THE DAUGAVET PROPERTY FOR SPACES OF LIPSCHITZ FUNCTIONS

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Abstract
For a compact metric space $K$ the space $\text{Lip}(K)$ has the Daugavet property if and only if the norm of every $f \in \text{Lip}(K)$ is attained locally. If $K$ is a subset of an $L_p$-space, $1 < p < \infty$, this is equivalent to the convexity of $K$.

1. Introduction
A Banach space $X$ is said to have the Daugavet property if

\[(1.1) \quad \| \text{Id} + T \| = 1 + \| T \| \]

for every rank-1 operator $T: X \to X$; then (1.1) also holds for all weakly compact operators on $X$ and even all operators that do not fix copies of $\ell_1$. The Daugavet property was introduced in [5] and further studied in [10] and [6], but examples of spaces having the Daugavet property have long been known; e.g., $C[0, 1]$, $L_1[0, 1]$, $L_\infty[0, 1]$, the disk algebra, $H_\infty$, etc.

In this paper we shall investigate the Daugavet property for spaces of Lipschitz functions. Throughout, $(K, \rho)$ stands for a complete metric space that is not reduced to a singleton. The space of all Lipschitz functions on $K$ will be equipped with the seminorm

\[\| f \| = \sup \left\{ \frac{|f(t_1) - f(t_2)|}{\rho(t_1, t_2)} : t_1 \neq t_2 \in K \right\}.\]

If one quotients out the kernel of this seminorm, i.e., the constant functions, one obtains the Banach space $\text{Lip}(K)$, whose norm will also be denoted by $\| \cdot \|$. Equivalently, one can fix a point $t_0 \in K$ and consider the Banach space $\text{Lip}_0(K)$ consisting of all Lipschitz functions on $K$ that vanish at $t_0$, with the Lipschitz constant as an actual norm. It is easily seen that $\text{Lip}(K)$ and $\text{Lip}_0(K)$...
are isometrically isomorphic. In this paper we prefer the first point of view, but will refer to the elements of Lip$(K)$ as functions rather than equivalence classes, as is familiar with $L_p$-spaces.

Since Lip$(0, 1)$ is isometric to $L_{\infty}[0, 1]$ via differentiation almost everywhere, it is clear that Lip$(0, 1)$ has the Daugavet property. On the other hand the Hölder space $H^\alpha[0, 1]$, being the dual of a space with the RNP [13, p. 83], fails the Daugavet property by the results of [16]; $H^\alpha[0, 1]$ is just the Lipschitz space for $K = [0, 1]$ with the metric $\rho_\alpha(s, t) = |s - t|^\alpha$. But for the unit square $Q = [0, 1] \times [0, 1]$ with the Euclidean metric it is far from obvious whether the Daugavet property holds for Lip$(Q)$; in fact, this will turn out to be true as a special case of Theorem 3.1 below. The validity of the Daugavet property of Lip$(Q)$ was asked in [15].

Whereas for the “classical” function spaces the validity of the Daugavet property is equivalent to a nonatomicity condition ([3] for $C(S)$ and $L_1(\mu)$, [16] for function algebras, [14] for $L_1$-preduals and [8] for the noncommutative case), in the setting of Lipschitz spaces it is a locality condition that plays a similar role, for in Theorem 3.3 we will show for a compact metric space $K$ that the Daugavet property of Lip$(K)$ is equivalent to the fact that every Lipschitz function on $K$ almost attains its norm at close-by points; see Definition 2.2(a) for precision. We also characterise compact “local” metric spaces by a condition that is reminiscent of metric convexity (Proposition 2.8) and is sometimes even equivalent to it, e.g., for compact subsets of $L_p$, $1 < p < \infty$ (Proposition 2.9). As a result, for a compact subset of $L_p$, $1 < p < \infty$, the Daugavet property of Lip$(K)$ is equivalent to the convexity of $K$.

An important tool to construct Lipschitz functions is McShane’s extension theorem saying that if $M \subset K$ and $f: M \to \mathbb{R}$ is a Lipschitz function, then there is an extension to a Lipschitz function $F: K \to \mathbb{R}$ with the same Lipschitz constant; see [1, p. 12/13]. This will be used several times.

We will also make use of the following geometric characterisations of the Daugavet property from [5] and [2]. Part (iii) is particularly useful when one doesn’t have full access to the dual space. As for notation, we denote the closed unit ball (resp. sphere) of a Banach space $X$ by $B_X$ (resp. $S_X$) and the closed ball with centre $t$ and radius $r$ in a metric space $K$ by $B_K(t, r)$.

**Lemma 1.1.** The following assertions are equivalent:

(i) $X$ has the Daugavet property.

(ii) For every $y \in S_X$, $x^* \in S_{X^*}$ and $\varepsilon > 0$ there exists some $x \in S_X$ such that $x^*(x) \geq 1 - \varepsilon$ and $\|x + y\| \geq 2 - \varepsilon$.

(iii) For every $\varepsilon > 0$ and for every $y \in S_X$ the closed convex hull of the set

\[ \{u \in (1 + \varepsilon)B_X : \|y + u\| \geq 2 - \varepsilon\} \]

contains $S_X$. 

2. Local metric spaces

Let us recall that a metric space $K$ is called \textit{metrically convex} if for any two points $t_1, t_2 \in K$ two closed balls $B_K(t_1, r_1)$ and $B_K(t_2, r_2)$ intersect if and only if $\rho(t_1, t_2) \leq r_1 + r_2$.

Clearly, convex subsets of normed spaces are metrically convex, and $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is metrically convex for the geodesic metric, but not for the Euclidean metric.

We shall need the following lemma.

\textbf{Lemma 2.1.} A complete metric space $K$ is metrically convex if and only if for every two distinct points $t, \tau \in K$ there is an isometric embedding $\phi : [0, a] \to K$ (where $a = \rho(t, \tau)$) such that $\phi(0) = t$, $\phi(a) = \tau$. In other words, $K$ is metrically convex if and only if every two points of $K$ can be connected by an isometric copy of a linear segment.

