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THE CONNECTION BETWEEN THE CEGRELL CLASSES AND COMPLIANT FUNCTIONS

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Abstract

In this article the connection between the Cegrell classes and compliant functions is studied. A suitable norm is constructed which makes the compliant functions into a Banach space. As an application a characterization of the Dirichlet problem for pluriharmonic functions is achieved. Explicit examples of non-compliant functions will be constructed and a sufficient condition for compliance will be proved.

1. Introduction

Throughout this article let $\Omega \subseteq \mathbf{C}^n$ be a hyperconvex domain, i.e., a connected, open set that admits a negative plurisubharmonic exhaustion function. Furthermore it is assumed that Ω is bounded. Recall that the Perron-Bremermann envelope for a given function $f : \partial\Omega \to \mathbf{R}$ is defined by

$$(1)\quad PB_f(z) = \sup\left\{ w(z) : w \in \mathcal{PSH}(\Omega), \limsup_{\substack{\zeta\to\xi \\ \zeta\in\Omega}} w(\zeta) \leq f(\xi) \forall\xi \in \partial\Omega \right\},$$

where $\mathcal{PSH}(\Omega)$ is the class of all plurisubharmonic functions defined on Ω. If $f : \partial\Omega \to \mathbf{R}$ is a continuous function, then $PB_f \in \mathcal{PSH}(\Omega)$ since a hyperconvex domain viewed as a set in $\mathbf{R}^{2n}$ is regular to the Laplace operator. Consider the following two assertions:

$$\text{P1:}\quad \lim_{\substack{z\to\xi \\ z\in\Omega}} \big(PB_f + PB_{-f}\big)(z) = 0 \qquad \text{for every} \quad \xi \in \partial\Omega,$$

$$\text{P2:}\quad \int_\Omega (dd^c(PB_f + PB_{-f}))^n < +\infty,$$

where $(dd^c\,\cdot)^n$ is the complex Monge-Ampère operator. A continuous function $f : \partial\Omega \to \mathbf{R}$ which satisfies P1 and P2 is called a *compliant function*. The compliant functions first arose in [3] when some of the Cegrell classes

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given by a continuous function were introduced. Let $\mathcal{CP}(\partial\Omega)$ denote the class of compliant functions defined on $\partial\Omega$. If $n = 1$, then the set of compliant functions and continuous functions defined on $\partial\Omega$ coincide, therefore it will be assumed that $n \geq 2$ throughout this article. The special case when Ω is a hyperconvex product domain $\Omega = \Omega_{n_1} \times \cdots \times \Omega_{n_k}$, $n = n_1 + \cdots + n_k$, $k \geq 3$, was studied in [7].

The Cegrell classes were first introduced in [3]. In [4] and [15] new classes were added to the Cegrell family. Let $\mathcal{E}_0(f)$ be the class of plurisubharmonic functions defined in [3] (see also Definiton 2.1). Example 2.4 shows that there exists a function $u \in \mathcal{E}_0(f)$ such that

$$\int_\Omega (dd^c u)^n = +\infty.$$

This cannot occur if f is compliant (Lemma 2.5). Let $u \in \mathcal{E}_0(f)$. Then f is compliant if, and only if, $(u + PB_{-f}) \in \mathcal{E}_0(0)$. A more thorough study about this property in the different Cegrell classes will be made in Section 2. By using a convexity property of $\mathcal{E}_0(f)$ it is proved that a continuous function $f : \partial\Omega \to \mathbb{R}$ is compliant if, and only if, $\mathcal{E}_0(f) \oplus \mathcal{E}_0(-f) \subseteq \mathcal{E}_0(0)$, where $\oplus$ is the sum of two sets (Theorem 2.7).

Let $\|\cdot\| : \mathcal{CP}(\partial\Omega) \to \mathbb{R}$ be defined by

$$\|f\| = \|f\|_\infty + \left(\int_\Omega (dd^c(PB_f + PB_{-f}))^n\right)^{\frac{1}{n}}, \tag{2}$$

where $\|f\|_\infty = \sup\{|f(\xi)| : \xi \in \partial\Omega\}$. The aim of Section 3 is to prove that $(\mathcal{CP}(\partial\Omega), \|\cdot\|)$ is a Banach space (Theorem 3.3). Let $\mathcal{PH}(\partial\Omega)$ denote those continuous functions $\partial\Omega \to \mathbb{R}$ which can be extended to a pluriharmonic function in Ω. A considerable amount of results concerning the Dirichlet problem for pluriharmonic functions exist, see e.g. [1], [2], [7], [8], [9], [10], [11] and the references therein. As an application of Theorem 3.3 it is proved that $\mathcal{PH}(\partial\Omega)$ is equivalent to the closed subspace, $\mathcal{CP}_0(\partial\Omega)$, of $\mathcal{CP}(\partial\Omega)$ that contains functions for which $-PB_f = PB_{-f}$ on Ω (Theorem 3.5).

