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SUB-SOLUTIONS AND MEAN-VALUE OPERATORS FOR ULTRAPARABOLIC EQUATIONS ON LIE GROUPS

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Abstract

The aim of this paper is to provide a theory of sub-solutions for a class of hypoelliptic ultraparabolic operators $\mathscr{L}$, by using mean-value operators on the level sets of the fundamental solution of $\mathscr{L}$.

1. Introduction

In this paper we investigate several questions in Potential Theory related to a class of hypoelliptic ultraparabolic operators $\mathscr{L}$ with underlying homogeneous Lie group structures. Our class is contained in the one of the Hörmander operators and was singled out by Kogoj and Lanconelli in [9]. It contains, e.g., the heat operators on Carnot groups and the Kolmogorov type operators studied in [12]. We are mainly interested in a characterization of $\mathscr{L}$-subharmonic functions in terms of suitable mean-value operators and in representation formulas. We also characterize the bounded-above $\mathscr{L}$-subharmonic functions in R^{N+1} and their related $\mathscr{L}$-Riesz measures. These results extend to our class of operators several theorems proved by Watson in [14], [15], [16] for temperatures and subtemperatures. They also extend some results first proved in [7], [8] for classical parabolic operators with smooth coefficients.

We consider operators of the following type

$$\mathscr{L} = \sum_{j=1}^{m} X_j^2 + X_0 - \partial_t \qquad \text{in} \quad \mathsf{R}^{N+1}, \tag{1.1}$$

where the X_j's are smooth vector fields on R^N, i.e. denoting $z = (x, t)$ the point in R^{N+1}

$$X_j(x) = \sum_{k=1}^{N} a_j^k(x)\partial_{x_k}, \qquad j = 0, \dots, m,$$

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where any a_j^k is a C^∞ function. For our purposes, in the sequel we also consider the X_j's as vector fields in R^{N+1}. We denote by Y the vector field in R^{N+1}

$$Y := X_0 - \partial_t,$$

and by $\mathscr{L}_0$ the operator in R^N

$$\mathscr{L}_0 := \sum_{j=1}^{m} X_j^2 + X_0.$$

We next state our main assumptions:

(H.1) there exists a homogeneous Lie group $\mathsf{L} = (\mathsf{R}^{N+1}, \circ, d_\lambda)$ such that
  (i) $X_1, \dots, X_m, Y$ are left invariant on L;
  (ii) $X_1, \dots, X_m$ are d_λ-homogeneous of degree one and Y is d_λ-homogeneous of degree two;

(H.2) for every $(x, t), (\xi, \tau) \in \mathsf{R}^{N+1}$ with $t > \tau$, there exists an $\mathscr{L}$-admissible path $\eta : [0, T] \longrightarrow \mathsf{R}^{N+1}$ such that $\eta(0) = (x, t)$, $\eta(T) = (\xi, \tau)$. The curve η is called $\mathscr{L}$-admissible if it is absolutely continuous and satisfies

$$\eta'(s) = \sum_{j=1}^{m} \lambda_j(s) X_j(\eta(s)) + \lambda_0(s) Y(\eta(s)), \qquad \text{a.e. in} \quad [0, T],$$

for suitable piecewise constant real functions $\lambda_0, \lambda_1, \dots \lambda_m$, $\lambda_0 \geq 0$.

The operators of the form (1.1) with the assumptions (H.1) and (H.2) have been introduced by Kogoj and Lanconelli in [9].

The aim of this paper is to provide a theory of $\mathscr{L}$-subharmonic functions by using mean-value operators on the level sets of Γ, the fundamental solution of $\mathscr{L}$. More precisely, the contents of the paper are the following ones. After collecting in Section 2 some basic results of Potential Theory for $\mathscr{L}$, in Section 3 we recall some mean-value representation formulas and we present some properties of $\mathscr{L}$-harmonic functions with respect to mean-value integral operators $\mathscr{M}_r$ and M_r. In Section 4, we show some results on upper semi-continuous functions satisfying solid sub-mean property. In Section 5, we prove some characterizations of $\mathscr{L}$-subharmonic functions in terms of averaging operators $\mathscr{M}_r$ and M_r. Finally, in Section 6 we give a characterization of the bounded-above $\mathscr{L}$-subharmonic functions in R^{N+1} and their related $\mathscr{L}$-Riesz measures.

2. Basic potential theory for $\mathscr{L}$

We first recall some consequences of hypotheses (H.1) and (H.2). One of these is the Hörmander condition:

$$\text{rank Lie}\{X_1, \ldots, X_m, Y\}(z) = N + 1, \qquad \forall\, z \in \mathsf{R}^{N+1};$$

hence $\mathscr{L}$ and $\mathscr{L}_0$ are hypoelliptic operators in R^{N+1} and in R^N respectively (see [9, Proposition 10.1]). From (H.1) and (H.2) it follows also that the composition law $\circ$ is euclidean in the "time" variable, i.e.

$$(x, t) \circ (\xi, \tau) = (S(x, t, \xi, \tau), t + \tau)$$

for a suitable smooth function S, and the dilation d_λ takes the following form

$$d_\lambda(x, t) = (D_\lambda(x), \lambda^2 t) = (\lambda^{\sigma_1} x_1, \ldots, \lambda^{\sigma_N} x_N, \lambda^2 t).$$

The natural number

$$Q = \sum_{k=1}^{N} \sigma_k + 2$$

is the *homogeneous dimension* of L. We shall assume that $Q \geq 5$ so that $Q - 2$, the homogeneous dimension of R^N with respect to D_λ, will be ≥ 3. We shall denote by $|\cdot|$ a fixed d_λ*-homogeneous norm* on L, that is a function $|\cdot| : \mathsf{R}^{N+1} \longrightarrow [0, \infty[$ with the following properties: $|\cdot| \in C^\infty(\mathsf{R}^{N+1} \setminus \{(0,0)\}) \cap C(\mathsf{R}^{N+1})$, $|d_\lambda(z)| = \lambda|z|$, $|z^{-1}| = |z|$, $|z| = 0$ iff $z = 0$.

In [9] it is proved that $\mathscr{L}$ has a global fundamental solution $\Gamma \in C^\infty(\mathsf{R}^{N+1} \setminus \{(0,0)\})$ such that $\mathscr{L}\,\Gamma = -\delta$. Moreover $\Gamma(x, t) > 0$ iff $t > 0$. If we put

$$\Gamma(z, \zeta) := \Gamma(\zeta^{-1} \circ z),$$

since $\mathscr{L}$ is left translation invariant, we have $\mathscr{L}\,\Gamma(\cdot, \zeta) = -\delta_\zeta$ for every $\zeta \in \mathsf{R}^{N+1}$.

Integrating Γ with respect to the t variable one obtains a fundamental solution γ with pole at $x = 0$ for the operator $\mathscr{L}_0$ (see [9, Section 3]):

$$\gamma(x) := \int_0^\infty \Gamma(x, t)\, \mathrm{d}t;$$

γ is smooth and strictly positive out of the origin.

Throughout the paper, Ω will always denote an open subset of R^{N+1}, even if we do not mention this. We call $\mathscr{L}$*-harmonic* in Ω every smooth function $u : \Omega \longrightarrow \mathsf{R}$ such that $\mathscr{L}u = 0$. We shall denote by $\mathscr{H}^{\mathscr{L}}(\Omega)$ the linear space of $\mathscr{L}$-harmonic functions in Ω.

We say that a bounded open set $V \subset \mathsf{R}^{N+1}$ is *$\mathscr{L}$-regular* if for any $\varphi \in C(\partial V)$ there exists a unique function $H_\varphi^V \in \mathscr{H}^{\mathscr{L}}(V)$ such that

$$\lim_{z \to z_0} H_\varphi^V(z) = \varphi(z_0), \qquad \text{for every} \quad z_0 \in \partial V,$$

and $H_\varphi^V \geq 0$ whenever $\varphi \geq 0$, as the classical Picone's maximum principle holds for $\mathscr{L}$ (see [9, Proposition 2.1]). Then, if V is $\mathscr{L}$-regular, for every fixed $z \in V$ the map

$$C(\partial V) \ni \varphi \mapsto H_\varphi^V(z) \in \mathsf{R}$$

defines a linear positive functional on $C(\partial V)$. As a consequence, there exists a Radon measure μ_z^V supported in ∂V, such that

$$H_\varphi^V(z) = \int_{\partial V} \varphi(\zeta)\, \mathrm{d}\mu_z^V(\zeta), \qquad \text{for every} \quad \varphi \in C(\partial V).$$

We call μ_z^V the *$\mathscr{L}$-harmonic measure* related to V and z.

