THE CONNECTION BETWEEN THE CEGRELL
CLASSES AND COMPLIANT FUNCTIONS

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Abstract

In this article the connection between the Cegrell classes and compliant functions is studied. A
suitable norm is constructed which makes the compliant functions into a Banach space. As an
application a characterization of the Dirichlet problem for pluriharmonic functions is achieved.
Explicit examples of non-compliant functions will be constructed and a sufficient condition for
compliance will be proved.

1. Introduction

Throughout this article let $\Omega \subseteq \mathbb{C}^n$ be a hyperconvex domain, i.e., a connected,
open set that admits a negative plurisubharmonic exhaustion function. Furthermore it is assumed that $\Omega$ is bounded. Recall that the Perron-Bremermann envelope for a given function $f : \partial \Omega \to \mathbb{R}$ is defined by

\[
(PB_f(z) = \sup \left\{ w(z) : w \in \mathcal{PH}(\Omega), \limsup_{\zeta \to \xi, \zeta \in \Omega} w(\zeta) \leq f(\xi) \forall \xi \in \partial \Omega \right\},
\]

where $\mathcal{PH}(\Omega)$ is the class of all plurisubharmonic functions defined on $\Omega$. If $f : \partial \Omega \to \mathbb{R}$ is a continuous function, then $PB_f \in \mathcal{PH}(\Omega)$ since a hyperconvex domain viewed as a set in $\mathbb{R}^{2n}$ is regular to the Laplace operator. Consider the following two assertions:

P1: $\lim_{z \to \xi} (PB_f + PB_{-f})(z) = 0$ for every $\xi \in \partial \Omega$,

P2: $\int_{\Omega} (dd^c (PB_f + PB_{-f}))^n < +\infty$,

where $(dd^c \cdot)^n$ is the complex Monge-Ampère operator. A continuous function $f : \partial \Omega \to \mathbb{R}$ which satisfies P1 and P2 is called a compliant function. The compliant functions first arose in [3] when some of the Cegrell classes

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given by a continuous function were introduced. Let $\mathcal{C}(\partial \Omega)$ denote the class of compliant functions defined on $\partial \Omega$. If $n = 1$, then the set of compliant functions and continuous functions defined on $\partial \Omega$ coincide, therefore it will be assumed that $n \geq 2$ throughout this article. The special case when $\Omega$ is a hyperconvex product domain $\Omega = \Omega_{n_1} \times \cdots \times \Omega_{n_k}$, $n = n_1 + \cdots + n_k$, $k \geq 3$, was studied in [7].

The Cegrell classes were first introduced in [3]. In [4] and [15] new classes were added to the Cegrell family. Let $E_0(f)$ be the class of plurisubharmonic functions defined in [3] (see also Definition 2.1). Example 2.4 shows that there exists a function $u \in E_0(f)$ such that $\int_{\Omega} (dd^c u)^n = +\infty$. This cannot occur if $f$ is compliant (Lemma 2.5). Let $u \in E_0(f)$. Then $f$ is compliant if, and only if, $(u + PB_{-f}) \in E_0(0)$. A more thorough study about this property in the different Cegrell classes will be made in Section 2. By using a convexity property of $E_0(f)$ it is proved that a continuous function $f : \partial \Omega \to \mathbb{R}$ is compliant if, and only if, $E_0(f) \oplus E_0(-f) \subseteq E_0(0)$, where $\oplus$ is the sum of two sets (Theorem 2.7).

Let $\| \cdot \| : \mathcal{C}(\partial \Omega) \to \mathbb{R}$ be defined by

\[
\| f \| = \| f \|_{\infty} + \left( \int_{\Omega} (dd^c (PB_f + PB_{-f}))^n \right) \frac{1}{n},
\]

where $\| f \|_{\infty} = \sup \{ |f(\xi)| : \xi \in \partial \Omega \}$. The aim of Section 3 is to prove that $(\mathcal{C}(\partial \Omega), \| \cdot \|)$ is a Banach space (Theorem 3.3). Let $PH(\partial \Omega)$ denote those continuous functions $\partial \Omega \to \mathbb{R}$ which can be extended to a pluriharmonic function in $\Omega$. A considerable amount of results concerning the Dirichlet problem for pluriharmonic functions exist, see e.g. [1], [2], [7], [8], [9], [10], [11] and the references therein. As an application of Theorem 3.3 it is proved that $PH(\partial \Omega)$ is equivalent to the closed subspace, $\mathcal{C}(\partial \Omega)$, of $\mathcal{C}(\partial \Omega)$ that contains functions for which $-PB_f = PB_{-f}$ on $\Omega$ (Theorem 3.5).