Example 4.1 shows that it is not enough to assume that f is a C^1-function to ensure that P2 is true, even if Ω is the unit ball. Example 4.2 shows that even if P2 is true and $f \in C^\infty(\partial\Omega)$, it may happen that P1 is false. If Ω is a strictly pseudoconvex domain in $\mathbb{C}^n$ with C^2-boundary and $f : \partial\Omega \to \mathbb{R}$ is a C^2-function, then f is a compliant function (Proposition 4.3).

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2. The connection between the Cegrell classes and compliant functions

In this section assume that $f : \partial\Omega \to \mathsf{R}$ is a continuous function such that

$$\lim_{z\to\xi} PB_f(z) = f(\xi), \tag{3}$$

for every $\xi \in \partial\Omega$, hence $PB_f \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega})$ by Walsh's theorem (see [13]). A bounded plurisubharmonic function φ defined on Ω belongs to $\mathcal{E}_0$ $(= \mathcal{E}_0(\Omega))$ if $\lim_{z\to\xi} \varphi(z) = 0$ for every $\xi \in \partial\Omega$ and

$$\int_\Omega (dd^c\varphi)^n < +\infty.$$

The class $\mathcal{E}_0$ has a role similar to that of the test functions, $C_0^\infty(\Omega)$, in the theory of distributions. Before the connection between the Cegrell classes and compliant functions will be discussed the definition of the Cegrell classes given by a continuous function will be stated. For the definition of $\mathcal{F}_p$ and $\mathcal{E}_p$ see [3] and for $\mathcal{F}$, $\mathcal{E}$ see [4]. For further information about the Cegrell classes see e.g. [6] and the references therein.

Definition 2.1. Let $\mathcal{K} \in \{\mathcal{E}_0, \mathcal{F}_p, \mathcal{E}_p, \mathcal{F}, \mathcal{E}\}$ and $f : \partial\Omega \to \mathsf{R}$ a continuous function that satisfies (3). A plurisubharmonic function u defined on Ω belongs to $\mathcal{K}(f)$ $(= \mathcal{K}(\Omega, f))$ if there exists a function $\varphi \in \mathcal{K}$ such that

$$PB_f \geq u \geq \varphi + PB_f.$$

Remark. Let $\mathcal{K} \in \{\mathcal{E}_0, \mathcal{F}_p, \mathcal{E}_p, \mathcal{F}, \mathcal{E}\}$, then $\mathcal{K}(0) = \mathcal{K}$.

Proposition 2.2. *Let $\mathcal{K} \in \{\mathcal{E}_0, \mathcal{F}_p, \mathcal{E}_p, \mathcal{F}\}$ and let $u \in \mathcal{K}(f)$. If f is a compliant function, then $(u + PB_{-f}) \in \mathcal{K}(0)$.*

Proof. Let $\mathcal{K} \in \{\mathcal{E}_0, \mathcal{F}_p, \mathcal{E}_p, \mathcal{F}\}$ and $u \in \mathcal{K}(f)$. Definition 2.1 implies that $u \in \mathcal{PSH}(\Omega)$ and that there exists a function $\varphi \in \mathcal{K}$ such that $PB_f \geq u \geq \varphi + PB_f$, hence

$$0 \geq PB_f + PB_{-f} \geq u + PB_{-f} \geq \varphi + PB_f + PB_{-f}\,. \tag{4}$$

The function $(\varphi + PB_f + PB_{-f})$ belongs to $\mathcal{K}$, since $\mathcal{K}$ is a convex cone and f is compliant. By (4) it follows that $(u + PB_{-f}) \in \mathcal{K}(0)$.

Remark. The converse statement of Proposition 2.2 is true for $\mathcal{E}_0(f)$, i.e., if $u \in \mathcal{E}_0(f)$ and $(u + PB_{-f}) \in \mathcal{E}_0(0)$, then f is a compliant function. Let $\mathcal{K} \in \{\mathcal{F}_p, \mathcal{E}_p, \mathcal{F}\}$ and $u \in \mathcal{K}$. If $(u + PB_{-f}) \in \mathcal{K}(0)$, then $(PB_f + PB_{-f}) \in \mathcal{K}(0)$. But $(PB_f + PB_{-f}) \in \mathcal{K}(0)$ is generally not a sufficient condition for f to be compliant.