We say that $u : \Omega \longrightarrow [-\infty, \infty[$ is *$\mathscr{L}$-subharmonic* in Ω ($u \in \underline{\mathscr{S}}^{\mathscr{L}}(\Omega)$) if u is upper semi-continuous (u.s.c.), $u > -\infty$ in a dense subset of Ω, and for every open $\mathscr{L}$-regular set $V \subset \overline{V} \subset \Omega$ and for every $z \in V$,

$$u(z) \leq \int_{\partial V} u(\zeta)\, \mathrm{d}\mu_z^V(\zeta).$$

It is easy to prove that a function $u : \Omega \longrightarrow [-\infty, \infty[$ u.s.c. and finite in a dense subset of Ω is $\mathscr{L}$-subharmonic in Ω if

$$u \leq H_\varphi^V \qquad \text{in} \quad V,$$

for every V open $\mathscr{L}$-regular set, $\overline{V} \subset \Omega$, and for every $\varphi \in C(\partial V)$ such that $\varphi \geq u|_{\partial V}$. Proceeding as in [13, Theorem 1], we can obtain the following further characterization of $\mathscr{L}$-subharmonic functions.

PROPOSITION 2.1. *Let $u : \Omega \longrightarrow [-\infty, \infty[$ be an* u.s.c. *function. Then, if $u \in \underline{\mathscr{S}}^{\mathscr{L}}(\Omega)$, we have $u \in L^1_{loc}(\Omega)$ and $\mathscr{L}u \geq 0$ in the distribution sense.*

REMARK 2.2. By Proposition 2.1, if $u \in \underline{\mathscr{S}}^{\mathscr{L}}(\Omega)$ then there exists a Radon measure μ in Ω such that $\mathscr{L}u = \mu$. We shall call μ the *$\mathscr{L}$-Riesz measure* related to u.

We obviously have $\underline{\mathscr{S}}^{\mathscr{L}}(\Omega) \cap (-\underline{\mathscr{S}}^{\mathscr{L}}(\Omega)) = \mathscr{H}^{\mathscr{L}}(\Omega)$.

In the sense of the abstract Potential Theory (see, e.g., [6]), the map $\mathsf{R}^{N+1} \supseteq \Omega \mapsto \mathscr{H}^{\mathscr{L}}(\Omega)$ is a *harmonic sheaf* and $(\mathsf{R}^{N+1}, \mathscr{H}^{\mathscr{L}})$ is a $\mathfrak{B}$-*harmonic space*. The second statement is a consequence of the following properties:

- the $\mathscr{L}$-regular sets form a basis of the Euclidean topology (see [5, Corollary 5.2]);
- $\mathscr{H}^{\mathscr{L}}$ satisfies *the Doob convergence property*, i.e. the pointwise limit of any increasing sequence of $\mathscr{L}$-harmonic functions on any open set is $\mathscr{L}$-harmonic whenever it is finite on a dense set (see [9, Proposition 7.4]);
- for every fixed $\zeta = (\xi, \tau) \in \mathsf{R}^{N+1}$, the functions $z \mapsto -\Gamma(\zeta^{-1} \circ z)$ and $(x, t) \mapsto -\gamma(\xi^{-1} \circ x)$ are $\mathscr{L}$-subharmonic in R^{N+1} and it easy to show that the families $\{z \mapsto -\Gamma(\zeta^{-1} \circ z) \mid \zeta \in \mathsf{R}^{N+1}\}$, $\{(x, t) \mapsto -\gamma(\xi^{-1} \circ x) \mid \xi \in \mathsf{R}^N\}$ separate the points of R^{N+1}.

3. Mean-value formulas and $\mathscr{L}$-harmonic functions

Given $z \in \mathsf{R}^{N+1}$ and $r > 0$, we define the *$\mathscr{L}$-ball of center z and radius r* as follows:

$$\Omega_r(z) := \left\{\zeta \in \mathsf{R}^{N+1} \,\middle|\, \Gamma(\zeta^{-1} \circ z) > \frac{1}{r^{Q-2}}\right\}.$$

Obviously, $\Omega_r(z) = z \circ \Omega_r(0)$. The properties of the $\mathscr{L}$-balls stated in the next proposition directly follow from the properties of the fundamental solution Γ proved in [9].

PROPOSITION 3.1. *For every $z \in \mathsf{R}^{N+1}$, the $\mathscr{L}$-balls centered in z have the following properties:*

(i) *for every $r > 0$, $\Omega_r(z)$ is a bounded nonempty set;*

(ii) *$\Omega_r(z)$ shrinks to $\{z\}$ as r goes to 0, that is $\bigcap_{r>0} \overline{\Omega}_r(z) = \{z\}$;*

(iii) *if we denote by $|\Omega_r(z)|$ the Lebesgue measure of $\Omega_r(z)$, then*

$$\lim_{r\to 0^+} \frac{|\Omega_r(z)|}{r^{Q-2}} = 0;$$

(iv) *for almost every $r > 0$, $\partial\Omega_r(z)$ is a N-dimensional C^∞ manifold;*

(v) *if $z = (x, t)$, then $\bigcup_{r>0} \Omega_r(z) = \mathsf{R}^N \times]-\infty, t[$.*

If $\Omega \subseteq \mathsf{R}^{N+1}$ is an open set containing 0, and $v \in C^2(\Omega)$, we have

$$v(0) = \mathscr{M}_r(v)(0) - \mathscr{N}_r(\mathscr{L}v)(0), \qquad \text{for every} \quad \overline{\Omega}_r(0) \subseteq \Omega; \tag{3.1}$$

where

$$\mathcal{M}_r(v)(0) := \int_{\partial\Omega_r(0)} \mathcal{K}(\zeta)v(\zeta)\,\mathrm{d}\sigma(\zeta), \qquad \text{with} \quad \mathcal{K}(\zeta) := \frac{|\nabla_{\mathcal{L}}\Gamma(0,\zeta)|^2}{|\nabla_\zeta\Gamma(0,\zeta)|};$$

$$\mathcal{N}_r(\mathcal{L}v)(0) := \int_{\Omega_r(0)} \left(\Gamma(0,\zeta) - \frac{1}{r^{Q-2}}\right)\mathcal{L}v(\zeta)\,\mathrm{d}\zeta.$$

Hereafter we denote by $\nabla_{\mathcal{L}}$ the vector valued differential operator

$$\nabla_{\mathcal{L}} = (X_1, \dots, X_m),$$

and $\nabla_x = (\partial_{x_1}, \dots, \partial_{x_N})$.

Formula (3.1) is a particular version of the Green representation theorem for $\mathcal{L}$. In order to get it, we proceed as in [11, Theorem 1.5], using the properties of fundamental solution Γ showed in [9], in particular the inequality (5.1) and the identity (6.1), and writing $\mathcal{L}$ in the following divergence form

$$\mathcal{L} = \mathrm{div}(A\nabla_x) + Y,$$

where A is a suitable $N \times N$ matrix, and Y is divergence free.

