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ON SYMMETRIC WORDS IN THE SYMMETRIC GROUP OF DEGREE THREE

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Abstract

A word $w(x_1, x_2, \dots, x_n)$ from absolutely free group F_n is called symmetric n-word in a group G, if the equality $w(g_1, g_2, \dots, g_n) = w(g_{\sigma 1}, g_{\sigma 2}, \dots, g_{\sigma n})$ holds for all $g_1, g_2, \dots, g_n \in G$ and all permutations $\sigma \in S_n$. The set $\mathbf{S}^{(n)}(G)$ of all symmetric n-words is a subgroup of F_n. In this paper the groups of all symmetric 2-words and 3-words for the symmetric group of degree 3 are determined.

0. Introduction

Let $\mathscr{A} = (A, \mathbf{F})$ be an algebra with a family $\mathbf{F}$ of fundamental finitary operations on A and let $\mathbf{A}^{(n)}(\mathscr{A})$ denote the set of all n-ary polynamials of the algebra $\mathscr{A}$. An element $f \in \mathbf{A}^{(n)}(\mathscr{A})$ is called symmetric if the equality $f(a_{\sigma 1}, a_{\sigma 2}, \dots, a_{\sigma n}) = f(a_1, a_2, \dots, a_n)$ holds for all $a_1, a_2, \dots, a_n \in A$ and all permutations $\sigma \in S_n$. The set of all symmetric polynomials of $\mathscr{A}$ is denoted by $\mathbf{S}^{(n)}(\mathscr{A})$. In the case of a group G the set $\mathbf{A}^{(n)}(G)$ consists of all functions $G^n \ni (g_1, g_2, \dots, g_n) \longrightarrow w(g_1, g_2, \dots, g_n)$, where $w(x_1, x_2, \dots, x_n)$ is a word of n variables, i.e. an element of the free group $\mathscr{F}_n$ on free generators $x_1, x_2, \dots, x_n$. Let $\mathscr{V}_n(G) = \mathscr{V}_n$ be the subgroup of $\mathscr{F}_n$ consisting of all words w such that $w(g_1, g_2, \dots, g_n) = 1$ for all $g_1, g_2, \dots, g_n \in G$. For a permutation $\sigma \in S_n$ the mapping $x_i \longrightarrow x_{\sigma(i)}, 1 \le i \le n, \sigma \in S_n$, define an automorphism φ_σ of $\mathscr{F}_n$. Namely we have $\varphi_\sigma(w)(x_1, x_2, \dots, x_n) = w(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$. Since $\mathscr{V}_n$ is complete characteristic subgroup of $\mathscr{F}_n$, the automorphism φ_σ induces an automorphism $\overline{\varphi}_\sigma$ of the factor group $\mathscr{F}_n/\mathscr{V}_n$. Clearly two words $w, v \in \mathscr{F}_n$ yield the same element of $\mathbf{A}^{(n)}(G)$ if and only if $wv^{-1} \in \mathscr{V}_n$ and therefore the set $\mathbf{S}^{(n)}(G)$ of all n-ary symmetric operations in G can be identified with the subgroup of the group $\mathscr{F}_n/\mathscr{V}_n$ consisting of all elements which are stable under the action of all automorphisms $\overline{\varphi}_\sigma, \sigma \in S_n$. Thus

$$\begin{aligned}\mathbf{S}^{(n)}(G) &= \{w \cdot \mathscr{V}_n \in \mathscr{F}_n/\mathscr{V}_n : \overline{\varphi}_\sigma(w \cdot \mathscr{V}_n) = w \cdot \mathscr{V}_n \text{ for all } \sigma \in S_n\} \\ &= \{w \in \mathscr{F}_n/\mathscr{V}_n : \varphi_\sigma(w) \cdot w^{-1} \in \mathscr{V}_n \text{ for all } \sigma \in S_n\}.\end{aligned}$$

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The question of characterization of symmetric words of n variables (shortly n-words) in groups was initiated in [11] and [12]. It has been done for arbitrary nilpotent groups of class ≤ 3 and in the case of symmetric 2-words also for dihedral group of order $2p$. In the papers [2] and [9] 2-words are determined for free metabelian groups and soluble groups of derived length 3. A description of the groups $\mathbf{S}^{(2)}(G)$ and $\mathbf{S}^{(3)}(G)$ for free metabelian and free metabelian, nilpotent group G is given in [4]. The same for free nilpotent groups of class 4 and 5 has been done in [5], [6] and [7]. Very recently all symmetric n-words for free metabelian groups are characterized in [8]. Some applications of symmetric words one can find in [13]. In this note all symmetric 2-words for the symmetric group S_3 are listed and using this we determine the group $\mathbf{S}^{(3)}(S_3)$. Unexpectedly enough it turns out that it is isomorphic to the group $(\mathsf{Z}_3)^6$, whereas the group $\mathbf{S}^{(2)}(S_3)$ is non-Abelian. Moreover we prove that all groups $S^{(n)}(S_3)$ with the exception $n = 2$ are commutative.

1. Preliminaries

Let us denote $e = (1, 2, 3)$, $\mathbf{1} = (2, 3, 1)$, $\mathbf{2} = (3, 1, 2)$, $\varphi = (2, 1, 3)$, $\varphi\mathbf{1} = (3, 2, 1)$ and $\varphi\mathbf{2} = (1, 3, 2)$. We use standard notation:

$$x^{-1}yx = y^x, \quad x^{-1}y^{-1}xy = [x, y], \quad x^{y+\alpha z+\beta} = x^y(x^\alpha)^z x^\beta, \quad x^0 = e$$

for arbitrary group elements x, y, z and all integers α, β. We often shall make use the following simple

STATEMENTS. (i) *The following relations*

$$yx = xy[y, x], \quad [y, x]^{-1} = [x, y], \quad [xy, z] = [x, z][x, z, y][y, z],$$
$$[x, yz] = [x, z][x, y][x, y, z]$$

are identities in any group.