\textbf{Proof.} The property displayed in the lemma clearly implies the metric convexity of $K$. To prove the converse, let $K$ be metrically convex and let $t$ and $\tau$ be two points at a distance $a$; we shall label them $t_0$ and $t_a$. Then there is a point $t_{a/2} \in B_K(t_0, a/2) \cap B_K(t_a, a/2)$. It follows that $\rho(t_0, t_{a/2}) = \rho(t_{a/2}, t_a) = a/2$. Likewise, pick points $t_{a/4} \in B_K(t_0, a/4) \cap B_K(t_{a/2}, a/4)$ and $t_{3/4a} \in B_K(t_{a/2}, a/4) \cap B_K(t_a, a/4)$. Continuing in this manner, one obtains for each dyadic rational $d \in [0, 1]$ a point $t_{da} \in K$ such that $\rho(t_{da}, t_{d'a}) = |d - d'|a$. The mapping $da \mapsto t_{da}$ can now be extended to an isometric mapping $\phi : [0, a] \to K$, as requested.

The following definition is crucial for this paper.

\textbf{Definition 2.2.} Let $K$ be a metric space.

(a) The space $K$ is called \textit{local} if for every $\varepsilon > 0$ and for every function $f \in \text{Lip}(K)$ there are two distinct points $\tau_1, \tau_2 \in K$ such that $\rho(\tau_1, \tau_2) < \varepsilon$ and

\[ \frac{f(\tau_2) - f(\tau_1)}{\rho(\tau_1, \tau_2)} > \|f\| - \varepsilon. \]  

(b) Let $f \in \text{Lip}(K)$ and $\varepsilon > 0$. A point $t \in K$ is said to be an $\varepsilon$-\textit{point} of $f$ if in every neighbourhood $U \subset K$ of $t$ there are two points $\tau_1, \tau_2 \in U$ for which (2.1) holds true.

(c) The space $K$ is called \textit{spreadingly local} if for every $\varepsilon > 0$ and for every function $f \in \text{Lip}(K)$ there are infinitely many $\varepsilon$-points of $f$.

The next proposition provides a large class of examples.
**Proposition 2.3.** A metrically convex complete metric space $K$ is spreadingly local.

**Proof.** Fix an $\varepsilon > 0$ and a function $f \in \text{Lip}(K)$ with $\|f\| = 1$. Select $t, \tau \in K$ with $\rho(t, \tau) > 0$ such that

$$f(\tau) - f(t) > (1 - \varepsilon)\rho(t, \tau).$$

Denote $a = \rho(t, \tau)$ and apply Lemma 2.1 to this pair of points. The function $F = f \circ \phi: [0, a] \to \mathbb{R}$, where $\phi$ is from Lemma 2.1, is 1-Lipschitz. Hence $|F'| \leq 1$ a.e. on $[0, a]$ and

$$\int_0^a F'(r) \, dr = f(\tau) - f(t) > (1 - \varepsilon)a.$$

Therefore there are infinitely many points $r_i \in [0, a]$ with $F'(r_i) > 1 - \varepsilon$. Let us show that every point of the form $t_i = \phi(r_i)$ is an $\varepsilon$-point of $f$. By the definition of the derivative we have

$$\frac{F(r_i + \delta_i) - F(r_i)}{\delta_i} > 1 - \varepsilon.$$

for sufficiently small $\delta_i \in (0, \varepsilon)$. Denote $\tau_i = \phi(r_i + \delta_i)$. Then $\rho(t_i, \tau_i) = \delta_i$ and $f(\tau_i) - f(t_i) > (1 - \varepsilon)\delta_i$.

Actually this proposition applies to a slightly more general class of spaces $K$, defined by the requirement that for each pair of points $t, \tau \in K$ and each $\eta > 0$ there exists a curve of length $\leq \rho(t, \tau) + \eta =: a_\eta$ joining $t$ and $\tau$. In other words, there exists a 1-Lipschitz mapping (having arclength as parameter) $\phi: [0, a_\eta] \to K$ with $\phi(0) = t, \phi(a_\eta) = \tau$. Such spaces could be termed almost metrically convex. A variant of the above proof then shows that almost metrically convex spaces are spreadingly local.

**Example 2.4.** There is a (noncompact) almost metrically convex space that is not metrically convex. Indeed, let

$$M = \{f \in L_1[0, 1]: \|f\| = 1 \text{ a.e.}\};$$

this is a closed subset of $L_1$. Instead of the $L_1$-norm we shall use the following equivalent norm on $L_1$. Pick a total sequence of functionals $x_n^* \in S_{L_\infty}$ and put, for $f \in L_1$,

$$\|f\| = \|f\|_{L_1} + \left(\sum_{n=1}^{\infty} 2^{-n}\|x_n^*(f)\|^2\right)^{1/2}.$$

This norm is strictly convex. It follows that $M$, equipped with the metric $\rho(f, g) = \|f - g\|$, is not metrically convex since it is not convex; indeed,
if $f, g \in M$, then no nontrivial convex combination belongs to $M$ (unless $f = g$).

On the other hand, $(M, \rho)$ is almost metrically convex. To see this let $f \neq g$ be two functions in $M$. For a Borel set $A \subset [0, 1]$ define $h_A \in M$ by

$$h_A = f \chi_A + g \chi_{[0,1] \setminus A}.$$ 

Given $\varepsilon > 0$, pick $\varepsilon' \leq \varepsilon \|f - g\|$ and $N \in \mathbb{N}$ such that $2(\sum_{n>N} 2^{-n})^{1/2} \leq \varepsilon'$. Define a nonatomic vector measure taking values in $\mathbb{R}^N$ by

$$\mu(A) = \left(\int_A |f - g|, x_1^*((f - g)\chi_A), \ldots, x_N^*((f - g)\chi_A)\right).$$

By the Lyapunov convexity theorem [9, Th. 5.5] there exists a Borel set $\Delta$ such that $\mu(\Delta) = \frac{1}{2} \mu([0, 1])$. We then have, since $g - h_\Delta = (g - f)\chi_\Delta$

$$\|g - h_\Delta\| = \|g - h_\Delta\|_{L_1} + \left(\sum_{n=1}^{\infty} 2^{-n} |x_n^*(g - h_\Delta)|^2\right)^{1/2} \leq \int_\Delta |f - g| + \left(\sum_{n=1}^{N} 2^{-n} |x_n^*((f - g)\chi_\Delta)|^2\right)^{1/2} + \varepsilon' = \frac{1}{2} \int_0^1 |f - g| + \frac{1}{2} \left(\sum_{n=1}^{N} 2^{-n} |x_n^*(f - g)|^2\right)^{1/2} + \varepsilon' \leq \frac{1}{2} \|f - g\| + \varepsilon' \leq \left(\frac{1}{2} + \varepsilon\right)\|f - g\|$$

and likewise

$$\|f - h_\Delta\| \leq \left(\frac{1}{2} + \varepsilon\right)\|f - g\|.$$

Let $F_0 = f$, $F_1 = g$, $F_{1/2} = h_\Delta$. Now we reiterate the above construction, first applying it to $F_0$, $F_{1/2}$, and $\varepsilon/2$ and then to $F_{1/2}$, $F_1$, and $\varepsilon/2$ to obtain functions $F_{1/4}$, $F_{3/4} \in M$ such that

$$\max\{|\|F_0 - F_{1/4}\|, |\|F_{1/2} - F_{1/4}\|\} \leq \left(\frac{1}{2} + \frac{\varepsilon}{2}\right)\|F_0 - F_{1/2}\|,$$

$$\max\{|\|F_{1/2} - F_{3/4}\|, |\|F_1 - F_{1/4}\|\} \leq \left(\frac{1}{2} + \frac{\varepsilon}{2}\right)\|F_1 - F_{1/2}\|.$$ 