Let $z \in \mathbb{R}^{N+1}$. We apply (3.1) at the function $v_z(\zeta) = u(z \circ \zeta)$, and using the invariance of $\mathcal{L}$ w.r.t. the left translations on $\mathbb{L}$ we get

(3.2)
$$\begin{aligned} u(z) &= \int_{\partial\Omega_r(0)} \mathcal{K}(\zeta)u(z\circ\zeta)\,\mathrm{d}\sigma(\zeta) - \int_{\Omega_r(z)} \left(\Gamma(\zeta^{-1}\circ z) - \frac{1}{r^{Q-2}}\right)\mathcal{L}u(\zeta)\,\mathrm{d}\zeta \\ &=: \mathcal{M}_r(u)(z) - \mathcal{N}_r(\mathcal{L}u)(z), \qquad \text{for every} \quad \overline{\Omega}_r(z) \subseteq z\circ\Omega. \end{aligned}$$

Setting $r = l$ in (3.2), multiplying both sides by l^{Q-3} and integrating between 0 and r give

$$u(z)\frac{r^{Q-2}}{Q-2} = \int_0^r l^{Q-3}\mathcal{M}_l(u)(z)\,\mathrm{d}l - \int_0^r l^{Q-3}\mathcal{N}_l(\mathcal{L}u)(z)\,\mathrm{d}l, \tag{3.3}$$

then, by means of Federer's co-area formula, we obtain

(3.4)
$$\begin{aligned} u(z) &= \frac{1}{r^{Q-2}}\int_{\Omega_r(z)} K(\zeta^{-1}\circ z)u(\zeta)\,\mathrm{d}\zeta - \frac{Q-2}{r^{Q-2}}\int_0^r l^{Q-3}\mathcal{N}_l(\mathcal{L}u)(z)\,\mathrm{d}l \\ &=: M_r(u)(z) - N_r(\mathcal{L}u)(z), \qquad \text{for every} \quad \overline{\Omega}_r(z) \subseteq z\circ\Omega, \end{aligned}$$

where

$$K(\zeta^{-1}\circ z) = K(z,\zeta) := \frac{|\nabla_{\mathcal{L}}\Gamma(z,\zeta)|^2}{\Gamma^2(z,\zeta)}.$$

We explicitly note that the kernel K is invariant w.r.t. the left translation on L, unlike $\mathscr{K}$. Let $z = (x, t)$ be fixed. We have $K(z, \cdot) \geq 0$ in R^{N+1}, $K(z, \cdot) \in C^\infty(\{(\xi, \tau) \in \mathrm{R}^{N+1} \mid \tau < t\})$. By [9, Lemma 7.3], the set

$$\Sigma := \{\zeta = (\xi, \tau) \in \mathrm{R}^{N+1} \mid \tau < t, K(z, \zeta) = 0\}$$

does not contain interior points.

Now, let $\Omega \subseteq \mathrm{R}^{N+1}$ be an arbitrary open set. By comparing (3.3) with (3.4), we deduce that, if $u \in C^2(\Omega)$,

(3.5)
$$M_r(u)(z) = \frac{Q-2}{r^{Q-2}} \int_0^r l^{Q-3} \mathscr{M}_l(u)(z)\, \mathrm{d}l, \qquad \text{for every} \quad \overline{\Omega}_r(z) \subseteq \Omega.$$

We stress that, by a standard argument of approximation, (3.5) also holds if u is u.s.c.

Moreover, from (3.2) and (3.4) it follows that any $\mathscr{L}$-harmonic function in Ω satisfies the mean value formulas

$$u(z) = \mathscr{M}_r(u)(z) \qquad \text{and} \qquad u(z) = M_r(u)(z),$$

for every $z \in \Omega$ and $r > 0$ such that $\overline{\Omega}_r(z) \subseteq \Omega$. If $u \in C(\Omega)$, also the converse implication of this result is true. Indeed we have the following generalization to the classical Koebe theorem.

THEOREM 3.2. *Let $u \in C(\Omega)$ be such that*

$$u(z) = \mathscr{M}_r(u)(z), \qquad \textit{for every} \quad \overline{\Omega}_r(z) \subseteq \Omega. \tag{3.6}$$

Then $u \in C^\infty(\Omega)$ and $\mathscr{L}u = 0$. An analogous result holds if

$$u(z) = M_r(u)(z), \qquad \textit{for every} \quad \overline{\Omega}_r(z) \subseteq \Omega. \tag{3.7}$$

In order to prove this theorem, we need a lemma. Let $J \in C_0^\infty(\mathrm{R}^{N+1})$, $J \geq 0$ be such that $\operatorname{supp} J \subseteq B(0, 1)$ and $\int_{\mathrm{R}^{N+1}} J = 1$. Let $\Omega \subseteq \mathrm{R}^{N+1}$ be an open set, and let $u \in L^1_{\mathrm{loc}}(\Omega)$. For $\varepsilon > 0$, we define the ε-$\mathscr{L}$*-mollified* of u in Ω as follows

$$\begin{aligned} u_\varepsilon : D_\varepsilon^\Omega &\to \mathrm{R} \\ z &\mapsto \int_\Omega u(\zeta) J\big(d_{\varepsilon^{-1}}(z \circ \zeta^{-1})\big) \varepsilon^{-Q}\, \mathrm{d}\zeta, \end{aligned}$$

where $D_\varepsilon^\Omega = \left\{\zeta \in \mathrm{R}^{N+1} \;\middle|\; \overline{B}(\zeta^{-1}, \varepsilon) \subset \Omega^{-1}\right\}$. It is a standard matter to show that $u_\varepsilon \in C^\infty(D_\varepsilon^\Omega)$, and $u_\varepsilon \longrightarrow u$ in $L^1_{\mathrm{loc}}(\Omega)$ as $\varepsilon \to 0$. We next prove the so called solid sub-mean property of u_ε (see Section 4).

LEMMA 3.3. *Let* $u : \Omega \longrightarrow [-\infty, \infty[$ *be an* u.s.c. *function,* $u \in L^1_{\text{loc}}(\Omega)$. *If* $u(z) \leq M_r(u)(z)$ *for every* $\overline{\Omega}_r(z) \subseteq \Omega$, *then* $u_\varepsilon(z) \leq M_r(u_\varepsilon)(z)$ *for every* $\overline{\Omega}_r(z) \subseteq D^\Omega_\varepsilon$.

PROOF. For $\overline{\Omega}_r(z) \subseteq D^\Omega_\varepsilon$, we have

$$\begin{aligned}
&M_r(u_\varepsilon)(z)\\
&\quad = \int_{\overline{B}(0,\varepsilon)} J(d_{\varepsilon^{-1}}(\eta))\,\varepsilon^{-Q}\left(\frac{1}{r^{Q-2}}\int_{\Omega_r(z)} K(\zeta^{-1}\circ z)u(\eta^{-1}\circ\zeta)\,\mathrm{d}\zeta\right)\mathrm{d}\eta\\
&\quad = \int_{\overline{B}(0,\varepsilon)} J(d_{\varepsilon^{-1}}(\eta))\varepsilon^{-Q} M_r(u)(\eta^{-1}\circ z)\,\mathrm{d}\eta\\
&\qquad\qquad\qquad\qquad (\overline{\Omega}_r(\eta^{-1}\circ z) \subseteq \overline{B}(0,\varepsilon)\circ D^\Omega_\varepsilon \subseteq \Omega)\\
&\quad \geq \int_{\overline{B}(0,\varepsilon)} J(d_{\varepsilon^{-1}}(\eta))\varepsilon^{-Q} u(\eta^{-1}\circ z)\,\mathrm{d}\eta = u_\varepsilon(z),
\end{aligned}$$

and the assertion follows.

PROOF OF THEOREM 3.2. It easily follows from (3.5) that (3.6) is equivalent to (3.7). Now, if $u \in C^\infty(\Omega)$ satisfies (3.6), then from (3.2) we obtain

$$0 = \mathcal{N}_r(\mathcal{L}u)(z) = \int_{\Omega_r(z)} \left(\Gamma(\zeta^{-1}\circ z) - \frac{1}{r^{Q-2}}\right)\mathcal{L}u(\zeta)\,\mathrm{d}\zeta,$$

and so $\mathcal{L}u = 0$. Then, it suffices to prove that u is smooth. If we show that $\mathcal{L}u = 0$ in the distribution sense on Ω, the assertion follows from the hypoellipticity of $\mathcal{L}$. From (3.7) and Lemma 3.3, we get $u_\varepsilon(z) = M_r(u_\varepsilon)(z)$ for every $\overline{\Omega}_r(z) \subseteq D^\Omega_\varepsilon$. So the sequence of $(1/n)$-$\mathcal{L}$-mollified $\{u_{1/n}\}_n$ is such that $\mathcal{L}u_{1/n} = 0$ in $D^\Omega_{1/n}$ and $u_{1/n} \longrightarrow u$ per $n \to \infty$ uniformly on compact subsets of Ω. For every $\varphi \in C^\infty_0(\Omega)$, we have

$$\int_\Omega u(\zeta)\mathcal{L}^*\varphi(\zeta)\,\mathrm{d}\zeta = \lim_{n\to\infty}\int_{D^\Omega_{1/n}\cap\operatorname{supp}\varphi} \mathcal{L}\,u_{1/n}(\zeta)\varphi(\zeta)\,\mathrm{d}\zeta = 0,$$

and the smoothness of u is proved.

