n}\right)^s\right) \prod_{k=1}^{n} \left(1 + \frac{1}{\lambda_k^2}\right)
\end{aligned}$$

$$\leq \lambda_n \left( \left( \frac{\lambda_n}{\lambda_{n-1}} \right)^s - \left( \frac{\lambda_{n+1}}{\lambda_n} \right)^s \right) \prod_{k=1}^{n} \left( 1 + \frac{1}{k} \right)$$

$$= (n+1)\lambda_n \left( \left( \frac{\lambda_n}{\lambda_{n-1}} \right)^s - \left( \frac{\lambda_{n+1}}{\lambda_n} \right)^s \right),$$

where we have used $\sqrt{k} \leq \lambda_k$ from Lemma 3.3 part 1.

Using that $(x^s - y^s) \leq s(x-y)$ for $1 < y < x$ and $0 < s \leq 1$, we get

$$\left| \psi^{\circ n}(a_n(s)) - \psi^{\circ n}(b_n(s)) \right| \leq s(n+1)\lambda_n \left( \frac{\lambda_n}{\lambda_{n-1}} - \frac{\lambda_{n+1}}{\lambda_n} \right),$$

and by (3.6) we finally get

$$\left| \psi^{\circ n}(a_n(s)) - \psi^{\circ n}(b_n(s)) \right|$$

$$\leq \frac{1}{2} s(n+1)\lambda_n \left( \left( 1 + \sqrt{1 + \tfrac{4}{\lambda_{n-1}^2}} \right) - \left( 1 + \sqrt{1 + \tfrac{4}{\lambda_n^2}} \right) \right)$$

$$= \frac{1}{2} s(n+1)\lambda_n \left( \sqrt{1 + \tfrac{4}{\lambda_{n-1}^2}} - \sqrt{1 + \tfrac{4}{\lambda_n^2}} \right)$$

$$= \frac{2s(n+1)\lambda_n \left( \frac{1}{\lambda_{n-1}^2} - \frac{1}{\lambda_n^2} \right)}{\sqrt{1 + \frac{4}{\lambda_{n-1}^2}} + \sqrt{1 + \frac{4}{\lambda_n^2}}} \leq \frac{s(n+1)}{\lambda_n \lambda_{n-1}^2} (\lambda_n^2 - \lambda_{n-1}^2),$$

which tends to zero by part 2, 3 and 4 of Lemma 3.3.

For each real number $s$, we define the sequence $(\lambda_n(s))_n$ by $\lambda_0(s) = s$ and

$$\lambda_{n+1}(s) = \frac{\lambda_n(s) + \sqrt{\lambda_n(s)^2 + 4}}{2}, \qquad n \geq 0. \tag{3.9}$$

Notice that $\lambda_{n+1}(s)$ is the positive root of $z^2 - \lambda_n(s)z - 1 = 0$ and that

$$\psi(\lambda_{n+1}(s)) = \lambda_n(s). \tag{3.10}$$

Therefore, if $s \in Y_l$ then $\lambda_n(s) \in Y_{l+n}$, and for $s = 0$ we have $\lambda_n(0) = \lambda_n$, $n \geq 0$. Furthermore, $\lambda_n(\lambda_l(s)) = \lambda_{n+l}(s)$.

DEFINITION 3.5. For integers $k, l \geq 0$ we denote by $r(k, l)$ the unique solution $x \in \{1, 2, \ldots, 2^l\}$ of the congruence equation $x \equiv k \mod 2^l$.

LEMMA 3.6. *For $p \geq 1$, $k = 1, 2, \ldots, 2^p$ we have*

(i) $\psi(\alpha_{p,k}) = \alpha_{p-1, r(k, p-1)}$.

(ii) $\psi^{\circ l}(\alpha_{p,k}) = \alpha_{p-l,r(k,p-l)}$ *for* $l = 0, 1, \ldots, p$.

PROOF. Since $\psi(Y_p) = Y_{p-1}$ and $\psi$ is strictly increasing mapping $]-\infty, 0[$ onto $\mathsf{R}$, we see that

$$\psi(\alpha_{p,k}) = \alpha_{p-1,k}, \qquad k = 1, 2, \ldots, 2^{p-1},$$

and since similarly $\psi$ maps $]0, \infty[$ onto $\mathsf{R}$ we get

$$\psi(\alpha_{p,k}) = \alpha_{p-1,j}, \qquad k = 2^{p-1} + j, \quad j = 1, 2, \ldots, 2^{p-1}.$$

In the first case $k = r(k, p-1)$ and in the second case $j = r(k, p-1)$ so the assertion (i) follows.