Continuing in this manner, we can assign to each dyadic rational $d \in [0, 1]$ a function $F_d \in M$ such that the curve $[0, 1] \to M, t \mapsto F_t$, obtained from this
by continuous extension, has a length that can be estimated from above by
\[
\sup_n \left( \frac{1}{2} + \frac{\varepsilon}{2^{n-1}} \right) \left( \frac{1}{2} + \frac{\varepsilon}{2^{n-2}} \right) \cdots \left( \frac{1}{2} + \varepsilon \right) 2^n \leq \exp(2^{2-n} \varepsilon + \cdots + 2 \varepsilon) \leq e^{4 \varepsilon}.
\]
Therefore \( M \) is almost metrically convex.

We will need a lemma in order to control the Lipschitz constant of a function by the Lipschitz constant of some restriction under highly technical assumptions that we shall meet later. In the following, \( \sqcup \) is used to indicate a disjoint union.

**Lemma 2.5.** Let \( A = B \sqcup C \) be a metric space, \( r \in (0, 1/4] \), \( \delta < r^2/16 \), \( \rho(B, C) > r \). Suppose \( \tilde{C} \subset C \) is a \( \delta \)-net of \( C \) such that every two points of \( \tilde{C} \) are at least \( r \)-distant, and let \( f : A \to \mathbb{R} \) be a function that is 1-Lipschitz on \( B \sqcup \tilde{C} \) and also 1-Lipschitz on every ball \( B_{A}(t, \delta) \) for \( t \in \tilde{C} \). Then \( f \) is \( (1 + r/2) \)-Lipschitz on the whole space \( A \).

**Proof.** Consider arbitrary points \( s_1 \neq s_2 \in A \). We have to prove that
\[
|f(s_2) - f(s_1)| \leq \frac{1 + r}{2}.
\]

We have to distinguish three cases: firstly, when \( s_1, s_2 \in B \); secondly, when \( s_1, s_2 \in C \); and thirdly, when one of the points (say, \( s_1 \)) belongs to \( B \) and the other belongs to \( C \).

In the first case (2.2) holds true even with 1 on the right hand side by assumption on \( f \). Consider the second case. If \( s_1, s_2 \) belong to the same ball of the form \( B_{A}(t, \delta) \) for \( t \in \tilde{C} \), then the job is likewise done. If not, let \( t_1 \neq t_2 \in \tilde{C} \) be points such that \( \rho(t_1, s_1) \leq \delta \) and \( \rho(t_2, s_2) \leq \delta \). Then
\[
|f(s_2) - f(s_1)| \leq |f(s_2) - f(t_2)| + |f(t_2) - f(t_1)| + |f(t_1) - f(s_1)|
\]
\[
\leq \frac{\delta}{\rho(s_1, s_2)} + \frac{\rho(t_2, t_1)}{\rho(s_1, s_2)} + \frac{\delta}{\rho(s_1, s_2)}
\]
\[
\leq \frac{2\delta}{r - 2\delta} + \frac{\rho(t_2, t_1)}{\rho(t_2, t_1) - 2\delta}
\]
\[
\leq \frac{2\delta}{r - 2\delta} + 1 + \frac{2\delta}{\rho(t_2, t_1) - 2\delta}
\]
\[
\leq 1 + \frac{4\delta}{r - 2\delta} \leq 1 + r/2.
\]
In the last case find $t_2 \in \tilde{C}$ such that $\rho(t_2, s_2) \leq \delta$. Then
\[
\left| \frac{f(s_2) - f(s_1)}{\rho(s_1, s_2)} \right| \leq \left| \frac{f(s_2) - f(t_2)}{\rho(s_1, s_2)} \right| + \left| \frac{f(t_2) - f(s_1)}{\rho(s_1, s_2)} \right|
\]
\[
\leq \frac{\delta}{\rho(s_1, s_2)} + \frac{\rho(t_2, s_1)}{\rho(s_1, s_2)}
\]
\[
\leq \frac{\delta}{\rho(t_2, s_1)} + \frac{\rho(t_2, s_1) - \delta}{\rho(t_2, s_1)}
\]
\[
\leq \frac{\delta}{r} + \frac{\delta}{r - \delta} = 1 + \frac{\delta}{r} + \frac{\delta}{r - \delta} \leq 1 + r/2.
\]
This completes the proof of the lemma.

Obviously, a spreadingly local space is local. In the compact case the converse is valid, too, as will be pointed out now.

**Lemma 2.6.** If $K$ is compact and local, then it is spreadingly local.

**Proof.** We will prove by induction on $n$ that for every $f \in \text{Lip}(K)$ and for every $\varepsilon > 0$ there are $n \varepsilon$-points of $f$.

Thanks to the compactness of $K$ every function $f \in \text{Lip}(K)$ has a “0-point”, i.e., a point that is an $\varepsilon$-point for every $\varepsilon > 0$. Indeed, take a sequence of pairs $t_n, \tau_n \in K$ satisfying Definition 2.2 with $\varepsilon = 1/n, n = 1, 2, \ldots$, and take an arbitrary limit point of $(t_n)$. So the start of the induction holds true.

Now assume the statement for a fixed $n$ and let us prove it for $n+1$.