Proposition 2.2 yields an easy method to transform questions about the Cegrell classes given by a continuous function to the classes with zero boundary values. The classes with zero boundary values is much easier to handle and therefore the question of finding a natural characterization of the compliant functions is of importance. If $u \in \mathcal{E}(f)$, then $(u + PB_{-f})$ always belongs to $\mathcal{E}$ without the assumption that f is compliant. Example 2.4 is a slightly modified version of Example 5.6 in [5] and it shows that there exists a function $u \in \mathcal{E}_0(f)$, $f \in C^\infty(\partial\Omega)$, such that the total mass of $(dd^c u)^n$ is infinite. This cannot occur if f is a compliant function (see Lemma 2.5).

LEMMA 2.3. *If $f, g \in \mathcal{F}$, then*

$$\left(\int_\Omega (dd^c(u+v))^n\right)^{\frac{1}{n}} \leq \left(\int_\Omega (dd^c u)^n\right)^{\frac{1}{n}} + \left(\int_\Omega (dd^c v)^n\right)^{\frac{1}{n}}.$$

PROOF. See Lemma 2.5 in [5].

EXAMPLE 2.4. Let P be the unit polydisc in $\mathbb{C}^2$, i.e., $P = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\}$. Let $f : \partial P \to \mathbb{R}$ be defined by $f(z_1, z_2) = |z_2|^2$. The function f is not compliant, $f \in C^\infty(\partial P)$ and $PB_f(z_1, z_2) = |z_2|^2$. For each $j \in \mathbb{N}$ define the function $\varphi_j : P \to \mathbb{R}$ by $\varphi_j(z) = \varphi_j(z_1, z_2) = \max(a_j \log|z_1|, b_j \log|z_2|, c_j)$, where $a_j, b_j, c_j \in \mathbb{R}$, $a_j, b_j > 0$ and $c_j < 0$. Then $\varphi_j \in \mathcal{PSH}(P) \cap C(\bar{P})$, $\lim_{(z_1,z_2)\to(\xi_1,\xi_2)} \varphi_j(z_1, z_2) = 0$ for every $(\xi_1, \xi_2) \in \partial P$ and

$$\int_P (dd^c\varphi_j)^2 = (2\pi)^2 a_j b_j < +\infty, \tag{5}$$

hence $\varphi_j \in \mathcal{E}_0$. Let $v_k : P \to \mathbb{R}$ be defined by $v_k = \sum_{j=1}^k \varphi_j$. From this definition it follows that $v_k \in \mathcal{E}_0$ and that $[v_k]$ is a decreasing sequence on P. Lemma 2.3 and (5) yields that

$$\int_P (dd^c v_k)^2 \leq \left(\sum_{j=1}^k \left(\int_P (dd^c\varphi_j)^2\right)^{\frac{1}{2}}\right)^2 \leq (2\pi)^2 \left(\sum_{j=1}^k (a_j b_j)^{\frac{1}{2}}\right)^2. \tag{6}$$

Assume that

$$\sum_{j=1}^\infty (a_j b_j)^{\frac{1}{2}} < +\infty \qquad \text{and} \qquad \sum_{j=1}^\infty c_j > -\infty \tag{7}$$

and let $v(z) = \lim_{k\to\infty} v_k(z)$. The construction of the function v implies that $\lim_{(z_1,z_2)\to(\xi_1,\xi_2)} v(z_1, z_2) = 0$ for every $(\xi_1, \xi_2) \in \partial P$. The assumptions in (7) imply that $v \in \mathcal{PSH}(P) \cap L^\infty(P)$ and by inequality (6) it follows that $v \in \mathcal{E}_0$. Let $u : P \to \mathbb{R}$ be defined by $u = v + PB_f$, hence $u = (v + |z_2|^2) \in \mathcal{E}_0(f)$. Then it follows that

$$\begin{aligned}
\int_P (dd^c(v_k + |z_2|^2))^2 &= \int_P (dd^c v_k)^2 + 4i \int_P (dd^c v_k) \wedge dz_2 \wedge d\bar{z}_2 \\
&= \int_P (dd^c v_k)^2 + 32 \int_P \frac{\partial^2 v_k}{\partial z_1 \partial \bar{z}_1}\, dV((z_1, z_2)) \\
&= \int_P (dd^c v_k)^2 + 32 \int_P \sum_{j=1}^{k} \frac{\partial^2 \varphi_j}{\partial z_1 \partial \bar{z}_1}\, dV((z_1, z_2)) \\
&\geq 32 \sum_{j=1}^{k} \int_P \frac{\partial^2 \varphi_j}{\partial z_1 \partial \bar{z}_1}\, dV((z_1, z_2)), \qquad (8)
\end{aligned}$$