Example 4.1 shows that it is not enough to assume that $f$ is a $C^1$-function to ensure that P2 is true, even if $\Omega$ is the unit ball. Example 4.2 shows that even if P2 is true and $f \in C^\infty(\partial \Omega)$, it may happen that P1 is false. If $\Omega$ is a strictly pseudoconvex domain in $\mathbb{C}^n$ with $C^2$-boundary and $f : \partial \Omega \to \mathbb{R}$ is a $C^2$-function, then $f$ is a compliant function (Proposition 4.3).

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2. The connection between the Cegrell classes and compliant functions

In this section assume that \( f : \partial \Omega \to \mathbb{R} \) is a continuous function such that

\[
\lim_{z \to \xi} PB_f(z) = f(\xi),
\]

for every \( \xi \in \partial \Omega \), hence \( PB_f \in \mathcal{PH}(\Omega) \cap C(\overline{\Omega}) \) by Walsh’s theorem (see [13]). A bounded plurisubharmonic function \( \varphi \) defined on \( \Omega \) belongs to \( \mathcal{E}_0(\Omega) \) if\( \lim_{z \to \xi} \varphi(z) = 0 \) for every \( \xi \in \partial \Omega \) and

\[
\int_{\Omega} (dd^c \varphi)^n < +\infty.
\]

The class \( \mathcal{E}_0 \) has a role similar to that of the test functions, \( C^\infty(\Omega) \), in the theory of distributions. Before the connection between the Cegrell classes and compliant functions will be discussed the definition of the Cegrell classes given by a continuous function will be stated. For the definition of \( F_p \) and \( E_p \) see [3] and for \( F, E \) see [4]. For further information about the Cegrell classes see e.g. [6] and the references therein.

**Definition 2.1.** Let \( \mathcal{K} \in \{ \mathcal{E}_0, F_p, E_p, F, E \} \) and \( f : \partial \Omega \to \mathbb{R} \) a continuous function that satisfies (3). A plurisubharmonic function \( u \) defined on \( \Omega \) belongs to \( \mathcal{K}(f)(= \mathcal{K}(\Omega, f)) \) if there exists a function \( \varphi \in \mathcal{K} \) such that

\[
PB_f \geq u \geq \varphi + PB_f.
\]

**Remark.** Let \( \mathcal{K} \in \{ \mathcal{E}_0, F_p, E_p, F, E \} \), then \( \mathcal{K}(0) = \mathcal{K} \).

**Proposition 2.2.** Let \( \mathcal{K} \in \{ \mathcal{E}_0, F_p, E_p, F, E \} \) and let \( u \in \mathcal{K}(f) \). If \( f \) is a compliant function, then \( (u + PB_{-f}) \in \mathcal{K}(0) \).

**Proof.** Let \( \mathcal{K} \in \{ \mathcal{E}_0, F_p, E_p, F, E \} \) and \( u \in \mathcal{K}(f) \). Definition 2.1 implies that \( u \in \mathcal{PH}(\Omega) \) and that there exists a function \( \varphi \in \mathcal{K} \) such that \( PB_f \geq u \geq \varphi + PB_f \), hence

\[
0 \geq PB_f + PB_{-f} \geq u + PB_{-f} \geq \varphi + PB_f + PB_{-f}.
\]

The function \( (\varphi + PB_f + PB_{-f}) \) belongs to \( \mathcal{K} \), since \( \mathcal{K} \) is a convex cone and \( f \) is compliant. By (4) it follows that \( (u + PB_{-f}) \in \mathcal{K}(0) \).

**Remark.** The converse statement of Proposition 2.2 is true for \( \mathcal{E}_0(f) \), i.e., if \( u \in \mathcal{E}_0(f) \) and \( (u + PB_{-f}) \in \mathcal{E}_0(0) \), then \( f \) is a compliant function. Let \( \mathcal{K} \in \{ F_p, E_p, F \} \) and \( u \in \mathcal{K} \). If \( (u + PB_{-f}) \in \mathcal{K}(0) \), then \( (PB_f + PB_{-f}) \in \mathcal{K}(0) \). But \( (PB_f + PB_{-f}) \) is generally not a sufficient condition for \( f \) to be compliant.
Proposition 2.2 yields an easy method to transform questions about the Cegrell classes given by a continuous function to the classes with zero boundary values. The classes with zero boundary values is much easier to handle and therefore the question of finding a natural characterization of the compliant functions is of importance. If \( u \in \mathcal{E}(f) \), then \( (u + PB - f) \) always belongs to \( \mathcal{E} \) without the assumption that \( f \) is compliant. Example 2.4 is a slightly modified version of Example 5.6 in [5] and it shows that there exists a function \( u \in \mathcal{E}_0(f), f \in C^\infty(\partial \Omega) \), such that the total mass of \( (dd^c u)^n \) is infinite. This cannot occur if \( f \) is a compliant function (see Lemma 2.5).