We also show another property of the ε-$\mathcal{L}$-mollified.

PROPOSITION 3.4. *Let* $u \in \underline{\mathcal{S}}^{\mathcal{L}}(\Omega)$. *Then* u_ε *is* $\mathcal{L}$*-subharmonic in* D^Ω_ε.

PROOF. Since $u_\varepsilon \in C^\infty(D^\Omega_\varepsilon)$, it is enough to prove that $\mathcal{L}u_\varepsilon \geq 0$ in the weak sense of distribution. Let $\varphi \in C_0^\infty(D_\varepsilon^\Omega)$, $\varphi \geq 0$. We have

$$\begin{aligned}\langle \mathscr{L} u_\varepsilon, \varphi\rangle &= \int_{D_\varepsilon^\Omega} u_\varepsilon(z)\mathscr{L}^*\varphi(z)\,\mathrm{d}z \\ &= \int_{\overline{B}(0,\varepsilon)} \left(\int_{D_\varepsilon^\Omega} u(\eta^{-1}\circ z)\mathscr{L}^*\varphi(z)\,\mathrm{d}z\right) J(d_{\varepsilon^{-1}}(\eta))\varepsilon^{-Q}\,\mathrm{d}\eta \\ &= \int_{\overline{B}(0,\varepsilon)} \left(\int_{\Omega} u(\zeta)\mathscr{L}^*[\varphi(\eta\circ\zeta)]\,\mathrm{d}\zeta\right) J(d_{\varepsilon^{-1}}(\eta))\varepsilon^{-Q}\,\mathrm{d}\eta \geq 0,\end{aligned}$$

and the assertion is proved.

As a straightforward consequence, we have the following smoothing result.

COROLLARY 3.5. *Let $u \in \underline{\mathscr{S}}^{\mathscr{L}}(\Omega)$. There exists a sequence of smooth $\mathscr{L}$-subharmonic functions which tends to u in $L^1_{\mathrm{loc}}(\Omega)$.*

4. Sub-mean functions

We say that an u.s.c. function $u : \Omega \longrightarrow [-\infty, \infty[$ satisfies the *surface (solid) sub-mean property* if

$$u(z) \leq \mathscr{M}_r(u)(z) \quad \big(u(z) \leq M_r(u)(z)\big), \qquad \text{for every} \quad \overline{\Omega}_r(z) \subseteq \Omega.$$

Next theorem shows that solid sub-mean functions satisfy a weak maximum principle.

THEOREM 4.1. *Let $u : \Omega \longrightarrow [-\infty, \infty[$ be an* u.s.c. *function satisfying the solid sub-mean property. We have:*

(i) *if Ω is bounded and $\limsup_{\Omega\ni z\to\zeta} u(z) \leq 0$ for every $\zeta \in \partial\Omega$ then $u \leq 0$ in Ω;*

(ii) *if Ω is unbounded and*

$$\limsup_{\Omega\ni z\to\zeta} u(z) \leq 0 \quad \textit{for every } \zeta \in \partial\Omega, \qquad \limsup_{z\in\Omega, |z|\to\infty} u(z) \leq 0,$$

then $u \leq 0$ in Ω.

PROOF. (i) Let $z_0 \in \overline{\Omega}$ be such that $\sup_\Omega u = \sup_{\Omega\cap V} u$ for every $V \in \mathscr{U}_{z_0}$, where $\mathscr{U}_{z_0}$ is the set of all the neighborhoods of z_0. If $z_0 \in \partial\Omega$, by the hypothesis we have

$$0 \geq \limsup_{\Omega\ni z\to z_0} u(z) = \inf_{V\in\mathscr{U}_{z_0}} \sup_{\Omega\cap(V\setminus\{z_0\})} u = \inf_{V\in\mathscr{U}_{z_0}} \sup_{\Omega\cap V} u = \sup_\Omega u,$$

whence $u \leq 0$ on Ω.

Let us suppose $z_0 \in \Omega$. By the upper semicontinuity of u, $u(z_0) = \inf_{V \in \mathcal{U}_{z_0}} \sup_{\Omega \cap V} u = \sup_\Omega u$, whence $u(z_0) = \max_\Omega u$. We may consider $u(z_0) \neq -\infty$, otherwise the claim is obvious. Since Ω is an open set, there exists $r > 0$ such that $\overline{\Omega}_r(z_0) \subseteq \Omega$. By the solid sub-mean property of u,

$$0 \leq \frac{1}{r^{Q-2}} \int_{\Omega_r(z_0)} (u(\zeta) - u(z_0)) K(\zeta^{-1} \circ z_0) \, \mathrm{d}\zeta. \tag{4.1}$$

Thus, as $K(z_0, \cdot) \geq 0$ in R^{N+1} and $u(\zeta) \leq u(z_0)$,

$$K(\zeta^{-1} \circ z_0)(u(\zeta) - u(z_0)) = 0 \qquad \text{a.e. in} \quad \Omega_r(z_0).$$

On the other hand, $K(z_0, \cdot) > 0$ in a dense open subset of $\{(x, t) \in \mathsf{R}^{N+1} \mid t < t_0\}$ and u is a u.s.c. function which attains in z_0 the maximum on Ω. This yields

$$u \equiv u(z_0) \text{ on } \Omega_r(z_0), \qquad \text{for every} \quad \overline{\Omega}_r(z_0) \subseteq \Omega. \tag{4.2}$$

As Ω is bounded and by (4.2), it is easy to show that there exists $r_0 \in]0, \infty[$ such that $u \equiv u(z_0)$ on $\Omega_{r_0}(z_0) \subseteq \Omega$, with $\overline{\Omega}_{r_0}(z_0) \nsubseteq \Omega$. Hence, for $\zeta \in \overline{\Omega}_{r_0}(z_0) \cap \partial\Omega$, we have

$$0 \geq \limsup_{\Omega \ni z \to \zeta} u(z) \geq \limsup_{\Omega_{r_0}(z_0) \ni z \to \zeta} u(z) = u(z_0) = \max_\Omega u,$$

and this prove (i).

(ii) By the hypothesis, for every $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that

$$\sup_{\Omega \setminus B(0,R)} u \leq \varepsilon, \qquad \text{for every} \quad R \geq R_\varepsilon. \tag{4.3}$$

We consider the bounded open set $\Omega_R := \Omega \cap B(0, R)$ for $R \geq R_\varepsilon + 1$, and let $\zeta \in \partial\Omega_R$. If $\zeta \in \partial\Omega \cap \overline{B}(0, R)$, it follows from the hypothesis that $\limsup_{\Omega_R \ni z \to \zeta} u(z) \leq \varepsilon$. Otherwise, if $\zeta \in \partial B(0, R) \cap \overline{\Omega}$, recalling (4.3) we obtain

$$\limsup_{\Omega_R \ni z \to \zeta} u(z) = \inf_{V \in \mathcal{U}_\zeta} \sup_{V \cap \Omega_R} u \leq \sup_{(\Omega \setminus B(0,R_\varepsilon)) \cap \Omega_R} u \leq \varepsilon.$$

Now, applying (i) at the function $u - \varepsilon$ on Ω_R, it follows $u \leq \varepsilon$ on Ω so that, letting $\varepsilon \to 0$, (ii) is proved.

We shall prove next proposition by using the properties of the kernel K.

PROPOSITION 4.2. *Let* $u : \mathsf{R}^{N+1} \longrightarrow [-\infty, \infty[$ *be an* u.s.c. *function satisfying the solid sub-mean property. If* u *is finite at* $z_0 = (x_0, t_0)$*, then* $u > -\infty$ *in a dense subset of* $\{(x, t) \in \mathsf{R}^{N+1} \mid t < t_0\}$.