Take an $f \in \text{Lip}(K)$ with $\|f\| = 1$ and $\varepsilon \in (0, 1/4]$. Due to our hypothesis there are $\varepsilon$-points $t_1, \ldots, t_n$ of $f$. Also, select two points $\tau_1, \tau_2 \in K$ distinct from all the $t_i$ and such that
\[
\frac{f(\tau_2) - f(\tau_1)}{\rho(\tau_1, \tau_2)} > 1 - \varepsilon/4.
\]

Let $r \in (0, \varepsilon/4]$ be a number so small that the balls $U_i = B_K(t_i, r), i = 1, \ldots, n$, are disjoint and contain neither $\tau_1$ nor $\tau_2$. Fix a $\delta < r^2/16$, denote the interior of $B_K(t_i, \delta)$ by $V_i$ and consider $\tilde{K} = (K \setminus \bigcup_{i=1}^n U_i) \cup \bigcup_{i=1}^n V_i$ as a subspace of the metric space $K$. Define $\tilde{f}: \tilde{K} \to \mathbb{R}$ as follows: $\tilde{f}(t) = f(t)$ for $t \in K \setminus \bigcup_{i=1}^n U_i$ and $\tilde{f}(t_i) = f(t_i)$ on the corresponding $V_i$. Lemma 2.5 implies that $\tilde{f}$ satisfies a Lipschitz condition on $\tilde{K}$ with the constant $1 + \varepsilon/2$. Extend $\tilde{f}$ to a function on $K$ preserving the Lipschitz constant, still denoted by $\tilde{f}$.

Take as $t_{n+1}$ an arbitrary 0-point of the function $g = f + \tilde{f}$. Since
\[
\|g\| \geq \frac{g(\tau_2) - g(\tau_1)}{\rho(\tau_1, \tau_2)} = 2 \frac{\tilde{f}(\tau_2) - \tilde{f}(\tau_1)}{\rho(\tau_1, \tau_2)} > 2 - \varepsilon/2,
\]
in every neighbourhood of $t_{n+1}$ there are points $s_1, s_2$ with

$$\frac{f(s_2) - f(s_1)}{\rho(s_1, s_2)} + \frac{\tilde{f}(s_2) - \tilde{f}(s_1)}{\rho(s_1, s_2)} > 2 - \epsilon/2. \tag{2.3}$$

This implies that $t_{n+1}$ cannot belong to any $V_i$ since in $V_i$ the second fraction of (2.3) is zero, but the first one is not greater than 1; hence $t_{n+1}$ differs from all the other $t_i$. On the other hand, by our construction $\|\tilde{f}\| \leq 1 + \epsilon/2$, so the second fraction of (2.3) is $\leq 1 + \epsilon/2$. Hence there is an estimate for the first fraction, namely

$$\frac{f(s_2) - f(s_1)}{\rho(s_1, s_2)} > 1 - \epsilon,$$

which means that $t_{n+1}$ is an $\epsilon$-point of $f$.

Next we are going to characterise local metric spaces intrinsically, at least in the compact case, using the following geometric property that we have chosen to give an ad-hoc name.

**Definition 2.7.** A metric space $K$ has property $(Z)$ if the following condition is met: Given $t, \tau \in K$ and $\epsilon > 0$, there is some $z \in K \setminus \{t, \tau\}$ satisfying

$$\rho(t, z) + \rho(z, \tau) \leq \rho(t, \tau) + \epsilon \min\{\rho(z, t), \rho(z, \tau)\}. \tag{2.4}$$

A compact space satisfying (2.4) with $\epsilon = 0$ is easily seen to be metrically convex. Thus, property $(Z)$ is “$\epsilon$-close” to metric convexity, and there are instances when $(Z)$ actually implies metric convexity; see Corollary 2.10 and Remark 2.11 below.

Here is the connection between locality and property $(Z)$.

**Proposition 2.8.** Let $K$ be a metric space.

(a) If $K$ is local, then $K$ has property $(Z)$.

(a) If $K$ is compact and has property $(Z)$, then $K$ is local.

**Proof.** (a) Assume that $K$ fails property $(Z)$, i.e., for some $t_0, \tau_0 \in K$ and $\epsilon_0 > 0$ there are no points $z \in K \setminus \{t_0, \tau_0\}$ as in (2.4). For a point $z \in K$ let $r(z) = \rho(z, t_0)$, $s(z) = \rho(z, \tau_0)$ and $d = \rho(t_0, \tau_0)$. Pick $\epsilon > 0$ with

$$\frac{\epsilon}{1 - \epsilon} < \frac{\epsilon_0}{4}.$$
Now define \( f: K \to \mathbb{R} \) by

\[
\begin{align*}
  f(z) &= \begin{cases} 
    \max\{d/2 - (1 - \varepsilon)s(z), 0\} & \text{if } r(z) \geq s(z), \ r(z) + (1 - 2\varepsilon)s(z) \geq d, \\
    -\max\{d/2 - (1 - \varepsilon)r(z), 0\} & \text{if } r(z) \leq s(z), \ (1 - 2\varepsilon)r(z) + s(z) \geq d.
  \end{cases}
\end{align*}
\]

This function is well defined, since for \( r(z) = s(z) \) both parts of the definition yield 0, and all points of \( K \) are covered in the two “if” cases by our assumption on \( K \); note that \( 2\varepsilon < \varepsilon_0 \).

Let us show that \( f \) is a Lipschitz function with \( \|f\| = 1 \). Indeed, the only critical case is to estimate \( f(z_2) - f(z_1) \) when \( f(z_2) > 0 \) and \( f(z_1) < 0 \); in this case

\[
\begin{align*}
  f(z_2) - f(z_1) &= \left( \frac{d}{2} - (1 - \varepsilon)s(z_2) \right) + \left( \frac{d}{2} - (1 - \varepsilon)r(z_1) \right) \\
  &\leq \left( \frac{r(z_2) + (1 - 2\varepsilon)s(z_2)}{2} - (1 - \varepsilon)s(z_2) \right) \\
  &\quad + \left( \frac{(1 - 2\varepsilon)r(z_1) + s(z_1)}{2} - (1 - \varepsilon)r(z_1) \right) \\
  &= \frac{1}{2} (r(z_2) - s(z_2)) + \frac{1}{2} (s(z_1) - r(z_1)) \\
  &\leq \rho(z_1, z_2);
\end{align*}
\]

also, the norm is attained at \( \tau_0, t_0 \), i.e., \( f(\tau_0) - f(t_0) = \rho(\tau_0, t_0) \).