**Lemma 2.3.** If \( f, g \in \mathcal{F} \), then

\[
\left( \int_{\Omega} (dd^c (u + v))^n \right)^{\frac{1}{n}} \leq \left( \int_{\Omega} (dd^c u)^n \right)^{\frac{1}{n}} + \left( \int_{\Omega} (dd^c v)^n \right)^{\frac{1}{n}}.
\]

**Proof.** See Lemma 2.5 in [5].

**Example 2.4.** Let \( P \) be the unit polydisc in \( \mathbb{C}^2 \), i.e., \( P = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\} \). Let \( f : \partial P \to \mathbb{R} \) be defined by \( f(z_1, z_2) = |z_2|^2 \). The function \( f \) is not compliant, \( f \in C^\infty(\partial P) \) and \( PB_f(z_1, z_2) = |z_2|^2 \). For each \( j \in \mathbb{N} \) define the function \( \varphi_j : P \to \mathbb{R} \) by \( \varphi_j(z_1, z_2) = \max(aj \log |z_1|, bj \log |z_2|, c_j) \), where \( a_j, b_j, c_j \in \mathbb{R}, a_j, b_j > 0 \) and \( c_j < 0 \). Then \( \varphi_j \in \mathcal{P} \mathcal{S} \mathcal{H} (P) \cap C(\bar{P}) \), \( \lim_{(z_1, z_2) \to (\xi_1, \xi_2)} \varphi_j(z_1, z_2) = 0 \) for every \( (\xi_1, \xi_2) \in \partial P \) and

\[
\int_{P} (dd^c \varphi_j)^2 = (2\pi)^2 a_j b_j < +\infty,
\]

hence \( \varphi_j \in \mathcal{E}_0 \). Let \( v_k : P \to \mathbb{R} \) be defined by \( v_k = \sum_{j=1}^{k} \varphi_j \). From this definition it follows that \( v_k \in \mathcal{E}_0 \) and that \( [v_k] \) is a decreasing sequence on \( P \). Lemma 2.3 and (5) yields that

\[
\int_{P} (dd^c v_k)^2 \leq \left( \sum_{j=1}^{k} \left( \int_{P} (dd^c \varphi_j)^2 \right)^{\frac{1}{2}} \right)^2 \leq (2\pi)^2 \left( \sum_{j=1}^{k} (a_j b_j)^{\frac{1}{2}} \right)^2.
\]

Assume that

\[
\sum_{j=1}^{\infty} (a_j b_j)^{\frac{1}{2}} < +\infty \quad \text{and} \quad \sum_{j=1}^{\infty} c_j > -\infty
\]

and let \( v(z) = \lim_{k \to \infty} v_k(z) \). The construction of the function \( v \) implies that \( \lim_{(z_1, z_2) \to (\xi_1, \xi_2)} v(z_1, z_2) = 0 \) for every \( (\xi_1, \xi_2) \in \partial P \). The assumptions in (7)


imply that $v \in \mathcal{P}(P) \cap L^\infty(P)$ and by inequality (6) it follows that $v \in \mathcal{E}_0$.

Let $u : P \to \mathbb{R}$ be defined by $u = v + PBf$, hence $u = (v + |z_2|^2) \in \mathcal{E}_0(f)$.

Then it follows that

$$
\int_P (dd^c(v_k + |z_2|^2))^2 = \int_P (dd^c v_k)^2 + 4i \int_P (dd^c v_k) \wedge dz_2 \wedge d\bar{z}_2 \\
= \int_P (dd^c v_k)^2 + 32 \int_P \frac{\partial^2 v_k}{\partial z_1 \partial \bar{z}_1} dV((z_1, z_2)) \\
= \int_P (dd^c v_k)^2 + 32 \int_P \sum_{j=1}^k \frac{\partial^2 \phi_j}{\partial z_1 \partial \bar{z}_1} dV((z_1, z_2)) \\
\geq 32 \sum_{j=1}^k \int_P \frac{\partial^2 \phi_j}{\partial z_1 \partial \bar{z}_1} dV((z_1, z_2)),
$$