where V is the Lebesgue measure on $\mathbb{C}^2$. Let $\varepsilon > 0$ be given such that $0 < \varepsilon < 1$ and let $D(0, r) = \{z \in \mathbb{C} : |z| < r\}$. Choose $\chi_1, \chi_2 \in C_0^\infty(D(0, 1))$ such that $0 \leq \chi_1, \chi_2 \leq 1$ and $\chi_1 = 1 = \chi_2$ on $D(0, 1 - \varepsilon)$. For fixed $|z_2| \leq \min\big(1 - \varepsilon, (1 - \varepsilon)^{\frac{a_j}{b_j}}\big)$ fix, it follows that

$$\int_{D(0,1)} \chi_1(z_1) \frac{\partial^2 \varphi_j}{\partial z_1 \partial \bar{z}_1}\, dV(z_1) = 8\pi a_j. \qquad (9)$$

Under the assumption that $a_j \geq b_j$ inequality (8) together with (9) yield that

$$\begin{aligned}
\int_P (dd^c(v_k + |z_2|^2))^2 &\geq 32 \sum_{j=1}^{k} \int_P (\chi_1(z_1)\chi_2(z_2)) \frac{\partial^2 \varphi_j}{\partial z_1 \partial \bar{z}_1}\, dV((z_1, z_2)) \\
&\geq c \sum_{j=1}^{k} a_j \left(\min\left(1 - \varepsilon, (1 - \varepsilon)^{\frac{a_j}{b_j}}\right)\right)^2, \qquad (10)
\end{aligned}$$

where $c > 0$ is a constant. Let $\varepsilon \to 0^+$, then (10) implies that

$$\int_P (dd^c(v_k + |z_2|^2))^2 \geq c \sum_{j=1}^{k} a_j.$$

Thus

$$\int_P (dd^c u)^2 = \lim_{k \to +\infty} \int_P (dd^c(v_k + |z_2|^2))^2 \geq c \sum_{j=1}^{\infty} a_j. \qquad (11)$$

Let the sequences $[a_j]$, $[b_j]$ and $[c_j]$ be defined by $a_j = 1/j$, $b_j = 1/j^3$ and $c_j = -1/j^2$. Thus the assumptions (7) and $a_j \geq b_j$ are satisfied, which implies that the function defined on P by

$$u(z_1, z_2) = \sum_{j=1}^{\infty} \max\left(\frac{1}{j}\log|z_1|, \frac{1}{j^3}\log|z_2|, -\frac{1}{j^2}\right) + |z_2|^2,$$

belongs to $\mathscr{E}_0(f)$ and $\int_P (dd^c u)^2 = +\infty$, by (11).

LEMMA 2.5. *If $f \in \mathscr{C}\mathscr{P}(\partial\Omega)$, then $\mathscr{F}(f) = \{u \in \mathscr{F}(f) : \int_\Omega (dd^c u)^n < +\infty\}$.*

PROOF. Let $u \in \mathscr{F}(f)$, i.e., $u \in \mathscr{PSH}(\Omega)$ and there exists a function $\varphi \in \mathscr{F}$ such that $PB_f \geq u \geq \varphi + PB_f$. Theorem 2.1 in [4] implies that there exists a decreasing sequence $[\varphi_j]$, $\varphi_j \in \mathscr{E}_0$, that converges pointwise to φ as $j \to +\infty$. Let the sequence $[u_j]$, $j \in \mathsf{N}$, be defined by $u_j = \max\big(u, \varphi_j + PB_f\big)$. Then the decreasing sequence $[u_j]$, $u_j \in \mathscr{E}_0(f)$, converges pointwise to u as $j \to +\infty$ and

$$(12)\quad \begin{aligned}(dd^c(\varphi_j + PB_f + PB_{-f}))^n &\geq (dd^c(\varphi_j + PB_f))^n + (dd^c PB_{-f})^n \\ &= (dd^c(\varphi_j + PB_f))^n,\end{aligned}$$

since $(PB_f + PB_{-f}) \in \mathscr{E}_0(0)$. The sequence $[(\varphi_j + PB_f + PB_{-f})]$ is decreasing and converges pointwise to $(\varphi + PB_f + PB_{-f}) \in \mathscr{F}$ as $j \to +\infty$. From Proposition 5.1 in [4] it follows that

$$\lim_{j\to+\infty} \int_\Omega (dd^c(\varphi_j + PB_f + PB_{-f}))^n = \int_\Omega (dd^c(\varphi + PB_f + PB_{-f}))^n,$$

hence

$$(13)\qquad \sup_j \int_\Omega (dd^c u_j)^n < +\infty,$$

by (12). Let $[\phi_s]$, $\phi_s \in C_0^\infty(\Omega)$, $\phi_s \geq 0$, be an increasing sequence which converges pointwise to 1 on Ω as $s \to +\infty$. For $s \in \mathsf{N}$ fixed, it follows from (13) that

$$(14)\qquad \int_\Omega \phi_s (dd^c u)^n = \lim_{j\to+\infty} \int_\Omega \phi_s (dd^c u_j)^n \leq \sup_j \int_\Omega (dd^c u_j)^n < +\infty.$$

Let $s \to +\infty$, this lemma then follows from (14) and the monotone convergence theorem.