Consider now points \( z_1, z_2 \in K \) where

\[
(2.5) \quad \frac{f(z_2) - f(z_1)}{\rho(z_2, z_1)} > 1 - \varepsilon;
\]

we shall show that then \( z_1 \) is close to \( t_0 \) and \( z_2 \) is close to \( \tau_0 \) so that their distance is necessarily big. Obviously, we must have \( f(z_2) > 0 \) and \( f(z_1) < 0 \) for (2.5) to subsist. In particular, we have

\[
(2.6) \quad \rho(z_1, t_0) < \rho(z_1, \tau_0); \quad \rho(z_2, \tau_0) < \rho(z_2, t_0).
\]
Hence
\[
(1 - \varepsilon)\rho(z_1, z_2) < f(z_2) - f(z_1)
\]
\[
= \left( \frac{d}{2} - (1 - \varepsilon)\rho(z_2, \tau_0) \right) - \left( \frac{d}{2} - (1 - \varepsilon)\rho(z_1, t_0) \right)
\]
\[
= d - (1 - \varepsilon)(\rho(z_2, \tau_0) + \rho(z_1, t_0));
\]
in other words
\[
(1 - \varepsilon)(\rho(z_1, z_2) + \rho(z_2, \tau_0) + \rho(z_1, t_0)) < d
\]
so that
\[
\rho(z_k, t_0) + \rho(z_k, \tau_0) < \frac{d}{1 - \varepsilon}, \quad k = 1, 2.
\]
By our choice of \(\varepsilon_0, t_0, \tau_0\) and (2.6)
\[
\rho(z_1, t_0) + \rho(z_1, \tau_0) \geq d + \varepsilon_0\rho(z_1, t_0)
\]
so that by (2.7)
\[
d + \varepsilon_0\rho(z_1, t_0) < \frac{d}{1 - \varepsilon}
\]
and hence \(\rho(z_1, t_0) < d/4\) by our choice of \(\varepsilon\). Likewise \(\rho(z_2, \tau_0) < d/4\) and consequently \(\rho(z_1, z_2) > d/2\). Therefore, \(K\) cannot be local.

(b) Assume that \(K\) is not local. Then there is a Lipschitz function \(f\) with \(\|f\| = 1\) for which (2.1) is impossible for \(\tau_1, \tau_2\) at small distance, viz. for \(\rho(\tau_1, \tau_2) < \varepsilon\). By a compactness argument one hence deduces the existence of points \(t, \tau \in K\) such that
\[
\frac{f(\tau) - f(t)}{\rho(\tau, t)} = 1
\]
and \(\rho(t, \tau)\) is minimal among all points as in (2.8). Now let \(\varepsilon_n \searrow 0\) and apply condition \((Z)\) to \(t, \tau\) and \(\varepsilon_n\). This yields a sequence of points \(z_n \in K \setminus \{t, \tau\}\) such that
\[
\rho(t, z_n) + \rho(z_n, \tau) \leq \rho(t, \tau) + \varepsilon_n \min\{\rho(z_n, t), \rho(z_n, \tau)\}.
\]
Passing to a subsequence we may assume that \((z_n)\) converges, say \(z_n \to z_0\), and that without loss of generality
\[
\rho(t, z_n) \leq \rho(\tau, z_n) \quad \forall n \geq 1.
\]
Note that

\[ (2.11) \quad \rho(t, z_0) + \rho(z_0, \tau) = \rho(t, \tau). \]

If \( z_0 \neq t \), then

\[
1 \geq \frac{f(z_0) - f(t)}{\rho(z_0, t)} = \frac{f(\tau) - f(t)}{\rho(\tau, t)} \frac{\rho(\tau, t)}{\rho(z_0, t)} - \frac{f(\tau) - f(z_0)}{\rho(\tau, z_0)} \frac{\rho(\tau, z_0)}{\rho(z_0, t)} \geq \frac{\rho(\tau, t)}{\rho(z_0, t)} - \frac{\rho(\tau, z_0)}{\rho(z_0, t)} = 1
\]

by (2.11), and thus \( f \) attains its norm at the pair \( z_0, t \). But by (2.10)

\[
\rho(t, z_0) \leq \frac{1}{2} (\rho(t, z_0) + \rho(\tau, z_0)) = \frac{1}{2} \rho(t, \tau),
\]

which contradicts the minimality condition imposed on the pair \( t, \tau \).

Therefore, \( z_n \to t \), and for sufficiently large \( n \) we have \( \rho(t, z_n) < \varepsilon \) along with (2.9). But then

\[
\frac{f(z_n) - f(t)}{\rho(z_n, t)} = \frac{f(\tau) - f(t)}{\rho(\tau, t)} \frac{\rho(\tau, t)}{\rho(t, z_n)} - \frac{f(\tau) - f(z_n)}{\rho(\tau, z_n)} \frac{\rho(\tau, z_n)}{\rho(t, z_n)} \geq \frac{\rho(\tau, t) - \rho(\tau, z_n)}{\rho(t, z_n)} \geq 1 - \varepsilon
\]

by (2.9), which contradicts our choice of \( f \), since \( \rho(t, z_n) < \varepsilon \).

The definition of locality immediately implies that a compact local space is connected; one just has to apply the definition with the indicator function of a set that is both open and closed. We will now present a class of compact metric spaces for which property \((Z)\) and hence locality implies (metric) convexity. Recall that a Banach space \((E, \| \cdot \|_E)\) is called \textit{locally uniformly rotund} if for each \( x \in S_E \) and \( \eta > 0 \) there is some \( \delta = \delta_x(\eta) > 0 \) such that \( \| x - y \|_E \leq \eta \) whenever \( y \in B_E \) and \( \frac{1}{2} (x + y) \) 

\[ E \geq 1 - \delta. \]

\textbf{Proposition 2.9.} \textit{Let \((E, \| \cdot \|_E)\) be a smooth locally uniformly rotund Banach space and let \( K \subset E \) be a compact subset with property \((Z)\). Then \( K \) is convex.}

\textbf{Proof.} By a result of Vlasov ([12], [11, Th. 2.2, p. 368]) a compact Chebyshev subset of a smooth Banach space is convex. If we assume that \( K \) is not convex, this means that there are two points \( P, Q \in K \) and a ball \( B \) whose interior does not intersect \( K \) with \( P, Q \in \partial B \); we may assume that \( B \) is centred at the origin, \( B = B_E(0, \alpha) \), and by scaling that \( \| P - Q \|_E = 1 \). Applying condition \((Z)\) to \( P, Q \) and an arbitrary \( \varepsilon > 0 \) yields some \( z = z(\varepsilon) \in K \setminus \{P, Q\} \)
as in (2.4). We may as well assume that \( z_0 = \lim_{\varepsilon \to 0} z(\varepsilon) \) exists; \( z_0 \) lies on the line segment \([P, Q]\) by strict convexity of \( E \). Thus \( z_0 = P \) or \( z_0 = Q \); without loss of generality let us assume the latter. Fix, for the time being, \( \varepsilon \) and \( z = z(\varepsilon) \) and put \( r = \|z - Q\|_E (<1/2) \).