The assertion (ii) is clear for $l = 0$ and $l = p$ and follows for $l = 1$ by (i). Assuming (ii) for some $l$ such that $1 \le l \le p-2$ we get by (i)

$$\psi^{\circ(l+1)}(\alpha_{p,k}) = \psi(\alpha_{p-l,r(k,p-l)}) = \alpha_{p-l-1,j},$$

where $j := r(r(k, p-l), p-l-1)$. By definition

$$k \equiv r(k, p-l) \mod 2^{p-l}, \qquad 1 \le r(k, p-l) \le 2^{p-l}$$
$$j \equiv r(k, p-l) \mod 2^{p-l-1}, \qquad 1 \le j \le 2^{p-l-1}.$$

The first congruence also holds $\mod 2^{p-l-1}$, hence $j \equiv k \mod 2^{p-l-1}$ and finally $j = r(k, p-l-1)$.

COROLLARY 3.7. *For a zero* $\xi_{p,k}$ *of* $f$ *we have*

(i) $f(\xi_{p,k} + l) = \alpha_{l,r(k,l)}$, $l = 0, 1, \ldots, p$,

(ii) $f(\xi_{p,k} + l) = \lambda_{l-p}(\alpha_{p,k})$, $l = p+1, p+2, \ldots$, *where* $\lambda_n(s)$ *is defined in* (3.9).

PROOF. We first prove (i) for $l = p$, i.e. that $f(\xi_{p,k} + p) = \alpha_{p,k}$ since $r(k, p) = k$. Note that by (3.2) we have

$$\psi^{\circ p}(f(\xi_{p,k} + p)) = f(\xi_{p,k}) = 0,$$

hence $f(\xi_{p,k} + p) \in Y_p$. On the other hand $\xi_{p,k} + p \in ]-1, 0[$, and since $f$ is strictly increasing satisfying $f(]-1, 0[) = ]-\infty, 0[$, we see that $f(\xi_{p,k} + p)$, $k = 1, 2, \ldots, 2^{p-1}$ describe $2^{p-1}$ negative numbers in $Y_p$ in increasing order. Therefore, $f(\xi_{p,k} + p) = \alpha_{p,k}$, $k = 1, 2, \ldots, 2^{p-1}$.

By Lemma 3.6 and (3.2) we then get for $0 \le l \le p$

$$f(\xi_{p,k} + l) = \psi^{\circ(p-l)}(f(\xi_{p,k} + p)) = \psi^{\circ(p-l)}(\alpha_{p,k}) = \alpha_{l,r(k,l)}.$$

Clearly $0 < f(\xi_{p,k} + p + 1) \in Y_{p+1}$ and $\alpha_{p,k} = \psi(f(\xi_{p,k} + p + 1))$, hence $f(\xi_{p,k} + p + 1) = \lambda_1(\alpha_{p,k})$ by definition of $\lambda_1(s)$. The assertion (ii) follows easily by induction.

THEOREM 3.8. *The numbers $\xi_{p,k}$, $\rho_{p,k}$, $p \geq 1$, $k = 1, \ldots, 2^{p-1}$ and $\rho_0$ from Theorem 1.4 are given by the following formulas:*

$$\xi_{p,k} = \lim_{N\to\infty} \sqrt{2N}\Bigg(\sum_{l=1}^{p} \frac{1}{\alpha_{l,r(k,l)}} + \sum_{l=1}^{N-p} \frac{1}{\lambda_l(\alpha_{p,k})} - \lambda_N\Bigg), \tag{3.11}$$

$$\rho_{p,k} = \prod_{l=1}^{p}\Bigg(1 + \frac{1}{\alpha^2_{l,r(k,l)}}\Bigg)^{-1} \lim_{N\to\infty} \sqrt{2N} \prod_{l=1}^{N}\Bigg(1 + \frac{1}{\lambda_l^2(\alpha_{p,k})}\Bigg)^{-1}, \tag{3.12}$$

$$\rho_0 = \lim_{N\to\infty} \sqrt{2N} \prod_{l=1}^{N}\Bigg(1 + \frac{1}{\lambda_l^2}\Bigg)^{-1}. \tag{3.13}$$

PROOF. By applying $N$ times the functional equation (1.9) for the function $f$ and using Corollary 3.7 , we have for $p < N$:

$$\begin{aligned}0 = f(\xi_{p,k}) &= f(\xi_{p,k} + N) - \sum_{l=1}^{N} \frac{1}{f(\xi_{p,k} + l)} \\ &= f(\xi_{p,k} + N) - \Bigg(\sum_{l=1}^{p} \frac{1}{\alpha_{l,r(k,l)}} + \sum_{l=1}^{N-p} \frac{1}{\lambda_l(\alpha_{p,k})}\Bigg).\end{aligned}$$