PROOF. Let $z_0 \in \mathsf{R}^{N+1}$ be such that $u(z_0) > -\infty$. By contradiction we assume that $E := \{(x,t) \in \mathsf{R}^{N+1} \mid t < t_0,\ u(x,t) = -\infty\}$ has non-empty interior. Then, there exists $r > 0$ and an open set $\Omega \subseteq E$ such that $\Omega \subseteq \Omega_r(z_0)$. Since $K(z_0,\cdot) > 0$ in a dense open subset of $\{(x,t) \in \mathsf{R}^{N+1} \mid t < t_0\}$ and by the continuity of K, we deduce that there exists an open set $\Omega' \subseteq \Omega$ with $K(z_0,\cdot) > 0$ on Ω'. But this is in contradiction with

$$-\infty < u(z_0) \le M_r(u)(z_0) = \frac{1}{r^{Q-2}} \int_{\Omega_r(z_0)} K(\zeta^{-1} \circ z_0) u(\zeta)\, \mathrm{d}\zeta,$$

and the assertion follows.

5. Some characterizations of $\mathscr{L}$-subharmonic functions

The aim of this section is to give some characterizations of $\mathscr{L}$-subharmonic functions in terms of the averaging operators $\mathscr{M}_r$ and M_r.

For any Radon measure μ in R^{N+1}, we define the *$\mathscr{L}$-potential* Γ_μ of μ by

$$\Gamma_\mu(z) := -\int_{\mathsf{R}^{N+1}} \Gamma(\zeta^{-1} \circ z)\, \mathrm{d}\mu(\zeta), \qquad z \in \mathsf{R}^{N+1}.$$

If $\Gamma_\mu > -\infty$ in a dense subset of R^{N+1}, using [9, Theorem 2.7-(vi)] we easily get

$$\mathscr{L}\Gamma_\mu = \mu, \qquad \text{in the weak sense of distributions.}$$

An application for Fubini's theorem shows that Γ_μ is $\mathscr{L}$-subharmonic in R^{N+1}. Moreover, we have $\Gamma_\mu \in \mathscr{H}^{\mathscr{L}}(\mathsf{R}^{N+1} \setminus \operatorname{supp}\mu)$. Then, Remark 2.2 and the hypoellipticity of $\mathscr{L}$ yield the following theorem.

THEOREM 5.1. *Let $u \in \underline{\mathscr{S}}^{\mathscr{L}}(\Omega)$ and let $\mu = \mathscr{L}u$ be its $\mathscr{L}$-Riesz measure. For every bounded open set $V \subseteq \overline{V} \subseteq \Omega$ there exists $h \in \mathscr{H}^{\mathscr{L}}(V)$ such that, for almost every $z \in V$,*

$$u(z) = -\int_{\overline{V}} \Gamma(\zeta^{-1} \circ z)\, \mathrm{d}\mu(\zeta) + h(z). \tag{5.1}$$

In order to extend formula (3.4) to the class of $\mathscr{L}$-subharmonic functions in R^{N+1}, first we give a weak result holding almost everywhere. For this purpose, we proceed as in [11, Theorem 1.6], by using the inequality (5.1) of [9], Theorem 5.1 and Corollary 3.5.

THEOREM 5.2 (Poisson-Jensen-type formula). *Let $u \in \underline{\mathscr{S}}^{\mathscr{L}}(\Omega)$ and let $\mu = \mathscr{L}u$ be its related $\mathscr{L}$-Riesz measure. For almost every $z \in \Omega$ and $r > 0$*

with $\overline{\Omega}_r(z) \subseteq \Omega$, *we have*

(5.2)

$$u(z) = M_r(u)(z) - \frac{Q-2}{r^{Q-2}} \int_0^r l^{Q-3} \left(\int_{\Omega_l(z)} \left(\Gamma(\zeta^{-1} \circ z) - \frac{1}{l^{Q-2}} \right) \mathrm{d}\mu(\zeta) \right) \mathrm{d}l.$$

We will see later that (5.1) and (5.2) hold for all points of Ω. Now we can state our main characterization of $\mathscr{L}$-subharmonic functions.

THEOREM 5.3. *Let* $u : \Omega \longrightarrow [-\infty, \infty[$ *be an* u.s.c. *function finite in a dense subset of* Ω. *Then, the following statements are equivalent:*

(i) $u \in \underline{\mathscr{S}}^{\mathscr{L}}(\Omega)$*;*

(ii) *u satisfies the surface sub-mean property;*

(iii) *u satisfies the solid sub-mean property.*

PROOF. (i)$\Rightarrow$(iii): If $u \in C(\Omega)$, we get the assertion by formula (5.2) and by $\mu = \mathscr{L}u \geq 0$. If u is just $\mathscr{L}$-subharmonic, the claim follows from a standard approximation argument.

(i)$\Rightarrow$(ii): Let $z \in \Omega$ be such that (5.2) holds, and let $\overline{\Omega}_r(z) \subseteq \Omega$ for a suitable $r > 0$. As in the proof of [8, Theorem 1.6], we differentiate (5.2):

(5.3)

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}r} M_r(u)(z) = &-\frac{(Q-2)^2}{r^{Q-1}} \int_0^r l^{Q-3} \left(\int_{\Omega_l(z)} \left(\Gamma(\zeta^{-1} \circ z) - l^{2-Q} \right) \mathrm{d}\mu(\zeta) \right) \mathrm{d}l \\ &+ \frac{Q-2}{r} \int_{\Omega_r(z)} \left(\Gamma(\zeta^{-1} \circ z) - r^{2-Q} \right) \mathrm{d}\mu(\zeta). \end{aligned}$$

By Tonelli's theorem,

$$\begin{aligned} &\int_0^r l^{Q-3} \left(\int_{\Omega_l(z)} \left(\Gamma(\zeta^{-1} \circ z) - l^{2-Q} \right) \mathrm{d}\mu(\zeta) \right) \mathrm{d}l \\ &\quad = \int_{\Omega_r(z)} \left(\int_{(\Gamma(\zeta^{-1} \circ z))^{\frac{1}{2-Q}}}^r l^{Q-3} \left(\Gamma(\zeta^{-1} \circ z) - l^{2-Q} \right) \mathrm{d}l \right) \mathrm{d}\mu(\zeta) \\ &\quad = \int_{\Omega_r(z)} \left[\Gamma(\zeta^{-1} \circ z) \frac{l^{Q-2}}{Q-2} - \ln l \right]_{l=(\Gamma(\zeta^{-1} \circ z))^{\frac{1}{2-Q}}}^{l=r} \mathrm{d}\mu(\zeta) \\ &\quad = \frac{1}{Q-2} \left(r^{Q-2} \int_{\Omega_r(z)} \Gamma(\zeta^{-1} \circ z) \, \mathrm{d}\mu(\zeta) \right. \\ &\qquad \left. - \int_{\Omega_r(z)} \mathrm{d}\mu(\zeta) - \int_{\Omega_r(z)} \ln\left(r^{Q-2} \Gamma(\zeta^{-1} \circ z) \right) \mathrm{d}\mu(\zeta) \right). \end{aligned}$$

We insert this result in (5.3) and simplify, obtaining

$$\frac{\mathrm{d}}{\mathrm{d}r}M_r(u)(z) = \frac{Q-2}{r^{Q-1}}\int_{\Omega_r(z)} \ln\bigl(r^{Q-2}\Gamma(\zeta^{-1}\circ z)\bigr)\,\mathrm{d}\mu(\zeta) \geq 0.$$

Hence, for a.e. $z \in \Omega$, the function $r \mapsto M_r(u)(z)$ is monotone non-decreasing. As $z \mapsto M_r(u)(z)$ is continuous, $M_r(u)(z)$ is non-decreasing w.r.t. r for every $z \in \Omega$. On the other hand, by (3.5) we see that $r \mapsto M_r(u)(z)$ is locally absolutely continuous for $r > 0$. Thus,

(5.4)

$$\text{for every } z \in \Omega, \text{ and } r > 0 \text{ with } \overline{\Omega}_r(z) \subseteq \Omega,\ \frac{\mathrm{d}}{\mathrm{d}r}M_r(u)(z) \text{ exists and is} \geq 0.$$

As a consequence, using again (3.5) we get

$$\frac{\mathrm{d}}{\mathrm{d}r}\,M_r(u)(z) = -\frac{(Q-2)^2}{r^{Q-1}}\int_0^r l^{Q-3}\mathcal{M}_l(u)(z)\,\mathrm{d}l + \frac{Q-2}{r}\mathcal{M}_r(u)(z),$$

whence, by (5.4),

$$M_r(u)(z) \leq \mathcal{M}_r(u)(z), \qquad \text{for every} \quad z \in \Omega.$$

The assertion follows from the previous implication (i)⇒(iii).