Now consider \( Q(\lambda) = \lambda P + (1 - \lambda)Q \), \( 0 \leq \lambda \leq 1 \). Let us estimate \( \|z - Q(\lambda)\|_E \) in order to derive a contradiction. On the one hand we have, since \( z \in K \) and thus \( \|z\|_E \geq \alpha \),

\[
\|z - Q(\lambda)\|_E \geq \|z\|_E - \|Q(\lambda)\|_E \geq \alpha - \|Q(\lambda)\|_E =: \varphi(\lambda).
\]

Now \( \varphi \) is a concave function of \( \lambda \) with \( \varphi(0) = 0 \) and

\[
\varphi(1/2) = \alpha - \left\| \frac{1}{2} (P + Q) \right\| > 0
\]

by strict convexity. Hence with \( \sigma = 2\varphi(1/2) \)

\[
(2.12) \quad \|z - Q(r)\|_E \geq \varphi(r) \geq \sigma r.
\]

On the other hand, (2.4) means that \( z \in B_E(P, 1 - r + \varepsilon r) \); therefore the point \( w = \frac{1}{2}(z + Q(r)) \) also belongs to this ball, but \( w \notin \text{int} B_E(Q, r - \varepsilon r) \). In other words,

\[
(2.13) \quad \left\| \frac{(Q - z) + (Q - Q(r))}{2} \right\|_E = \left\| Q - z + \frac{Q(r)}{2} \right\|_E \geq r - \varepsilon r.
\]

Specifically, let \( \eta = \sigma/2 \) and \( 0 < \varepsilon < \delta_{P-Q}(\eta) \). Then (2.13) and local uniform rotundity (note that \( (Q - z)/r, (Q - Q(r))/r \in B_E \)) imply that

\[
\|z - Q(r)\|_E \leq r\eta < r\sigma
\]

contradicting (2.12).

Proposition 2.9 applies in particular to \( L_p \)-spaces for \( 1 < p < \infty \) and most particularly to Hilbert spaces.

We can sum up the previous results as follows.

**Corollary 2.10.** Let \( K \) be a compact metric space. Then the following are equivalent:

1. \( K \) is local;
2. \( K \) is spreadingly local;
3. \( K \) has property (Z).

If \( K \) is a subset of a smooth locally uniformly rotund Banach space, then a further equivalent condition is:
(4) $K$ is convex.

Another link between locality and metric convexity is provided by the following technical remark.

Remark 2.11. Let us say that $K$ satisfies $(Z')$ if in addition to (2.4) in Definition 2.7 we require that

$$\rho(z, \tau) \leq \rho(z, t).$$

Since one can exchange the roles of $t$ and $\tau$ here, this means that there is one point as in (2.4) that is closer to $\tau$ than to $t$ and another one that is closer to $t$ than to $\tau$. It is then possible to show that $(Z')$ implies metric convexity for compact spaces; see below. Hence locality implies metric convexity for those compact metric spaces that are symmetric in the sense that for any two points in $K$ there is an isometry on $K$ swapping these two points.

To prove this remark, we rephrase property $(Z')$ by saying that for every $\varepsilon > 0$ and every $t, \tau \in K$ there exists some $z \in K \setminus \{\tau\}$ such that

$$\begin{align*}
(1 - \varepsilon)\rho(\tau, z) + \rho(t, z) &\leq \rho(t, \tau), \\
\rho(\tau, z) &\leq \rho(t, z).
\end{align*}$$

The strategy of the proof will be to infer from this in the compact case that for every $\varepsilon > 0$ and every $t, \tau \in K$ there exists some $z \in K$ for which (2.14) holds and

$$\begin{align*}
\frac{1}{10} \rho(t, \tau) &\leq \rho(\tau, z) \leq \frac{9}{10} \rho(t, \tau).
\end{align*}$$

If we let $\varepsilon \to 0$ and consider a limit point $z_0$ of the $z = z(\varepsilon)$ satisfying (2.14) and (2.16), then we can be certain that $z_0 \neq t$ and $z_0 \neq \tau$, but

$$\rho(t, z_0) + \rho(z_0, \tau) = \rho(t, \tau).$$

As remarked earlier this implies the metric convexity of the compact space $K$.

Let us now come to the details. Fix $t, \tau$ and $\varepsilon$; we may suppose that $\rho(t, \tau) = 1$. Assume for a contradiction that we cannot achieve (2.14) and (2.16) simultaneously. Let

$$K_0 = \{z \in K: (2.14) \text{ and } (2.15) \text{ hold}\}.$$

Since $K_0 \neq \{\tau\}$ by property $(Z')$, there is some $u \in K_0$ such that $\rho(u, t) < 1$, and therefore $\alpha := \min\{\rho(z, t): z \in K_0\}$ is attained at some $u_0 \in K_0 \setminus \{\tau\}$. Then $(1 - \varepsilon)\rho(\tau, u_0) + \rho(u_0, t) \leq 1$ by (2.14). Now define $0 \leq \tilde{\varepsilon} \leq \varepsilon$ by

$$\begin{align*}
(1 - \tilde{\varepsilon})\rho(\tau, u_0) + \rho(u_0, t) &= 1.
\end{align*}$$
If \( \tilde{\varepsilon} = 0 \), we have already found a point as in (2.17), and we are done. So we assume that \( \tilde{\varepsilon} > 0 \) in the sequel. Then we can apply (2.14) and (2.15), i.e., property \((Z)\), with \( t, u_0 \) and \( \tilde{\varepsilon} \) in place of \( t, \tau \) and \( \epsilon \). This yields some \( \tilde{z} \in K \setminus \{ u_0 \} \) with

\[
(2.19) \quad (1 - \tilde{\varepsilon})\rho(u_0, \tilde{z}) + \rho(t, \tilde{z}) \leq \rho(t, u_0),
\]

\[
(2.20) \quad \rho(u_0, \tilde{z}) \leq \rho(t, \tilde{z}).
\]

Next, add (2.18) and (2.19) to obtain

\[
(2.21) \quad (1 - \tilde{\varepsilon})(\rho(\tau, u_0) + \rho(u_0, \tilde{z})) + \rho(t, \tilde{z}) \leq 1.
\]