Writing

$$y_{N,p,k} = \sum_{l=1}^{p} \frac{1}{\alpha_{l,r(k,l)}} + \sum_{l=1}^{N-p} \frac{1}{\lambda_l(\alpha_{p,k})},$$

we get $f(\xi_{p,k} + N) = y_{N,p,k}$. For $N \to \infty$ it follows by part 6 of Lemma 3.3 that $y_{N,p,k} \sim \sqrt{2N}$. Since $f$ is a strictly increasing bijection of $(-1, +\infty)$ onto $\mathsf{R}$, we can consider its inverse $f^{-1}$. Then we have $N = f^{-1}(\lambda_N)$, hence $\xi_{p,k} = f^{-1}(y_{N,p,k}) - f^{-1}(\lambda_N)$. Since $\xi_{p,k}$ is negative and $f$ is increasing, we deduce that $y_{N,p,k} < \lambda_N$. This gives for a certain number $\sigma_{N,p,k} \in\, ]y_{N,p,k}, \lambda_N[$ that

$$\begin{aligned}\xi_{p,k} = f^{-1}(y_{N,p,k}) - f^{-1}(\lambda_N) &= (f^{-1})'(\sigma_{N,p,k})(y_{N,p,k} - \lambda_N) \\ &= \frac{y_{N,p,k} - \lambda_N}{f'(\eta_{N,p,k})},\end{aligned}$$

where we have written $\eta_{N,p,k} = f^{-1}(\sigma_{N,p,k})$. Clearly $\eta_{N,p,k} \in ]\xi_{p,k} + N, N[$.

Taking into account that $\lim_{s\to\infty} f'(s)\sqrt{2s} = 1$ (part 7 of Lemma 3.3), we have

$$\xi_{p,k} = \lim_N \sqrt{2N}\left(y_{N,p,k} - \lambda_N\right),$$

that is, (3.11) holds.

The number $f'(\xi_{p,k})$ can be computed as follows: Deriving the functional equation (1.9) for $f$, we get

$$f'(z) = f'(z+1)\left(1 + \frac{1}{f^2(z+1)}\right)$$

hence by iteration

$$f'(z) = f'(z+N)\prod_{l=1}^{N}\left(1 + \frac{1}{f^2(z+l)}\right). \tag{3.14}$$

Using Corollary 3.7 and $\lim_{s\to\infty} f'(s)\sqrt{2s} = 1$, (Lemma 3.3, part 7) we get for $z = \xi_{p,k}$

$$f'(\xi_{p,k}) = \prod_{l=1}^{p}\left(1 + \frac{1}{\alpha^2_{l,r(k,l)}}\right)\lim_{N\to\infty}\frac{1}{\sqrt{2N}}\prod_{l=1}^{N}\left(1 + \frac{1}{\lambda_l^2(\alpha_{p,k})}\right),$$

and since $\rho_{p,k} = 1/f'(\xi_{p,k})$ by (1.12), we see that (3.12) holds.

Applying (3.14) for $z = 0$, we get

$$f'(0) = f'(N)\prod_{l=1}^{N}\left(1 + \frac{1}{\lambda_l^2}\right),$$

and (3.13) follows by (1.12) and $\lim_{N\to\infty} f'(N)\sqrt{2N} = 1$.

We give some values of the numbers of Theorem 3.8:

| $\rho_0 = 0.68\ldots$ | $\xi_0 = 0$ |
|---|---|
| $\rho_{1,1} = 0.14\ldots$ | $\xi_{1,1} = -1.46\ldots$ |
| $\rho_{2,1} = 0.06\ldots$ | $\xi_{2,1} = -2.61\ldots$ |
| $\rho_{2,2} = 0.05\ldots$ | $\xi_{2,2} = -2.33\ldots$ |

THEOREM 3.9. *The density* $\mathscr{D}$ *given by* (1.15) *satisfies*

$$\mathscr{D}(t) \sim \frac{1}{\sqrt{2\pi(1-t)}} \quad \textit{for} \quad t \to 1.$$

PROOF. By formula (1.8) and Lemma 3.3 part 6 we get

$$F(s) = \int_0^1 t^s \mathscr{D}(t)\,dt \sim \frac{1}{\sqrt{2s}}, \qquad s \to \infty,$$

or

$$\int_0^\infty e^{-us} \mathscr{D}(e^{-u}) e^{-u}\,du \sim \frac{1}{\sqrt{2s}}, \qquad s \to \infty.$$

By the Karamata Tauberian theorem, cf. [12, Theorem 1.7.1′], we get

$$\int_0^t \mathscr{D}(e^{-u}) e^{-u}\,du \sim \sqrt{\frac{2t}{\pi}}, \qquad t \to 0,$$

and since $\mathscr{D}$ is increasing we can use the Monotone Density theorem, cf. [12, Theorem 1.7.2b], to conclude that

$$\mathscr{D}(e^{-u}) e^{-u} \sim \frac{1}{\sqrt{2\pi u}}, \qquad u \to 0,$$

which is equivalent to the assertion.