where $V$ is the Lebesgue measure on $\mathbb{C}^2$. Let $\varepsilon > 0$ be given such that $0 < \varepsilon < 1$ and let $D(0, r) = \{z \in \mathbb{C} : |z| < r\}$. Choose $\chi_1, \chi_2 \in C_0^\infty(D(0, 1))$ such that $0 \leq \chi_1, \chi_2 \leq 1$ and $\chi_1 = \chi_2 = 1$ on $D(0, 1 - \varepsilon)$. For fixed $|z_2| \leq \min(1 - \varepsilon, (1 - \varepsilon)^{\frac{a_j}{b_j}})$ fix, it follows that

$$
\int_{D(0, 1)} \chi_1(z_1) \frac{\partial^2 \phi_j}{\partial z_1 \partial \bar{z}_1} dV(z_1) = 8\pi a_j.
$$

Under the assumption that $a_j \geq b_j$ inequality (8) together with (9) yield that

$$
\int_P (dd^c(v_k + |z_2|^2))^2 \geq 32 \sum_{j=1}^k \int_P \left(\chi_1(z_1) \chi_2(z_2)\right) \frac{\partial^2 \phi_j}{\partial z_1 \partial \bar{z}_1} dV((z_1, z_2)) \\
\geq c \sum_{j=1}^k a_j \left(\min\left(1 - \varepsilon, (1 - \varepsilon)^{\frac{a_j}{b_j}}\right)\right)^2,
$$

where $c > 0$ is a constant. Let $\varepsilon \to 0^+$, then (10) implies that

$$
\int_P (dd^c(v_k + |z_2|^2))^2 \geq c \sum_{j=1}^k a_j.
$$

Thus

$$
\int_P (dd^c u)^2 = \lim_{k \to +\infty} \int_P (dd^c(v_k + |z_2|^2))^2 \geq c \sum_{j=1}^\infty a_j.
$$
Let the sequences \([a_j], [b_j]\) and \([c_j]\) be defined by \(a_j = 1/j\), \(b_j = 1/j^3\) and \(c_j = -1/j^2\). Thus the assumptions (7) and \(a_j \geq b_j\) are satisfied, which implies that the function defined on \(P\) by
\[
u(z_1, z_2) = \sum_{j=1}^{\infty} \max \left( \frac{1}{j} \log |z_1|, \frac{1}{j^3} \log |z_2|, -\frac{1}{j^2} \right) + |z_2|^2,
\]
begins to \(\mathcal{E}_0(f)\) and \(\int_P (dd^c u)^2 = +\infty\), by (11).

**Lemma 2.5.** If \(f \in \mathcal{CP}(\partial U\Omega_\alpha)\), then \(F(f) = \{u \in F(f) : \int_{\Omega} (dd^c u)^n < +\infty\}\).

**Proof.** Let \(u \in F(f)\), i.e., \(u \in \mathcal{PH}(\Omega)\) and there exists a function \(\varphi \in \mathcal{F}\) such that \(PB_f \geq u \geq \varphi + PB_f\). Theorem 2.1 in [4] implies that there exists a decreasing sequence \([\varphi_j]\), \(\varphi_j \in \mathcal{E}_0\), that converges pointwise to \(\varphi\) as \(j \to +\infty\). Let the sequence \([u_j]\), \(j \in \mathbb{N}\), be defined by \(u_j = \max \{u, \varphi_j + PB_f\}\). Then the decreasing sequence \([u_j]\), \(u_j \in \mathcal{E}_0(f)\), converges pointwise to \(u\) as \(j \to +\infty\) and
\[
(dd^c (\varphi_j + PB_f + PB_{-f}))^n \geq (dd^c (\varphi_j + PB_f))^n + (dd^c PB_{-f})^n
= (dd^c (\varphi_j + PB_f))^n,
\]
since \((PB_f + PB_{-f}) \in \mathcal{E}_0(0)\). The sequence \([\varphi_j + PB_f + PB_{-f}]\) is decreasing and converges pointwise to \((\varphi + PB_f + PB_{-f}) \in \mathcal{F}\) as \(j \to +\infty\). From Proposition 5.1 in [4] it follows that
\[
\lim_{j \to +\infty} \int_{\Omega} (dd^c (\varphi_j + PB_f + PB_{-f}))^n = \int_{\Omega} (dd^c (\varphi + PB_f + PB_{-f}))^n,
\]
hence
\[
\sup_j \int_{\Omega} (dd^c u_j)^n < +\infty,
\]
by (12). Let \([\phi_s]\), \(\phi_s \in C_0^\infty(\Omega), \phi_s \geq 0\), be an increasing sequence which converges pointwise to 1 on \(\Omega\) as \(s \to +\infty\). For \(s \in \mathbb{N}\) fixed, it follows from (13) that
\[
\int_{\Omega} \phi_s (dd^c u)^n = \lim_{j \to +\infty} \int_{\Omega} \phi_s (dd^c u_j)^n \leq \sup_j \int_{\Omega} (dd^c u_j)^n < +\infty.
\]
Let \(s \to +\infty\), this lemma then follows from (14) and the monotone convergence theorem.
LEMMA 2.6. Assume that $f,g : \partial \Omega \rightarrow \mathbb{R}$ are continuous functions such that
\[
\lim_{z \to \xi} PB_f(z) = f(\xi) \quad \text{and} \quad \lim_{z \to \xi} PB_g(z) = g(\xi),
\]
for every $\xi \in \partial \Omega$ and in addition assume that $f$ is a compliant function. Let $\mathcal{H} \in \{\mathcal{E}_0, \mathcal{F}_p, \mathcal{E}_p, \mathcal{F}\}$. If $u \in \mathcal{H}(f)$ and $v \in \mathcal{H}(g)$, then $(\alpha u + \beta v) \in \mathcal{H}(\alpha f + \beta g)$, where $\alpha, \beta \in \mathbb{R}$, $\alpha, \beta \geq 0$.