But \( \rho(t, \tilde{z}) < \rho(t, u_0) = \alpha \), since \( \tilde{z} \neq u_0 \) in (2.19); hence \( \tilde{z} \notin K_0 \). Now the previous inequality, (2.21) and \( \tilde{\varepsilon} \leq \epsilon \) show that \( \tilde{z} \) satisfies (2.14); therefore it must fail (2.15), i.e.,

\[
(2.22) \quad \rho(\tau, \tilde{z}) > \rho(t, \tilde{z}).
\]

Also, recall that \( u_0 \) satisfies (2.14) and that we have assumed that (2.14) and (2.16) do not hold simultaneously. This implies that

\[
\rho(\tau, u_0) < \frac{1}{10} \quad \text{or} \quad \rho(\tau, u_0) > \frac{9}{10}
\]

and

\[
\rho(\tau, \tilde{z}) < \frac{1}{10} \quad \text{or} \quad \rho(\tau, \tilde{z}) > \frac{9}{10}.
\]

If \( \rho(\tau, u_0) > \frac{9}{10} \), then \( \rho(t, u_0) > \frac{9}{10} \) by (2.15); recall that \( u_0 \in K_0 \). Then (2.18) furnishes the contradiction

\[
1 = (1 - \tilde{\varepsilon})\rho(\tau, u_0) + \rho(u_0, t) > (2 - \tilde{\varepsilon})\frac{9}{10} > 1
\]

if, say, \( \varepsilon \leq 1/4 \). The conclusion at this point is

\[
(2.23) \quad \rho(\tau, u_0) < \frac{1}{10}.
\]

On the other hand, if \( \rho(\tau, \tilde{z}) < \frac{1}{10} \), then \( \rho(t, \tilde{z}) > \frac{9}{10} \) by the triangle inequality, which contradicts (2.22). Consequently

\[
(2.24) \quad \rho(\tau, \tilde{z}) > \frac{9}{10}.
\]

If we now use that \( \tilde{z} \) satisfies (2.19) and (2.20), we derive, for \( \varepsilon \leq 1/4 \), that

\[
\rho(u_0, \tilde{z}) \leq \rho(t, \tilde{z}) \leq 1 - (1 - \varepsilon)\rho(\tau, \tilde{z}) \leq \frac{13}{40}
\]
and hence the contradiction
\[\rho(\tau, t) \leq \rho(\tau, u_0) + \rho(u_0, \tilde{z}) + \rho(\tilde{z}, t) < 1.\]

This completes the proof of the remark.

We do not know any example of a compact space with \((Z)\) that is not metrically convex.

3. Locality and the Daugavet property

We can now prove a sufficient criterion for Lip\((K)\) to have the Daugavet property. In particular it turns out that for closed convex subsets of Banach spaces Lip\((K)\) has the Daugavet property.

**Theorem 3.1.** If \(K\) is a spreadingly local metric space (in particular if \(K\) is a metrically convex metric space or a compact local metric space), then Lip\((K)\) has the Daugavet property.

**Proof.** For short write \(X = \text{Lip}(K)\). Due to Lemma 1.1 it is sufficient to prove that for every \(\varepsilon \in (0, 1/4]\), and for every \(f, g \in S_x\) the closed convex hull of the set \(W = \{u \in (1 + \varepsilon)B_X : \|f + u\| \geq 2 - \varepsilon\}\) contains \(g\).

In order to do this fix an \(n \in \mathbb{N}\) and select \(\varepsilon/2\)-points \(s_1, \ldots, s_n\) of \(f\). Let \(r \in (0, \varepsilon/4]\) be a number so small that the balls \(U_i = B_K(s_i, r), i = 1, \ldots, n\), are disjoint. Fix a \(\delta < r^2/16\), and select \(t_i, \tau_i \in B_K(s_i, \delta)\) such that

\[(3.1)\]
\[f(\tau_i) - f(t_i) > (1 - \varepsilon/2)\rho(t_i, \tau_i).\]

Consider \(K_i = (K \setminus U_i) \cup \{t_i, \tau_i\}\) as a subspace of the metric space \(K\). Define \(u_i : K_i \to \mathbb{R}\) as follows: \(u_i(t_i) = g(t_i), u_i(\tau_i) = g(t_i) + f(\tau_i) - f(t_i)\) and \(u_i(s) = g(s)\) on the rest of \(K_i\). It follows from Lemma 2.5 that \(u_i\) satisfies a Lipschitz condition on \(K_i\) with the constant \(1 + r/2 < 1 + \varepsilon/2\). Extend \(u_i\) to a function on \(K\) preserving the Lipschitz constant, still denoted by \(u_i\).

Note that each \(u_i\) belongs to \(W\). In fact \(\|u_i\| \leq 1 + \varepsilon\) by construction and

\[\|f + u_i\| \geq \frac{(f + u_i)(\tau_i) - (f + u_i)(t_i)}{\rho(\tau_i, t_i)} = 2 \frac{f(\tau_i) - f(t_i)}{\rho(\tau_i, t_i)} > 2 - \varepsilon.\]

On the other hand the arithmetic mean of the \(u_i\) (the simplest convex combination) approximates \(g\), for

\[\left\|g - \frac{1}{n} \sum_{i=1}^{n} u_i\right\| = \frac{1}{n} \left\| \sum_{i=1}^{n} (u_i - g) \right\| \leq \frac{4 + 2\varepsilon}{n}.\]
The last inequality follows from the fact that each $u_i - g$ has norm $\leq \|u_i\| + \|g\| \leq 2 + \varepsilon$ and their supports $U_i$ are disjoint.

Finally we address the question in how far our locality conditions are necessary for the Daugavet property; for compact spaces, this will turn out to be the case (Theorem 3.3 below). The bulk of the technical work will be done in the following lemma.

**Lemma 3.2.** Suppose $\text{Lip}(K)$ has the Daugavet property. Then for every $t_1, t_2 \in K$ with $\rho(t_1, t_2) = a > 0$, for every $f \in S_{\text{Lip}(K)}$ with $f(t_2) - f(t_1) = a$ (i.e., $f$ attains its norm at the pair $t_1, t_2$) and for every $\varepsilon > 0$ there are $\tau_1 = \tau_1(\varepsilon), \tau_2 = \tau_2(\varepsilon) \in K$ with the following properties:

1. $f(\tau_2) - f(\tau_1) \geq (1 - \varepsilon)\rho(\tau_1, \tau_2);$
2. $\rho(t_1, \tau_2) - \rho(t_1, \tau_1) \geq (1 - \varepsilon)\rho(\tau_1, \tau_2),$
   $\rho(t_2, \tau_1) - \rho(t_2, \tau_2) \geq (1 - \varepsilon)\rho(\tau_1, \tau_2);$
3. $\rho(\tau_1, \tau_2) \to 0$ as $\varepsilon \to 0$.