LEMMA 2.6. *Assume that $f, g : \partial\Omega \to \mathbb{R}$ are continuous functions such that*

$$\lim_{\substack{z\to\xi \\ z\in\Omega}} PB_f(z) = f(\xi) \qquad \textit{and} \qquad \lim_{\substack{z\to\xi \\ z\in\Omega}} PB_g(z) = g(\xi),$$

for every $\xi \in \partial\Omega$ and in addition assume that f is a compliant function. Let $\mathcal{K} \in \{\mathcal{E}_0, \mathcal{F}_p, \mathcal{E}_p, \mathcal{F}\}$. If $u \in \mathcal{K}(f)$ and $v \in \mathcal{K}(g)$, then $(\alpha u + \beta v) \in \mathcal{K}(\alpha f + \beta g)$, where $\alpha, \beta \in \mathbb{R}$, $\alpha, \beta \geq 0$.

PROOF. Let $\mathcal{K} \in \{\mathcal{E}_0, \mathcal{F}_p, \mathcal{E}_p, \mathcal{F}\}$, $u \in \mathcal{K}(f)$ and $v \in \mathcal{K}(g)$. Definition 2.1 implies that $u, v \in \mathcal{PSH}(\Omega)$ and that there exist functions $\varphi, \psi \in \mathcal{K}$ such that $PB_f \geq u \geq \varphi + PB_f$ and $PB_g \geq v \geq \psi + PB_g$. The definition of the Perron-Bremermann envelope yields that $PB_{\beta g} \geq PB_{\alpha f+\beta g} + PB_{-\alpha f}$ and therefore it follows that

$$\begin{aligned} PB_{\alpha f+\beta g} \geq \alpha u + \beta v &\geq \alpha\varphi + \beta\psi + PB_{\alpha f} + PB_{\beta g} \\ &\geq \alpha\varphi + \beta\psi + PB_{\alpha f} + PB_{\alpha f+\beta g} + PB_{-\alpha f} \\ &= \alpha\varphi + \beta\psi + \alpha(PB_f + PB_{-f}) + PB_{\alpha f+\beta g}. \end{aligned}$$

Thus $(\alpha u + \beta v) \in \mathcal{K}(\alpha f + \beta g)$, since $(\alpha\varphi + \beta\psi + \alpha(PB_f + PB_{-f})) \in \mathcal{K}$.

THEOREM 2.7. *Let $\Omega \subseteq \mathbb{C}^n$ be a bounded hyperconvex domain and let $f : \Omega \to \mathbb{R}$ be a continuous function. The function f is compliant if, and only if,*

$$\mathcal{E}_0(f) \oplus \mathcal{E}_0(-f) \subseteq \mathcal{E}_0(0), \tag{15}$$

where $\oplus$ is the sum of two sets. Moreover, equality holds in (15) *if, and only if, PB_f is pluriharmonic on Ω and continuous on $\bar{\Omega}$.*

PROOF. Assume that f is a compliant, hence $-f$ is compliant. Let $u \in \mathcal{E}_0(f)$ and $v \in \mathcal{E}_0(-f)$. Lemma 2.6 implies that $(u + v) \in \mathcal{E}_0(f + (-f)) = \mathcal{E}_0(0)$. The converse follows immediately since $PB_f \in \mathcal{E}_0(f)$ and $PB_{-f} \in \mathcal{E}_0(-f)$. For the second statement first assume that equality holds in (15). Then there exist $u \in \mathcal{E}_0(f)$ and $v \in \mathcal{E}_0(-f)$ such that $u + v = 0$, hence u is pluriharmonic on Ω and $u = PB_f$. Walsh's theorem concludes that PB_f is continuous on $\bar{\Omega}$. This proof ends with noticing that if PB_f is pluriharmonic on Ω and continuous on $\bar{\Omega}$, then $PB_{-f} = PB_f$ and for $u \in \mathcal{E}_0$ it follows that $(u + PB_f) \in \mathcal{E}_0(f)$ and $u = (u + PB_f) + PB_{-f}$.