PROOF. Let $\mathcal{H} \in \{\mathcal{E}_0, \mathcal{F}_p, \mathcal{E}_p, \mathcal{F}\}, u \in \mathcal{H}(f)$ and $v \in \mathcal{H}(g)$. Definition 2.1 implies that $u, v \in \mathcal{P}R\mathcal{H}(\Omega)$ and that there exist functions $\varphi, \psi \in \mathcal{H}$ such that $PB_f \geq u \geq \varphi + PB_f$ and $PB_g \geq v \geq \psi + PB_g$. The definition of the Perron-Bremermann envelope yields that $PB_{\alpha f + \beta g} \geq PB_{\alpha f} + PB_{\beta g}$ and therefore it follows that
\[
PB_{\alpha f + \beta g} \geq \alpha u + \beta v \geq \alpha \varphi + \beta \psi + PB_{\alpha f} + PB_{\beta g} \\
\geq \alpha \varphi + \beta \psi + PB_{\alpha f} + PB_{\beta g} + PB_{\alpha f} + PB_{\beta g} \\
= \alpha \varphi + \beta \psi + \alpha (PB_f + PB_f) + PB_{\alpha f + \beta g}.
\]
Thus $(\alpha u + \beta v) \in \mathcal{H}(\alpha f + \beta g)$, since $(\alpha \varphi + \beta \psi + \alpha (PB_f + PB_{-f})) \in \mathcal{H}$.

THEOREM 2.7. Let $\Omega \subseteq \mathbb{C}^n$ be a bounded hyperconvex domain and let $f : \Omega \rightarrow \mathbb{R}$ be a continuous function. The function $f$ is compliant if, and only if,
\[
(15) \quad \mathcal{E}_0(f) \oplus \mathcal{E}_0(-f) \subseteq \mathcal{E}_0(0),
\]
where $\oplus$ is the sum of two sets. Moreover, equality holds in (15) if, and only if, $PB_f$ is pluriharmonic on $\Omega$ and continuous on $\Omega$.

PROOF. Assume that $f$ is a compliant, hence $-f$ is compliant. Let $u \in \mathcal{E}_0(f)$ and $v \in \mathcal{E}_0(-f)$. Lemma 2.6 implies that $(u + v) \in \mathcal{E}_0(f + (-f)) = \mathcal{E}_0(0)$. The converse follows immediately since $PB_f \in \mathcal{E}_0(f)$ and $PB_{-f} \in \mathcal{E}_0(-f)$. For the second statement first assume that equality holds in (15). Then there exist $u \in \mathcal{E}_0(f)$ and $v \in \mathcal{E}_0(-f)$ such that $u + v = 0$, hence $u$ is pluriharmonic on $\Omega$ and $u = PB_f$. Walsh’s theorem concludes that $PB_f$ is continuous on $\Omega$. This proof ends with noticing that if $PB_f$ is pluriharmonic on $\Omega$ and continuous on $\Omega$, then $PB_{-f} = PB_f$ and for $u \in \mathcal{E}_0$ it follows that $(u + PB_f) \in \mathcal{E}_0(f)$ and $u = (u + PB_f) + PB_{-f}$.

Corollary 2.8 is a direct consequence of Theorem 2.7 and Lemma 2.5.