(ii)⇒(iii): We suppose $u(z) \leq \mathcal{M}_r(u)(z)$ for every $\overline{\Omega}_r(z) \subseteq \Omega$. By a direct integration and (3.5),

$$u(z) \leq \frac{Q-2}{r^{Q-2}}\int_0^r l^{Q-3}\mathcal{M}_l(u)(z)\,\mathrm{d}l = M_r(u)(z).$$

(iii)⇒(i): Let $V \subset \overline{V} \subset \Omega$ be an open $\mathscr{L}$-regular set and $\varphi \in C(\partial V)$ with $\varphi \geq u|_{\partial V}$. Then the function $u - H_\varphi^V$ is u.s.c. and it satisfies the solid sub-mean property. Moreover, $\limsup_{V \ni z \to \zeta \in \partial V}(u - H_\varphi^V)(z) \leq 0$. Thus we can apply Theorem 4.1-(i) and we get $u \leq H_\varphi^V$ on V, so that $u \in \underline{\mathcal{S}}^{\mathscr{L}}(\Omega)$.

We also provide another characterization of $\mathscr{L}$-subharmonicity.

THEOREM 5.4. *Let $u : \Omega \longrightarrow [-\infty, \infty[$ be an* u.s.c. *function finite in a dense subset of Ω. The following statements are equivalent:*

(i) $u \in \underline{\mathcal{S}}^{\mathscr{L}}(\Omega)$;

(ii) *for every $z \in \Omega$, $r \mapsto M_r(u)(z)$ is monotone non-decreasing for $0 < r < \sup\{\rho > 0 \mid \overline{\Omega}_\rho(z) \subseteq \Omega\}$ and*

$$u(z) = \lim_{r \to 0^+} M_r(u)(z). \tag{5.5}$$

PROOF. (i)⇒(ii): The first statement follows from (5.4). However, the proof is analogous to that of [11, Corollary 1.7].

(ii)⇒(i): Since $r \mapsto M_r(u)(z)$ is monotone non-decreasing and from (5.5), we get

$$u(z) \le M_r(u)(z) \qquad \text{if} \quad 0 < r < \sup\{\rho > 0 \mid \overline{\Omega}_\rho(z) \subseteq \Omega\}.$$

Then u satisfies the solid sub-mean property so that, by Theorem 5.3, $u \in \underline{\mathcal{S}}^{\mathcal{L}}(\Omega)$.

As a remarkable consequence of the property (ii) in the previous theorem, we have:

THEOREM 5.5. *Let $u, v \in \underline{\mathcal{S}}^{\mathcal{L}}(\Omega)$. If $u \le v$ almost everywhere in Ω, then $u \le v$ in Ω. Consequently, if $u = v$ at all points where both functions are finite, then $u \equiv v$.*

PROOF. Let $\overline{\Omega}_r(z) \subseteq \Omega$. By integrating the inequality $u \le v$ which holds a.e. in $\Omega_r(z)$, we get $M_r(u)(z) \le M_r(v)(z)$, whence $u(z) = \lim_{r\to 0^+} M_r(u)(z) \le \lim_{r\to 0^+} M_r(v)(z) = v(z)$. The second assertion is a consequence of the first one, recalling that $u, v \in L^1_{\mathrm{loc}}(\Omega)$.

Now we can prove that the statement of Theorem 5.1 holds for every $z \in \Omega$.

THEOREM 5.6 (Riesz's Representation for $\underline{\mathcal{S}}^{\mathcal{L}}(\Omega)$). *Let $u \in \underline{\mathcal{S}}^{\mathcal{L}}(\Omega)$ and let $\mu = \mathcal{L}u$ be the $\mathcal{L}$-Riesz measure related to u. For every bounded open set $V \subseteq \overline{V} \subseteq \Omega$, there exists $h \in \mathcal{H}^{\mathcal{L}}(V)$ such that*

$$u(z) = -\int_{\overline{V}} \Gamma(\zeta^{-1} \circ z)\, \mathrm{d}\mu(\zeta) + h(z), \qquad z \in V. \tag{5.6}$$

Moreover the couple (μ, h) is unique in V.

With Theorem 5.6 in hand, proceeding in the same way we have obtained Theorem 5.2 (see the proof of [11, Theorem 1.6]), we show that the Poisson-Jensen-type formula (5.2) is valid at every point z. For sake of clearness, we state the following

THEOREM 5.7 (Poisson-Jensen's formula). *Let $u \in \underline{\mathcal{S}}^{\mathcal{L}}(\Omega)$ and let $\mu = \mathcal{L}u$ be its related $\mathcal{L}$-Riesz measure. For every $z \in \Omega$ and $r > 0$ with $\overline{\Omega}_r(z) \subseteq \Omega$, we have*

$$u(z) = M_r(u)(z) - \frac{Q-2}{r^{Q-2}} \int_0^r l^{Q-3} \left(\int_{\Omega_l(z)} \left(\Gamma(\zeta^{-1} \circ z) - \frac{1}{l^{Q-2}} \right) \mathrm{d}\mu(\zeta) \right) \mathrm{d}l. \tag{5.7}$$

We end this section with a proposition which says that the least $\mathscr{L}$-harmonic majorant of a $\mathscr{L}$-potential Γ_μ is the zero function.

PROPOSITION 5.8. *Let μ be a Radon measure in R^{N+1}, and let Γ_μ be finite in a dense subset of R^{N+1}. If $h \in \mathscr{H}^{\mathscr{L}}(\mathsf{R}^{N+1})$ is such that $h \le -\Gamma_\mu$, then $h \le 0$ in R^{N+1}. In particular,* $\sup_{\mathsf{R}^{N+1}} \Gamma_\mu = 0$.

PROOF. We consider a sequence $\{K_j\}_j$ of compact sets with $K_j \subseteq K_{j+1}$ and $\bigcup_j K_j = \mathsf{R}^{N+1}$. Since $\mu|_{K_j}$ is a compactly supported Radon measure, $\Gamma_{\mu|_{K_j}}$ is finite a.e. in R^{N+1}, so $\Gamma_{\mu|_{\mathsf{R}^{N+1}\setminus K_j}} = \Gamma_\mu - \Gamma_{\mu|_{K_j}} > -\infty$ in a dense subset of R^{N+1} and it is $\mathscr{L}$-subharmonic. Then,

$$\underline{\mathscr{S}}^{\mathscr{L}}(\mathsf{R}^{N+1}) \ni v(z) := h(z) + \Gamma_{\mu|_{\mathsf{R}^{N+1}\setminus K_j}}(z)$$
$$\le -\Gamma_{\mu|_{K_j}}(z) \le \mu(K_j) \cdot \sup_{\zeta \in K_j} \Gamma(\zeta^{-1} \circ z) \longrightarrow 0$$

as $|z| \to \infty$, by [9, Proposition 2.8-(ii)]. Theorem 4.1-(ii) now gives $v \le 0$ in R^{N+1}, whence

$$h(z) \le \int_{\mathsf{R}^{N+1}} \chi_{\mathsf{R}^{N+1}\setminus K_j}(\zeta)\, \Gamma(\zeta^{-1} \circ z)\, \mathrm{d}\mu(\zeta), \qquad z \in \mathsf{R}^{N+1}. \tag{5.8}$$

For every z in the dense set where Γ_μ is finite, by dominated convergence from (5.8) it follows that $h(z) \le 0$. As h is $\mathscr{L}$-harmonic and so continuous in R^{N+1}, we have $h \le 0$ everywhere.

Finally we show that $m := \inf_{\mathsf{R}^{N+1}}(-\Gamma_\mu) = 0$. Obviously $m \le -\Gamma_\mu$ and the constant function $h \equiv m$ is $\mathscr{L}$-harmonic in R^{N+1}. From the first part of the proof we get $m \le 0$, and the claim is proved.