**Proof.** We shall abbreviate $\text{Lip}(K)$ by $X$. Consider the following functions $y_i \in X$:

$$y_1 = f, \quad y_2(t) = \rho(t_1, t), \quad y_3(t) = -\rho(t_2, t).$$

For all these functions we have

$$\|y_i\| = 1.$$  \hfill (3.2)

Then the arithmetic mean $y = (y_1 + y_2 + y_3)/3$ is of norm 1 as well. Consider $x^* \in X^*$, with the action

$$x^*(x) = \frac{1}{a}(x(t_2) - x(t_1)).$$  \hfill (3.3)

Clearly $\|x^*\| = 1$. Due to the Daugavet property of $X$ there is, by Lemma 1.1, an $x \in S_X$ such that $x^*(x) > 1 - \varepsilon$, i.e.,

$$x(t_2) - x(t_1) > (1 - \varepsilon)a,$$ \hfill (3.4)

and at the same time $\|x - y\| > 2 - \varepsilon/3$. The last condition means that there are two distinct points $\tau_1, \tau_2 \in K$ for which

$$(x - y)(\tau_1) - (x - y)(\tau_2) > (2 - \varepsilon/3)\rho(\tau_1, \tau_2),$$

i.e.,

$$\frac{1}{3} \sum_{i=1}^{3} ((x - y_i)(\tau_1) - (x - y_i)(\tau_2)) > (2 - \varepsilon/3)\rho(\tau_1, \tau_2).$$
Since neither of these three summands exceeds $2\rho(\tau_1, \tau_2)$, we get the following three inequalities:

\[(3.5) \quad (x - y_i)(\tau_1) - (x - y_i)(\tau_2) > (2 - \varepsilon)\rho(\tau_1, \tau_2), \quad i = 1, 2, 3.\]

Taking into account $x(\tau_1) - x(\tau_2) \leq \rho(\tau_1, \tau_2)$ we deduce that

\[(3.6) \quad y_i(\tau_2) - y_i(\tau_1) > (1 - \varepsilon)\rho(\tau_1, \tau_2), \quad i = 1, 2, 3.\]

The case $i = 1$ gives us the requested property (1), and the cases $i = 2, 3$ of (3.6) immediately provide property (2). Finally, substituting the Lipschitz conditions $x(\tau_1) \leq x(t_1) + \rho(t_1, \tau_1)$ and $x(\tau_2) \geq x(t_2) - \rho(t_2, \tau_2)$ into (3.5) and applying (3.4) we obtain

\[(2 - \varepsilon)\rho(\tau_1, \tau_2) < x(t_1) - x(t_2) + \rho(t_1, \tau_1) + \rho(t_2, \tau_2) + y_i(\tau_2) - y_i(\tau_1)\]

\[\leq -(1 - \varepsilon)\rho(t_1, t_2) + \rho(t_1, \tau_1) + \rho(t_2, \tau_2) + \rho(\tau_1, \tau_2),\]

so

\[(1 - \varepsilon)\rho(t_1, t_2) < \rho(t_1, \tau_1) + \rho(t_2, \tau_2) - (1 - \varepsilon)\rho(\tau_1, \tau_2)\]

\[\leq (2 - \varepsilon) (\rho(t_1, \tau_1) + \rho(t_2, \tau_2)) - (1 - \varepsilon)\rho(t_1, t_2)\]

by the triangle inequality; hence

\[2\rho(t_1, \tau_1) + 2\rho(t_2, \tau_2) > 4(1 - \varepsilon)/(2 - \varepsilon)\rho(t_1, t_2).\]

Adding to this inequality both inequalities from property (2) we obtain

\[\rho(t_1, \tau_1) + \rho(t_2, \tau_2) + \rho(t_1, \tau_2) + \rho(t_2, \tau_1)\]

\[\geq 4(1 - \varepsilon)/(2 - \varepsilon)\rho(t_1, t_2) + 2(1 - \varepsilon)\rho(\tau_1, \tau_2).\]

Since the left hand side is not greater than $2\rho(t_1, t_2)$ we deduce

\[2(1 - \varepsilon)\rho(\tau_1, \tau_2) \leq \left(2 - \frac{4(1 - \varepsilon)}{2 - \varepsilon}\right)\rho(t_1, t_2)\]

which gives property (3).

We can now deduce the main theorem of this paper.

**Theorem 3.3.** If $K$ is a compact metric space, then $\text{Lip}(K)$ has the Daugavet property if and only if $K$ is local.

**Proof.** The “if” part has already been proved in Theorem 3.1. Let us prove the “only if” part. Assume $K$ is not local. Then there is a function $f \in \text{Lip}(K)$, $\|f\| = 1$, and there is an $r > 0$ such that

\[(3.7) \quad f(\tau_2) - f(\tau_1) < (1 - r)\rho(\tau_1, \tau_2)\]
for every $\tau_1, \tau_2 \in K$ with $\rho(\tau_1, \tau_2) < r$. Hence by a compactness argument there is a pair of points $t_1, t_2 \in K$ with $\rho(t_1, t_2) > 0$ on which $f$ attains its norm, i.e., with $f(t_2) - f(t_1) = \rho(t_1, t_2)$. If nevertheless Lip$(K)$ has the Daugavet property, then applying Lemma 3.2 to $f$ and these $t_1, t_2$ with $\varepsilon \to 0$ entails a contradiction between (3.7) and properties (1) and (3) from the lemma.

The space Lip$(K)$ has a canonical predual, called the Arens-Eells space in [13] and the Lipschitz free space in [4] and [7]. Since we have used in (3.3), in the proof of Lemma 3.2, a functional from that predual, i.e., a weak* open slice, the lemma works under the assumption that the Lipschitz free space on $K$ has the Daugavet property. Consequently, for a compact metric space Lip$(K)$ has the Daugavet property if and only if its Lipschitz free space has.

In the setting of subsets of certain Banach spaces like $L_p$, $1 < p < \infty$, we can rephrase Theorem 3.3 as follows, using Corollary 2.10.

**Corollary 3.4.** *If $K$ is a compact subset of a smooth locally uniformly rotund Banach space, then Lip$(K)$ has the Daugavet property if and only if $K$ is convex.*

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