## 4. Miscellaneous about the fixed point

The fixed point sequence $(m_n)_n$ given by (1.3) satisfies $m_{n+1} = \Phi(m_n)$ with

$$\Phi(x) = \frac{\sqrt{4x^2+1}-1}{2x}, \qquad x > 0.$$

This makes it possible to express $(m_n)_n$ as iterates of $\Phi$, viz.

$$m_n = \Phi^{\circ n}(1).$$

From Lemma 3.3 part 4 we get the asymptotic behaviour of $m_n$ as

$$m_n \sim \frac{1}{\sqrt{2n}}, \qquad n \to \infty.$$

This behaviour can also be deduced from a general result about iteration, cf. [13, p. 175]. The authors want to thank Bruce Reznick for this reference as well as the following description of $(m_n)_n$.

PROPOSITION 4.1. *Define* $h_n \in ]0, \pi/4]$ *by* $\tan h_n = m_n$ *and let*

$$G(x) = \frac{1}{2}\arctan(2\tan x), \qquad |x| < \frac{\pi}{2}.$$

*Then*

$$h_n = G^{\circ n}\left(\frac{\pi}{4}\right).$$

PROOF. We have

$$\tan h_n = m_n = \frac{m_{n+1}}{1-m_{n+1}^2} = \frac{\tan h_{n+1}}{1-\tan^2 h_{n+1}} = \frac{1}{2}\tan(2h_{n+1}),$$

hence $h_{n+1} = G(h_n)$ and the assertion follows.

A Hausdorff moment sequence $(a_n)_n$ is called *infinitely divisible* if $(a_n^\alpha)_n$ is a Hausdorff moment sequence for all $\alpha > 0$. If $a_n = \int_0^1 t^n\,d\nu(t)$, $n \geq 0$ then $(a_n)_n$ is infinitely divisible if and only if $\nu$ is infinitely divisible for the product convolution $\tau \diamond \nu$ of measures $[0, \infty[$ defined by

$$\int g\,d\tau \diamond \nu = \iint g(st)\,d\tau(s)\,d\nu(t).$$

For a general study of these concepts see [22], [5], [6]. In case the measure $\nu$ does not charge 0, the notion is the classical infinite divisibility on the locally compact group $]0, \infty[$ under multiplication.

PROPOSITION 4.2. *Hausdorff moment sequences of the form (1.1) are infinitely divisible.*

PROOF. Let $\nu \neq 0$ be a positive measure on $[0, 1]$ and let $a_n = \int t^n\,d\nu(t)$, $n \geq 0$ be the corresponding Hausdorff moment sequence. Let $\alpha > 0$ be fixed. We shall prove that $((a_0+a_1+\cdots+a_n)^{-\alpha})_n$ is a Hausdorff moment sequence.

For $0 < c < 1$ we denote by $\nu_c = \nu|[0, c[ + \nu(\{1\})\delta_c$, where the first term denotes the restriction of $\nu$ to $[0, c[$. Then $\lim_{c\to 1}\nu_c = \nu$ weakly and in particular for each $n \geq 0$

$$a_n(c) := \int_0^1 t^n\,d\nu_c(t) \to a_n \qquad \text{for} \quad c \to 1.$$

It therefore suffices to prove that

$$\big((a_0(c) + a_1(c) + \cdots + a_n(c))^{-\alpha}\big)_n \tag{4.1}$$

is a Hausdorff moment sequence. By a simple calculation we find

$$\begin{aligned}\left(\sum_{k=0}^{n} a_k(c)\right)^{-\alpha} &= \left(\int_0^1 \frac{1-t^{n+1}}{1-t}\,d\nu_c(t)\right)^{-\alpha} \\ &= \left(\int_0^1 \frac{d\nu_c(t)}{1-t} - \int_0^1 t^n \frac{t\,d\nu_c(t)}{1-t}\right)^{-\alpha} = H(\tau_n),\end{aligned}$$

where

$$\tau_n = \int_0^1 t^n \frac{t\, d\nu_c(t)}{1-t}, \qquad H(z) = \left(\int_0^1 \frac{d\nu_c(t)}{1-t} - z\right)^{-\alpha}.$$

The function $H$ is clearly holomorphic in

$$|z| < \int_0^1 \frac{d\nu_c(t)}{1-t}$$

with non-negative coefficients in the power series. Applying Lemma 2.1 in [9], shows that (4.1) is a Hausdorff moment sequence.