Corollary 2.8 is a direct consequence of Theorem 2.7 and Lemma 2.5.

COROLLARY 2.8. *If $f \in \mathcal{CP}(\partial\Omega)$ is a compliant function, then $\mathcal{E}_0(f) = \{u \in \mathcal{E}_0(f) : \int_\Omega (dd^c u)^n < +\infty\}$ and $\mathcal{E}_0(-f) = \{v \in \mathcal{E}_0(-f) : \int_\Omega (dd^c v)^n < +\infty\}$.*

3. The compliant functions as a Banach space

For convenience let u be the operator defined on a function $f : \partial\Omega \to \mathbb{R}$ by $u(f) = PB_f + PB_{-f}$. Then it immediately from (1) that if $f, g : \partial\Omega \to \mathbb{R}$ are two functions and $\alpha, \beta \in \mathbb{R}$, then

$$0 \geq u(\alpha f + \beta g) \geq |\alpha|\, u(f) + |\beta|\, u(g). \tag{16}$$

This inequality together with the fact that $\mathcal{E}_0$ is a convex cone yields Proposition 3.1.

Proposition 3.1. *Let $\Omega \subseteq \mathbb{C}^n$ be a bounded hyperconvex domain. The set of all compliant functions is a linear subspace of the real vector space containing the real-valued continuous functions defined on $\partial\Omega$.*

Lemma 3.2. *If $f, g \in \mathcal{CP}(\partial\Omega)$, then $|u(f)(z) - u(g)(z)| \leq 2\|f - g\|_\infty$ for all $z \in \Omega$.*

Proof. The definition of the Perron-Bremermann envelope implies that $PB_{f-g} \leq PB_f - PB_g$ and $PB_f - PB_g \leq -PB_{g-f}$ and from this it follows that

$$\min_{\partial\Omega}(g - f) \leq PB_{f-g} \leq PB_f - PB_g \leq -PB_{g-f} \leq -\min_{\partial\Omega}(f - g),$$

hence $|PB_f - PB_g| \leq \|f - g\|_\infty$ and therefore

$$|u(f)(z) - u(g)(z)| = |PB_f + PB_{-f} - PB_g - PB_{-g}| \leq 2\|f - g\|_\infty,$$

which concludes the proof.

Theorem 3.3. *If $\|\cdot\|$ is defined by* (2)*, then $(\mathcal{CP}(\partial\Omega), \|\cdot\|)$ is a Banach space.*

Proof. If f is identically 0, then it follows from (2) that $\|f\| = 0$. Moreover, if $\|f\| = 0$, then $\|f\|_\infty = 0$. Hence $f = 0$. Let $t \in \mathbb{R}$, then

$$\begin{aligned}\|tf\| &= \|tf\|_\infty + \left(\int_\Omega (dd^c u(tf))^n\right)^{\frac{1}{n}} = |t|\|f\|_\infty + |t|\left(\int_\Omega (dd^c u(f))^n\right)^{\frac{1}{n}} \\ &= |t|\|f\|.\end{aligned}$$

Thus $(\mathcal{CP}(\partial\Omega), \|\cdot\|)$ is a normed vector space, since the triangle inequality follows from Lemma 2.3. It remains to prove completeness. Now assume that $[f_j]$ is a Cauchy sequence in $(\mathcal{CP}(\partial\Omega), \|\cdot\|)$, hence it is a Cauchy sequence in $\|\cdot\|_\infty$ norm and therefore there exists a continuous function $f : \partial\Omega \to \mathbb{R}$ such that $[f_j]$ converges uniformly on $\partial\Omega$ to f as $j \to +\infty$. Because $[f_j]$ is a Cauchy sequence in $(\mathcal{CP}(\partial\Omega), \|\cdot\|)$ there exists an increasing sequence j_k such that $\|f_{j_{k+1}} - f_{j_k}\| \leq 2^{-k}$, hence

$$\int_\Omega (dd^c u(f_{j_{k+1}} - f_{j_k}))^n \leq 2^{-nk}, \tag{17}$$

by (2). Note that

$$f_{j_{k+1}} = f_{j_1} + \sum_{i=1}^{k} (f_{j_{i+1}} - f_{j_i})$$

and then by using (17) together with Lemma 2.3 it follows that

$$\left(\int_\Omega (dd^c u(f_{j_{k+1}}))^n\right)^{\frac{1}{n}} \leq \left(\int_\Omega (dd^c u(f_{j_1}))^n\right)^{\frac{1}{n}} + \sum_{i=1}^{k} \left(\int_\Omega (dd^c u(f_{j_{i+1}} - f_{j_i}))^n\right)^{\frac{1}{n}}$$
$$\leq \|f_{j_1}\| + 1.$$