6. Bounded-above $\mathscr{L}$-subharmonic functions in R^{N+1}

Let $u \in \underline{\mathscr{S}}^{\mathscr{L}}(\mathsf{R}^{N+1})$ be such that $u(z_0) > -\infty$ for a suitable $z_0 \in \mathsf{R}^{N+1}$ and $\mu = \mathscr{L}u$ be its related $\mathscr{L}$-Riesz measure. From the solid sub-mean property of u in z_0 and the Poisson-Jensen formula (5.7), we get $\int_0^r l^{Q-3}\big(\int_{\Omega_l(z_0)}(\Gamma(\zeta^{-1} \circ z_0) - l^{2-Q})\, \mathrm{d}\mu(\zeta)\big)\, \mathrm{d}l < \infty$ for $r > 0$, whence

$$\int_{\Omega_l(z_0)} \big(\Gamma(\zeta^{-1} \circ z_0) - l^{2-Q}\big)\, \mathrm{d}\mu(\zeta) < \infty, \qquad \text{for every} \quad l > 0. \tag{6.1}$$

If u and μ are as above and we set

$$n(z_0, t) := \int_{\Omega_t(z_0)} \mathrm{d}\mu(\zeta),$$

we obtain

$$\int_{\{\zeta\,|\,0<\Gamma(\zeta^{-1}\circ z_0)^{\frac{1}{2-Q}}<l\}} \left(\Gamma(\zeta^{-1}\circ z_0) - l^{2-Q}\right) \mathrm{d}\mu(\zeta) = \int_0^l (t^{2-Q} - l^{2-Q})\,\mathrm{d}n(z_0,t). \tag{6.2}$$

As $\left\{\zeta \in \mathsf{R}^{N+1} \mid 0 < \Gamma(\zeta^{-1}\circ z_0)^{\frac{1}{2-Q}} < l\right\} = \Omega_l(z_0)$, by (6.2) and (6.1),

$$\int_0^l (t^{2-Q} - l^{2-Q})\,\mathrm{d}n(z_0,t) < \infty, \qquad \text{for every} \quad l > 0. \tag{6.3}$$

Now, integrating by parts,

$$\begin{aligned} &\int_0^l (t^{2-Q} - l^{2-Q})\,\mathrm{d}n(z_0,t) \\ &\qquad = \lim_{\varepsilon\to 0^+}\left(-(\varepsilon^{2-Q} - l^{2-Q})n(z_0,\varepsilon) - \int_\varepsilon^l (2-Q)t^{1-Q}n(z_0,t)\,\mathrm{d}t\right) \\ &\qquad = (Q-2)\int_0^l \frac{n(z_0,t)}{t^{Q-1}}\,\mathrm{d}t. \end{aligned} \tag{6.4}$$

Indeed, by dominated convergence we have

$$n(z_0,0^+) := \lim_{t\to 0^+} n(z_0,t) = \int_{\mathsf{R}^{N+1}} \lim_{t\to 0^+} \chi_{\Omega_t(z_0)}(\zeta)\,\mathrm{d}\mu(\zeta) = 0,$$

so that

$$\begin{aligned} (\varepsilon^{2-Q} - l^{2-Q})\,n(z_0,\varepsilon) &= (\varepsilon^{2-Q} - l^{2-Q})\,(n(z_0,\varepsilon) - n(z_0,0^+)) \\ &= (\varepsilon^{2-Q} - l^{2-Q}) \lim_{t\to 0^+}\int_t^\varepsilon \mathrm{d}n(z_0,t) \\ &= \int_0^\varepsilon (\varepsilon^{2-Q} - l^{2-Q})\,\mathrm{d}n(z_0,t) \\ &\le \int_0^\varepsilon (t^{2-Q} - l^{2-Q})\,\mathrm{d}n(z_0,t) \longrightarrow 0 \quad \text{as } \varepsilon \to 0^+, \end{aligned}$$

where in the last limit we have used (6.3). Then, using (6.2) and (6.4) in the last term of (5.7), we obtain

$$\begin{aligned} \frac{Q-2}{r^{Q-2}}\int_0^r l^{Q-3}\left(\int_{\Omega_l(z_0)}\left(\Gamma(\zeta^{-1}\circ z_0) - \frac{1}{l^{Q-2}}\right)\mathrm{d}\mu(\zeta)\right)\mathrm{d}l \\ = \frac{(Q-2)^2}{r^{Q-2}}\int_0^r l^{Q-3}\left(\int_0^l \frac{n(z_0,t)}{t^{Q-1}}\,\mathrm{d}t\right)\mathrm{d}l. \end{aligned} \tag{6.5}$$

Now, replacing (6.5) in Poisson-Jensen's formula (5.7) we immediately get the following representation formula for $\mathscr{L}$-subharmonic functions in R^{N+1}.

THEOREM 6.1. *Let $u \in \underline{\mathscr{S}}^{\mathscr{L}}(\mathsf{R}^{N+1})$ be such that $u(z_0) > -\infty$ for a suitable $z_0 \in \mathsf{R}^{N+1}$ and $\mu = \mathscr{L}u$ be its related $\mathscr{L}$-Riesz measure. Then, for every $R > 0$, we have*

$$u(z_0) = M_R(u)(z_0) - (Q-2)^2 \int_0^1 \tau^{Q-3}\left(\int_0^{R\tau} \frac{n(z_0,t)}{t^{Q-1}}\,\mathrm{d}t\right)\mathrm{d}\tau. \tag{6.6}$$

Now we are ready to prove our main result.

THEOREM 6.2. *Let μ be a Radon measure in R^{N+1} and let $n(z,t)$ be defined as follows*

$$n(z,t) := \int_{\Omega_t(z)} \mathrm{d}\mu(\zeta), \qquad z \in \mathsf{R}^{N+1}.$$

Then, a necessary and sufficient condition for μ to be the $\mathscr{L}$-Riesz measure related to a bounded-above $\mathscr{L}$-subharmonic function u in R^{N+1} is that the following condition holds

$$\int_1^{\infty} \frac{n(z,t)}{t^{Q-1}}\,\mathrm{d}t < \infty, \tag{6.7}$$

for every z in a dense subset of R^{N+1}. If this condition is satisfied, then there exists $h \in \mathscr{H}^{\mathscr{L}}(\mathsf{R}^{N+1})$, $h \le 0$, such that

$$u(z) = U - \int_{\mathsf{R}^{N+1}} \Gamma(\zeta^{-1} \circ z)\,\mathrm{d}\mu(\zeta) + h(z), \qquad z \in \mathsf{R}^{N+1}, \tag{6.8}$$

where $U < \infty$ is the least upper bound of u.

PROOF. We prove the first statement of the theorem, beginning with the necessity part. Let $u \in \underline{\mathscr{S}}^{\mathscr{L}}(\mathsf{R}^{N+1})$ be such that $\sup_{\mathsf{R}^{N+1}} u = U < \infty$, and we choose $z_0 \in \mathsf{R}^{N+1}$ satisfying $u(z_0) > -\infty$. If we define $u^+ := \max\{u, 0\}$ and $u^- := \max\{-u, 0\}$, then for every $R > 0$ the following inequality holds

$$M_R(u^+)(z_0) \le \max\{U, 0\} \cdot M_R(1)(z_0) = \max\{U, 0\}.$$

From representation formula (6.6) we obtain

$$\begin{aligned}(Q-2)^2 \int_0^1 \tau^{Q-3}\left(\int_\tau^{R\tau} \frac{n(z_0,t)}{t^{Q-1}}\,\mathrm{d}t\right)\mathrm{d}\tau &= M_R(u^+)(z_0) - M_1(u^+)(z_0)\\ &\quad + M_1(u^-)(z_0) - M_R(u^-)(z_0) \le \max\{U, 0\} + M_1(u^-)(z_0),\end{aligned}$$

so that, by Beppo Levi's theorem,

$$(Q-2)^2 \int_0^1 \tau^{Q-3}\left(\int_\tau^\infty \frac{n(z_0,t)}{t^{Q-1}}\,\mathrm{d}t\right)\mathrm{d}\tau < \infty.$$

Hence, we get $\int_1^\infty t^{1-Q}\, n(z_0,t)\,\mathrm{d}t < \infty$ and, as the $\mathscr{L}$-subharmonic function u is finite in a dense subset of R^{N+1}, we obtain also (6.7).