Corollary 2.8. If $f \in \mathcal{CP}(\partial \Omega)$ is a compliant function, then $\mathcal{E}_0(f) = \{u \in \mathcal{E}_0(f) : \int_\Omega (dd^c u)^n < +\infty \}$ and $\mathcal{E}_0(-f) = \{v \in \mathcal{E}_0(-f) : \int_\Omega (dd^c v)^n < +\infty \}$. 

THE CONNECTION BETWEEN THE CEGRELL CLASSES...
3. The compliant functions as a Banach space

For convenience let $u$ be the operator defined on a function $f : \partial \Omega \to \mathbb{R}$ by $u(f) = PB_f + PB_{-f}$. Then it immediately from (1) that if $f, g : \partial \Omega \to \mathbb{R}$ are two functions and $\alpha, \beta \in \mathbb{R}$, then

$$0 \geq u(\alpha f + \beta g) \geq |\alpha| u(f) + |\beta| u(g).$$

This inequality together with the fact that $E_0$ is a convex cone yields Proposition 3.1.

**Proposition 3.1.** Let $U_{\Omega A} \subseteq C^0$ be a bounded hyperconvex domain. The set of all compliant functions is a linear subspace of the real vector space containing the real-valued continuous functions defined on $\partial U_{\Omega A}$.

**Lemma 3.2.** If $f, g \in CP(\partial U_{\Omega A})$, then $|u(f)(z) - u(g)(z)| \leq 2 \|f - g\|_{\infty}$ for all $z \in U_{\Omega A}$.

**Proof.** The definition of the Perron-Bremermann envelope implies that $PB_f - g \leq PB_f - PB_g$ and $PB_f - PB_g \leq -PB_{g-f}$ and from this it follows that

$$\min_{\partial \Omega} (g - f) \leq PB_{f - g} \leq PB_f - PB_g \leq -PB_{g-f} \leq -\min_{\partial \Omega} (f - g),$$

hence $|PB_f - PB_g| \leq \|f - g\|_{\infty}$ and therefore

$$|u(f)(z) - u(g)(z)| = |PB_f + PB_{-f} - PB_g - PB_{-g}| \leq 2 \|f - g\|_{\infty},$$

which concludes the proof.

**Theorem 3.3.** If $\| \cdot \|$ is defined by (2), then $(CP(\partial \Omega), \| \cdot \|)$ is a Banach space.

**Proof.** If $f$ is identically 0, then it follows from (2) that $\|f\| = 0$. Moreover, if $\|f\| = 0$, then $\|f\|_{\infty} = 0$. Hence $f = 0$. Let $t \in \mathbb{R}$, then

$$\|tf\| = \|tf\|_{\infty} + \left(\int_{\Omega} (dd^c u (tf))^n\right)^{\frac{1}{n}} = |t|\|f\|_{\infty} + |t| \left(\int_{\Omega} (dd^c u (f))^n\right)^{\frac{1}{n}} = |t|\|f\|.$$

Thus $(CP(\partial \Omega), \| \cdot \|)$ is a normed vector space, since the triangle inequality follows from Lemma 2.3. It remains to prove completeness. Now assume that $[f_j]$ is a Cauchy sequence in $(CP(\partial \Omega), \| \cdot \|)$, hence it is a Cauchy sequence in $\| \cdot \|_{\infty}$ norm and therefore there exists a continuous function $f : \partial \Omega \to \mathbb{R}$ such that $[f_j]$ converges uniformly on $\partial \Omega$ to $f$ as $j \to +\infty$. Because $[f_j]$ is
a Cauchy sequence in \((C\bar{P}(\partial \Omega), \| \cdot \|)\) there exists an increasing sequence \(j_k\) such that \(\| f_{j_{k+1}} - f_{j_k} \| \leq 2^{-k}\), hence

\[
(17) \quad \int_\Omega (dd^c u(f_{j_{k+1}} - f_{j_k}))^n \leq 2^{-nk}.
\]

by (2). Note that

\[f_{j_{k+1}} = f_{j_k} + \sum_{i=1}^k (f_{j_{i+1}} - f_{j_i})\]

and then by using (17) together with Lemma 2.3 it follows that

\[
\left( \int_\Omega (dd^c u(f_{j_{k+1}}))^n \right)^{\frac{1}{n}} \leq \left( \int_\Omega (dd^c u(f_{j_k}))^n \right)^{\frac{1}{n}} + \sum_{i=1}^k \left( \int_\Omega (dd^c u(f_{j_{i+1}} - f_{j_i}))^n \right)^{\frac{1}{n}} \leq \| f_{j_k} \| + 1.
\]