Thus, $\sup_k \int_\Omega (dd^c u(f_{j_{k+1}}))^n \leq (\|f_1\| + 1)^n < +\infty$. Lemma 3.2 now yields that the total mass of $(dd^c u(f))^n$ is bounded by $(\|f_{j_1}\| + 1)^n$, i.e., f satisfies P2. Now assume that P1 is false, then there exists a $\xi \in \partial\Omega$ and a sequence $[z_l]$ in Ω such that $[z_l]$ converges to ξ and $\lim_{l\to\infty} u(z_l) = -a < 0$, where $a \geq 0$ is a constant. The sequence $[f_j]$ converges uniformly to f on $\partial\Omega$, hence there exists $m \in \mathsf{N}$ such that $\|f_j - f\|_\infty \leq \frac{a}{4}$ for all $j \geq m$. Lemma 3.2 yields that $|u(f_j)(z_l) - u(f)(z_l)| \leq 2\|f_j - f\|_\infty < \frac{a}{2}$ and a contradiction has been achieved since

$$\left|\lim_{l\to\infty} (u(f_j)(z_l) - u(f)(z_l))\right| = a.$$

Thus, f is a compliant function and this proof is completed.

COROLLARY 3.4. *If $[f_j]$ is a sequence which converges in $(\mathcal{CP}(\partial\Omega), \|\cdot\|)$ to a function f, then $[(dd^c u(f_j))^n]$ converges to $(dd^c u(f))^n$ in the weak*-topology. In other words the map $\mathcal{CP}(\partial\Omega) \ni f \to (dd^c u(f))^n$ is continuous.*

THEOREM 3.5. *The set $\mathcal{CP}_0(\partial\Omega)$ is a closed subspace of $\mathcal{CP}(\partial\Omega)$. Moreover, $\mathcal{CP}_0(\partial\Omega) = \mathcal{PH}(\partial\Omega)$.*

PROOF. Inequality (16) implies that $\mathcal{CP}_0(\partial\Omega)$ is a subspace. Now assume that $[f_j]$ is a sequence in $\mathcal{CP}_0(\partial\Omega)$ which converges in norm to a function f. Corollary 3.4 implies that $[(dd^c u(f_j))^n]$ converges to $(dd^c u(f))^n$ in the weak*-topology. By assumption $u(f_j) = 0$, hence $(dd^c u(f_j))^n = 0$ and therefore it follows by Corollary 3.4 that $(dd^c u(f))^n = 0$. Hence $u(f) = 0$, i.e., $f \in \mathcal{CP}_0(\partial\Omega)$ which yields that $\mathcal{CP}_0(\partial\Omega)$ is closed.

To prove the second statement let $f \in \mathcal{CP}_0(\partial\Omega)$, i.e., $u(f)$ is identically 0. This together will Walsh's theorem implies that $PB_f, PB_{-f} \in \mathcal{PSH}(\Omega) \cap$

$C(\bar{\Omega})$ and $-PB_f = PB_{-f}$. Thus, $PB_f \in \mathcal{PH}(\Omega) \cap C(\bar{\Omega})$. On the other hand, if there exists a function $u \in \mathcal{PH}(\Omega) \cap C(\bar{\Omega})$ such that $u = f$ on $\partial\Omega$, then $dd^c u = 0$ and therefore $u = PB_f$. Similarly, $-u = PB_{-f}$. Thus, $u(f)$ is identically 0.

REMARK. It was proved in [7] that if D is a hyperconvex product domain $D = D_{n_1} \times \cdots \times D_{n_k}, n = n_1 + \cdots + n_k, k \geq 3$, then $\mathcal{CP}_0(\partial D) = \mathcal{CP}(\partial D) = \mathcal{PH}(\partial D)$. Consider $f(z_1, z_2) = |z_1|^2$ defined on the boundary of the unit ball in $\mathbb{C}^2$. This simple example shows by some straight forward calculations that $\mathcal{CP}_0(\partial B)$ is generally not equal to $\mathcal{CP}(\partial B)$.

4. Examples

Example 4.1 shows that it is not enough to assume that f is a C^1-function to ensure that P2 is true.