Let us now prove the sufficiency part. Let μ be a Radon measure on R^{N+1} satisfying (6.7) and consider the function

$$u(z) := -\int_{\mathsf{R}^{N+1}} \Gamma(\zeta^{-1}\circ z)\,\mathrm{d}\mu(\zeta) = \Gamma_\mu(z), \qquad z \in \mathsf{R}^{N+1}.$$

It is enough to prove that $u \in \underline{\mathscr{S}}^{\mathscr{L}}(\mathsf{R}^{N+1})$, $\mathscr{L}u = \mu$ in R^{N+1} and $\sup_{\mathsf{R}^{N+1}} u = 0$. If we show that u is finite in a dense subset of R^{N+1}, then the first two statements immediately follow from what we have seen at the beginning of Section 5, and Proposition 5.8 yields $\sup_{\mathsf{R}^{N+1}} u = 0$.

We consider $z_0 = (x_0, t_0)$ satisfying (6.7). For every fixed $R > 0$, we split u as follows

$$\begin{aligned} u(z) &= -\int_{\{\zeta \mid \Gamma(\zeta^{-1}\circ z_0)^{\frac{1}{2-Q}} \le R\}} \Gamma(\zeta^{-1}\circ z)\,\mathrm{d}\mu(\zeta) \\ &\quad - \int_{\{\zeta \mid R < \Gamma(\zeta^{-1}\circ z_0)^{\frac{1}{2-Q}} < +\infty\}} \Gamma(\zeta^{-1}\circ z)\,\mathrm{d}\mu(\zeta) \\ &\quad - \int_{\{\zeta \mid \Gamma(\zeta^{-1}\circ z_0) = 0\}} \Gamma(\zeta^{-1}\circ z)\,\mathrm{d}\mu(\zeta) \\ &=: u_1^{R,z_0}(z) + u_2^{R,z_0}(z) + u_3^{z_0}(z). \end{aligned}$$

The function u_1^{R,z_0} is $\mathscr{L}$-subharmonic in R^{N+1} and the same property holds for

$$u_\lambda^{R,z_0}(z) := -\int_{\{\zeta \mid R < \Gamma(\zeta^{-1}\circ z_0)^{\frac{1}{2-Q}} < \lambda\}} \Gamma(\zeta^{-1}\circ z)\,\mathrm{d}\mu(\zeta), \qquad \lambda > R.$$

We have $u_\lambda^{R,z_0} \downarrow u_2^{R,z_0}$ as $\lambda \uparrow \infty$, hence u_2^{R,z_0} is a u.s.c. function. Since u_λ^{R,z_0} satisfies the solid sub-mean property and using Beppo Levi's theorem, we get

$$u_2^{R,z_0}(z) \le M_r(u_2^{R,z_0})(z), \qquad \text{for every } r > 0 \text{ and } z \in \mathsf{R}^{N+1}.$$

Moreover

$$
\begin{aligned}
u_\lambda^{R,z_0}(z_0) &= -\int_R^\lambda t^{2-Q}\, \mathrm{d}n(z_0,t) \\
&= -\lambda^{2-Q} n(z_0,\lambda) + R^{2-Q} n(z_0,R) - (Q-2)\int_R^\lambda \frac{n(z_0,t)}{t^{Q-1}}\,\mathrm{d}t \\
&\geq -(Q-2)\int_R^\infty \frac{n(z_0,t)}{t^{Q-1}}\,\mathrm{d}t.
\end{aligned}
$$

Indeed, as the function $n(z_0,\cdot)$ is non-decreasing we have

$$
\lambda^{2-Q}\, n(z_0,\lambda) \leq (Q-2)\int_\lambda^\infty \frac{n(z_0,t)}{t^{Q-1}}\,\mathrm{d}t.
$$

Then, by (6.7), $u_2^{R,z_0}(z_0) = \lim_{\lambda\to\infty} u_\lambda^{R,z_0}(z_0) > -\infty$, so that, by Proposition 4.2, $u_2^{R,z_0} > -\infty$ in a dense subset of $\{(x,t) \in \mathsf{R}^{N+1} \mid t < t_0\}$. On the other hand, since Γ is supported in a half space, we have

$$
u_3^{z_0}(x,t) = 0 \qquad \text{in} \quad \mathsf{R}^N \times]-\infty, t_0].
$$

Thus the function u is finite in a dense subset of $\{(x,t) \in \mathsf{R}^{N+1} \mid t < t_0\}$, so that, by hypothesis (6.7), $u > -\infty$ in a dense subset of R^{N+1}. The first assertion of the theorem is so proved.

Now, let us consider a function $v \in \underline{\mathcal{S}}^{\mathcal{L}}(\mathsf{R}^{N+1})$ such that $\sup_{\mathsf{R}^{N+1}} v = U < \infty$. Let $\mu = \mathcal{L}v$ be its related $\mathcal{L}$-Riesz measure. By the first part of the theorem, also the function

$$
u(z) := U + \Gamma_\mu(z), \qquad z \in \mathsf{R}^{N+1}
$$

is $\mathcal{L}$-subharmonic in R^{N+1} with least upper bound U and related $\mathcal{L}$-Riesz measure μ. Then in the weak sense of distributions we have $\mathcal{L}(v-u) = 0$. Since $\mathcal{L}$ is hypoelliptic, there exists a function h, $\mathcal{L}$-harmonic in R^{N+1}, such that $h = v - u$ almost everywhere in R^{N+1}. So, we get $h \leq -\Gamma_\mu$ a.e. in R^{N+1}, hence everywhere as a consequence of Theorem 5.5. Now Proposition 5.8 yields $h \leq 0$ in R^{N+1}. This completes the proof of the theorem.

REMARK 6.3. We note that the hypotheses $\mathcal{L}h = 0$ in R^{N+1} and $h \leq 0$ in the previous Theorem 6.2 do not imply that h is a constant function, unlike the case of sub-Laplacians on Carnot groups. Indeed, for example, the function

$$
u(x,t) = -\exp(x_1 + \cdots + x_N + Nt), \qquad x \in \mathsf{R}^N,\ t \in \mathsf{R},
$$

is non positive, non constant and satisfies the classical heat equation $\sum_{j=1}^N \partial_{x_j}^2 u - \partial_t u = 0$ in R^{N+1}.

If we suppose for $|u|$ a suitable growth condition that enable us to get a Liouville-type theorem for $\mathscr{L}$ (see [10]), we obtain a global representation formula exactly analogous to (7.7) of [4].

COROLLARY 6.4. *Let u be a bounded-above $\mathscr{L}$-subharmonic function in R^{N+1}, and $\mu = \mathscr{L}u$ be its related $\mathscr{L}$-Riesz measure. If we suppose*

$$|u(0,t)| = O(t^m) \qquad \text{as} \quad t \longrightarrow \infty$$

for some $m \geq 0$, then

$$u(z) = U - \int_{\mathsf{R}^{N+1}} \Gamma(\zeta^{-1} \circ z)\, \mathrm{d}\mu(\zeta), \qquad z \in \mathsf{R}^{N+1},$$

where $U < \infty$ is the least upper bound of u.

PROOF. From (6.8) of Theorem 6.2 it follows that

$$u(z) = U + \Gamma_\mu(z) + h(z), \qquad z \in \mathsf{R}^{N+1},$$

where $h \leq 0$ is $\mathscr{L}$-harmonic in R^{N+1}. By Proposition 5.8 we have $\sup_{\mathsf{R}^{N+1}} \Gamma_\mu = 0$, so that $-h \leq U - u$ in R^{N+1}. In particular,

$$0 \leq -h(0,t) \leq U - u(0,t) = O(t^m) \qquad \text{as} \quad t \longrightarrow \infty.$$

Then, by [10, Theorem 1.1], $h =$ const. in R^{N+1}. But, since

$$\sup_{\mathsf{R}^{N+1}} (U + \Gamma_\mu + h) = \sup_{\mathsf{R}^{N+1}} u = U = \sup_{\mathsf{R}^{N+1}} (U + \Gamma_\mu),$$

we have $h \equiv 0$, and the assertion follows.

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