Thus, \(\sup_k \int_\Omega (dd^c u(f_{j_{k+1}}))^n \leq (\| f_k \| + 1)^n < +\infty\). Lemma 3.2 now yields that the total mass of \((dd^c u(f))^n\) is bounded by \((\| f_k \| + 1)^n\), i.e., \(f\) satisfies P2. Now assume that P1 is false, then there exists a \(\xi \in \partial \Omega\) and a sequence \([z_l]\) in \(\Omega\) such that \([z_l]\) converges to \(\xi\) and \(\lim_{l \to \infty} u(z_l) = -a < 0\), where \(a \geq 0\) is a constant. The sequence \([f_j]\) converges uniformly to \(f\) on \(\partial \Omega\), hence there exists \(m \in \mathbb{N}\) such that \(\| f_j - f \|_\infty \leq \frac{a}{2}\) for all \(j \geq m\). Lemma 3.2 yields that \(\| u(f_j)(z_l) - u(f)(z_l) \| \leq 2\| f_j - f \|_\infty < \frac{a}{2}\) and a contradiction has been achieved since

\[
\lim_{l \to \infty} (u(f_j)(z_l) - u(f)(z_l)) = a.
\]

Thus, \(f\) is a compliant function and this proof is completed.

**Corollary 3.4.** If \([f_j]\) is a sequence which converges in \((C\bar{P}(\partial \Omega), \| \cdot \|)\) to a function \(f\), then \([(dd^c u(f_j))^n]\) converges to \((dd^c u(f))^n\) in the weak∗-topology. In other words the map \(C\bar{P}(\partial \Omega) \ni f \to (dd^c u(f))^n\) is continuous.

**Theorem 3.5.** The set \(C\bar{P}_0(\partial \Omega)\) is a closed subspace of \(C\bar{P}(\partial \Omega)\). Moreover, \(C\bar{P}_0(\partial \Omega) = PH(\partial \Omega)\).

**Proof.** Inequality (16) implies that \(C\bar{P}_0(\partial \Omega)\) is a subspace. Now assume that \([f_j]\) is a sequence in \(C\bar{P}_0(\partial \Omega)\) which converges in norm to a function \(f\). Corollary 3.4 implies that \([(dd^c u(f_j))^n]\) converges to \((dd^c u(f))^n\) in the weak∗-topology. By assumption \(u(f_j) = 0\), hence \((dd^c u(f_j))^n = 0\) and therefore it follows by Corollary 3.4 that \((dd^c u(f))^n = 0\). Hence \(u(f) = 0\), i.e., \(f \in C\bar{P}_0(\partial \Omega)\) which yields that \(C\bar{P}_0(\partial \Omega)\) is closed.

To prove the second statement let \(f \in C\bar{P}_0(\partial \Omega)\), i.e., \(u(f)\) is identically 0. This together will Walsh’s theorem implies that \(PB_f, PB_{-f} \in PH(\Omega) \cap\)
Thus, $PB_f \in \mathcal{PH}(\Omega) \cap \mathcal{C}(\bar{\Omega})$. On the other hand, if there exists a function $u \in \mathcal{PH}(\Omega) \cap \mathcal{C}(\bar{\Omega})$ such that $u = f$ on $\partial\Omega$, then $dd^c u = 0$ and therefore $u = PB_f$. Similarly, $-u = PB_{-f}$. Thus, $u(f)$ is identically 0.

**Remark.** It was proved in [7] that if $D$ is a hyperconvex product domain $D = D_{n_1} \times \cdots \times D_{n_k}, n = n_1 + \cdots + n_k, k \geq 3$, then $\mathcal{CP}_0(\partial D) = \mathcal{CP}(\partial D) = \mathcal{PH}(\partial D)$. Consider $f(z_1, z_2) = |z_1|^2$ defined on the boundary of the unit ball in $\mathbb{C}^2$. This simple example shows by some straightforward calculations that $\mathcal{CP}_0(\partial B)$ is generally not equal to $\mathcal{CP}(\partial B)$.

### 4. Examples

Example 4.1 shows that it is not enough to assume that $f$ is a $C^1$-function to ensure that P2 is true.