EXAMPLE 4.1. Let $B \subseteq \mathbb{C}^n$ be the unit ball and $z = (z', z_n) \in B$. For fixed $0 < p < 1$, let $f_p : \partial B \to \mathbb{R}$ be the function defined by $f_p(z', z_n) = |z_n|^{2p}$. Then

$$PB_{f_p}(z', z_n) = |z_n|^{2p} \qquad \text{and} \qquad PB_{-f_p}(z', z_n) = -(1 - |z'|^2)^p.$$

Set $E = \{(z', z_n) \in B : z_n = 0\} \cup \{(z', z_n) \in B : |z'| = 1\}$. The set E is a pluripolar, hence

$$\begin{aligned}\int_\Omega (dd^c(PB_{f_p} + PB_{-f_p}))^n &= \int_{\Omega\setminus E} (dd^c(PB_{f_p} + PB_{-f_p}))^n \\ &= C \int_0^1 |r|^{2np-2n+1}(1 - r^2)^{n-1} dr,\end{aligned}$$

where $C > 0$ is a constant only depending on n and p. Thus f_p is not compliant if, and only if, $p \leq \frac{n-1}{n}$. For $n > 2$, the function $f_{\frac{n-1}{n}}$ belongs to $C^1(\partial B)$.

Example 4.2 shows that even if P2 is true and $f \in C^\infty(\partial\Omega)$, it may happen that P1 is false. This phenomenon cannot happen if Ω is a B-regular domain (see e.g. [12] for the definition and elementary properties of B-regular domains).

EXAMPLE 4.2. Let P be the unit polydisc in $\mathbb{C}^2$, i.e., $P = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\}$ and let the function $f : \partial P \to \mathbb{R}$ be defined by $f(z_1, z_2) = |z_2|^2$. Then $f \in C^\infty(\partial P)$ $PB_f(z_1, z_2) = |z_2|^2$. From (1) it follows that $PB_{-f}(z_1, z_2) \geq -1$. Fix $(z_0, w_0) \in P$ and let v_{z_0} be a function defined on the unit disc D in $\mathbb{C}$ by $v_{z_0}(\zeta) = PB_{-f}(z_0, \zeta)$, hence $v_{z_0} \in \mathcal{SH}(D)$ and $\limsup_{\zeta\to\xi} v_{z_0}(\zeta) \leq -1$ for every $\xi \in \partial D(0, 1)$, hence $v_{z_2} \leq -1$ on

$D(0,1)$ by the maximum principle for subharmonic functions and therefore $PB_{-f} \leq -1$ on P. Thus, $PB_{-f}(z_1, z_2) = -1$. Moreover,

$$\int_P (dd^c(PB_f + PB_{-f}))^2 = 0.$$

Thus P2 is true, but

$$\lim_{\substack{(z_1,z_2)\to(\xi_1,\xi_2)\\(z_1,z_2)\in P}} (PB_f(z_1,z_2) + PB_{-f}(z_1,z_2)) \neq 0,$$

when $(\xi_1, \xi_2) \in \partial P \setminus \{(w_1, w_2) \in \mathbf{C}^2 : |w_2| = 1\}$, hence P1 is false.

Proposition 4.3. *If $D \subseteq \mathbf{C}^n$ is a bounded, strictly pseudoconvex domain with C^2-boundary and $f \in C^2(\partial D)$, then f is a compliant function.*

Proof. The domain D is in particular B-regular and the function f is in particular continuous and therefore it follows that P1 is true. There exists an open neighbourhood U of D and a strictly plurisubharmonic C^2-function $\rho : U \to \mathbf{R}$ such that $\rho = 0$ on $\partial\Omega$, since D is a strictly pseudoconvex domain with C^2-boundary. By Theorem I in [14] there exists a C^2-function $\tilde{f} : \mathbf{C}^n \to \mathbf{R}$ such that $\tilde{f} = f$ on ∂D. Choose $A > 0$ such that $u = (\tilde{f} + A\rho) \in \mathcal{PSH}(D)$ and $B > 0$ such that $v = (-\tilde{f} + B\rho) \in \mathcal{PSH}(D)$. Hence, $u, v \in \mathcal{PSH}(U) \cap C^2(U)$, $u = -v = f$ on ∂D. Thus

$$\int_D (dd^c(u+v))^n = \int_D \sum_{k=0}^{n} \binom{n}{k} (dd^c u)^{n-k} \wedge (dd^c v)^k < +\infty. \tag{18}$$

The construction of PB_f and PB_{-f} implies that $u + v \leq PB_f + PB_{-f}$, hence

$$\int_D (dd^c(PB_f + PB_{-f}))^n \leq \int_D (dd^c(u+v))^n < +\infty,$$

by (18) and the comparison principle. Thus P2 holds.

Remark. Let $k \in \{0, 1, 2, \ldots\} \cup \{+\infty\}$. See [14] for the definition of how a function $f : \partial\Omega \to \mathbf{R}$ is of class C^k and their basic properties.

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