COROLLARY 4.3. *The fixed point sequence* $(m_n)_n$ *is infinitely divisible.*

REMARK 4.4. By Corollary 4.3 the fixed point measure $\mu$ is infinitely divisible for the product convolution. The image measure $\eta$ of $\mu$ under $\log(1/t)$ is an infinitely divisible probability measure in the ordinary sense, because $\log(1/t)$ maps products to sums. The measure $\eta$ has the density

$$\mathscr{D}(e^{-u})e^{-u} = \rho_0 e^{-u} + \sum_{p=1}^{\infty}\sum_{k=1}^{2^{p-1}} \rho_{p,k} e^{-u(1-\xi_{p,k})}, \qquad u > 0 \tag{4.2}$$

with respect to Lebesgue measure on the half-line. Since (4.2) is clearly a completely monotonic density, the infinite divisibility of $\eta$ is also a consequence of the Goldie-Steutel theorem, see [20, Theorem 10.7]. These remarks also show that Corollary 4.3 can be inferred from the complete monotonicity of (4.2) via the Goldie-Steutel theorem. The formula

$$\int_0^\infty e^{-us}\, d\eta(u) = \int_0^1 t^s\, d\mu(t) = F(s) = e^{-\log f(s+1)}, \qquad s \geq 0$$

shows that $\log f(s+1)$ is the Bernstein function associated with the convolution semigroup $(\eta_t)_{t>0}$ of probability measures on the half-line such that $\eta_1 = \eta$, see [10, p. 68].

REMARK 4.5. Let $\mathscr{H}_I$ denote the set of normalized infinitely divisible Hausdorff moment sequences. By Proposition 4.2 we have $T(\mathscr{H}) \subseteq \mathscr{H}_I$. We claim that this inclusion is proper. In fact, it is easy to see that $T : \mathscr{H} \to T(\mathscr{H})$ is one-to-one, and that

$$T^{-1}(\mathbf{b})_n = \frac{1}{b_n} - \frac{1}{b_{n-1}}, \qquad n \geq 1,$$

for $\mathbf{b} = (b_n)_n \in T(\mathscr{H})$. It follows that

$$T(\mathscr{H}) = \left\{ \mathbf{b} \in \mathscr{H} \;\middle|\; \left( \frac{1}{b_n} - \frac{1}{b_{n-1}} \right)_n \in \mathscr{H} \right\}.$$

(Here $1/b_n - 1/b_{n-1} = 1$ for $n = 0$.) Then $\mathbf{b} \in \mathscr{H}_I \setminus T(\mathscr{H})$ if we define $b_n = 1/(n+1)^2$.

The functions $f$, $F$ being holomorphic in $\operatorname{Re} z > -1$ with a pole at $z = -1$, they have power series expansions

$$F(z) = 1 + \sum_{n=1}^{\infty} a_n z^n, \quad f(z) = \sum_{n=1}^{\infty} b_n z^n, \qquad |z| < 1, \tag{4.3}$$

and the radius of convergence is 1 for both series.

PROPOSITION 4.6. *The coefficients in (4.3) are given for $n \geq 1$ by*

$$a_n = \frac{1}{n!} \int_0^1 (\log t)^n \, d\mu(t) = (-1)^n \left( \rho_0 + \sum_{p=1}^{\infty} \sum_{k=1}^{2^{p-1}} \frac{\rho_{p,k}}{(1 - \xi_{p,k})^{n+1}} \right),$$

$$\begin{aligned} b_n &= -\frac{1}{n!} \int_0^1 \frac{(\log t)^n}{1-t} \, d\mu(t) \\ &= (-1)^{n-1} \left( \rho_0 \zeta(n+1, 0) + \sum_{p=1}^{\infty} \sum_{k=1}^{2^{p-1}} \rho_{p,k} \zeta(n+1, -\xi_{p,k}) \right), \end{aligned}$$

*where*

$$\zeta(s, a) = \sum_{n=1}^{\infty} \frac{1}{(n+a)^s}, \qquad s > 1, \;\; a > -1$$

*is the Hurwitz zeta function.*

PROOF. The formula for $a_n$ follows from (1.7) and (1.13), and the formula for $b_n$ follows from (1.6) and (1.14).

ACKNOWLEDGEMENT. The authors wish to thank Henrik L. Pedersen for help with producing the graph of $f$.

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