**Example 4.1.** Let $B \subseteq \mathbb{C}^n$ be the unit ball and $z = (z', z_n) \in B$. For fixed $0 < p < 1$, let $f_p : \partial B \to \mathbb{R}$ be the function defined by $f_p(z', z_n) = |z_n|^{2p}$. Then

$$PB_{f_p}(z', z_n) = |z_n|^{2p} \quad \text{and} \quad PB_{-f_p}(z', z_n) = -(1 - |z'|^2)^p.$$

Set $E = \{(z', z_n) \in B : z_n = 0\} \cup \{(z', z_n) \in B : |z'| = 1\}$. The set $E$ is a pluripolar, hence

$$\int_{\Omega} (dd^c(PB_{f_p} + PB_{-f_p}))^n = \int_{\Omega \setminus E} (dd^c(PB_{f_p} + PB_{-f_p}))^n = C \int_0^1 |r|^{2np-2n+1}(1 - r^2)^{n-1}dr,$$

where $C > 0$ is a constant only depending on $n$ and $p$. Thus $f_p$ is not compliant if, and only if, $p \leq \frac{n-1}{n}$. For $n > 2$, the function $f_{\frac{n-1}{n}}$ belongs to $C^1(\partial B)$.

Example 4.2 shows that even if P2 is true and $f \in C^\infty(\partial \Omega)$, it may happen that P1 is false. This phenomenon cannot happen if $\Omega$ is a B-regular domain (see e.g. [12] for the definition and elementary properties of B-regular domains).

**Example 4.2.** Let $P$ be the unit polydisc in $\mathbb{C}^2$, i.e., $P = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\}$ and let the function $f : \partial P \to \mathbb{R}$ be defined by $f(z_1, z_2) = |z_2|^2$. Then $f \in C^\infty(\partial P) PB_f(z_1, z_2) = |z_2|^2$. From (1) it follows that $PB_{-f}(z_1, z_2) \geq -1$. Fix $(z_0, w_0) \in P$ and let $v_{z_0}$ be a function defined on the unit disc $D$ in $\mathbb{C}$ by $v_{z_0}(\zeta) = PB_{-f}(z_0, \zeta)$, hence $v_{z_0} \in \mathcal{PH}(D)$ and $\limsup_{\zeta \to \xi} v_{z_0}(\zeta) \leq -1$ for every $\xi \in \partial D(0, 1)$, hence $v_{z_2} \leq -1$ on
the connection between the Cegrell classes...  

$D(0, 1)$ by the maximum principle for subharmonic functions and therefore $PB_{-f} \leq -1$ on $P$. Thus, $PB_{-f}(z_1, z_2) = -1$. Moreover,

$$
\int_P (dd^c(PB_f + PB_{-f}))^2 = 0.
$$

Thus P2 is true, but

$$
\lim_{(z_1, z_2) \to (\xi_1, \xi_2)} (PB_f(z_1, z_2) + PB_{-f}(z_1, z_2)) \neq 0,
$$

when $(\xi_1, \xi_2) \in \partial P \setminus \{(w_1, w_2) \in C^2 : |w_2| = 1\}$, hence P1 is false.

**Proposition 4.3.** If $D \subseteq C^n$ is a bounded, strictly pseudoconvex domain with $C^2$-boundary and $f \in C^2(\partial D)$, then $f$ is a compliant function.

**Proof.** The domain $D$ is in particular B-regular and the function $f$ is in particular continuous and therefore it follows that P1 is true. There exists an open neighbourhood $U$ of $D$ and a strictly plurisubharmonic $C^2$-function $\rho : U \to \mathbb{R}$ such that $\rho = 0$ on $\partial \Omega$, since $D$ is a strictly pseudoconvex domain with $C^2$-boundary. By Theorem I in [14] there exists a $C^2$-function $\tilde{f} : C^n \to \mathbb{R}$ such that $\tilde{f} = f$ on $\partial D$. Choose $A > 0$ such that $u = (\tilde{f} + A\rho) \in \mathcal{P}H(D)$ and $B > 0$ such that $v = (-\tilde{f} + B\rho) \in \mathcal{P}H(D)$. Hence, $u, v \in \mathcal{P}H(U) \cap C^2(U), u = -v = f$ on $\partial D$. Thus

$$
(18) \quad \int_D (dd^c(u + v))^n = \int_D \sum_{k=0}^{n} \binom{n}{k} (dd^c u)^{n-k} \wedge (dd^c v)^k < +\infty.
$$

The construction of $PB_f$ and $PB_{-f}$ implies that $u + v \leq PB_f + PB_{-f}$, hence

$$
\int_D (dd^c(PB_f + PB_{-f}))^n \leq \int_D (dd^c(u + v))^n < +\infty,
$$

by (18) and the comparison principle. Thus P2 holds.

**Remark.** Let $k \in \{0, 1, 2, \ldots \} \cup \{+\infty\}$. See [14] for the definition of how a function $f : \partial \Omega \to \mathbb{R}$ is of class $C^k$ and their basic properties.

**REFERENCES**