(ii) *The following equations*

$$x^6 = 1, \tag{1}$$
$$[x, y]^3 = 1, \tag{2}$$
$$[x^2, [y, z]] = 1, \tag{3}$$
$$[[x, y], [z, u]] = 1 \tag{4}$$

are identities in S_3. The group S_3 is metabelian (comp. [10]) and therefore the Jacobi identity $J(x, y, z) = 1$, i.e. the equality

$$J(x, y, z) = [y, x]^{1-z}[z, x]^{y-1}[z, y]^{1-x} = 1$$

is an identity in S_3.

(iii) *For arbitrary group G the relation $s(x_1, x_2, \dots, x_n) \in \mathbf{S}^{(n)}(G)$ implies $s(x_1, x_2, \dots, x_{n-1}, 1) \in \mathbf{S}^{(n-1)}(G)$.*

(iv) *If two words $w, v \in \mathbf{A}^{(2)}(S_3)$ are equal on the pairs $(\mathbf{1}, e)$, $(e, \mathbf{1})$, $(\mathbf{1}, \varphi)$, $(\varphi, \mathbf{1})$, $(\varphi, \mathbf{1}\varphi)$ and $(\mathbf{1}\varphi, \varphi)$, then $w(x, y) = v(x, y)$ is an identity in S_3 ([11, Lemma]).*

2. Auxiliary results

We begin with

THEOREM 1. *The group $\mathbf{S}^{(2)}(S_3)$ consists of 18 elements*

$1 \cdot 1 \cdot 1 = 1$	$1 \cdot 1v = x^2y^2[y, x]^{x-y}$	$1 \cdot 1v^2 = x^4y^4[y, x]^{y-x}$
$s1 \cdot 1 = [y, x]^{x-y}$	$s1v = x^2y^2[y, x]^{y-x}$	$s1v^2 = x^4y^4$
$s^21 \cdot 1 = [y, x]^{y-x}$	$s^21v = x^2y^2$	$s^21v^2 = x^4y^4[y, x]^{x-y}$
$1t1 = x^3y^3[y, x]^{-x-y}$	$1tv = x^5y^5[y, x]^{1+x+y}$	$1tv^2 = xy[y, x]^{-1}$
$st1 = x^3y^3[y, x]^x$	$stv = x^5y^5[y, x]^{1-y}$	$stv^2 = xy[y, x]^{-1-x+y}$
$s^2t1 = x^3y^3[y, x]^y$	$s^2tv = x^5y^5[y, x]^{1-x}$	$s^2tv^2 = xy[y, x]^{-1+x-y}$

where

$$s = [y, x]^{x-y}, \quad t = x^3y^3[y, x]^{-x-y} \quad \textit{and} \quad v = x^2y^2[y, x]^{x-y}.$$

Thus the group $\mathbf{S}^{(2)}(S_3)$ is the direct product of subgroup $\mathrm{gp}\{s, t\} \cong S_3$ and cyclic group $\mathrm{gp}\{v\}$ of order 3.

PROOF. It was proved in [11, Theorem 2] that the words $w = xy[y, x]^{-1}$ and $u = x^2y^2$ are symmetric in the group S_3 and that the group $\mathbf{S}^{(2)}(S_3)$ is generated by this words. Therefore s, t and v are symmetric words in S_3. Using the identities from (i) and (ii) one can easily verify the relations $s^3 = 1, t^2 = 1, v^3 = 1, st = t^2s, sv = vs$ and $tv = vt$. Hence the group $\mathrm{gp}\{s, t\}$ is isomorphic to S_3, and thus $\mathrm{gp}\{s, t\} \times \{1, v, v^2\} = \mathbf{S}^{(2)}(S_3)$.

LEMMA 1. *If for integers a, b, c and d the equality*

$$[y, x]^{ax+by+cxy+d} = 1 \tag{5}$$

holds for all $x, y \in S_3$ then $a \equiv b \equiv c \equiv d \pmod 3$.

PROOF. We use (iv). Since the equality (5) holds for the pairs $(\mathbf{1}, e)$, $(e, \mathbf{1})$ and all integers a, b, c and d, we can restrict ourself to four pairs $(\mathbf{1}, \varphi)$, $(\varphi, \mathbf{1})$, $(b, \mathbf{1}\varphi)$ and $(\mathbf{1}\varphi, \varphi)$. By (iv) the equality (5) is an identity in the group S_3 if and only if the integers a, b, c and d satisfy the following system of equalities

$$\begin{cases} \mathbf{2}^a \cdot \mathbf{1}^b \cdot \mathbf{1}^c \cdot \mathbf{2}^d = e \\ \mathbf{2}^a \cdot \mathbf{1}^b \cdot \mathbf{2}^c \cdot \mathbf{1}^d = e \\ \mathbf{1}^a \cdot \mathbf{1}^b \cdot \mathbf{2}^c \cdot \mathbf{2}^d = e \\ \mathbf{2}^a \cdot \mathbf{2}^b \cdot \mathbf{1}^c \cdot \mathbf{1}^d = e \end{cases}$$

Since the mapping $e \longrightarrow 0$, $\mathbf{1} \longrightarrow 1$, $\mathbf{2} \longrightarrow 2$ is an isomorphism of the permutation group $(\{e, \mathbf{1}, \mathbf{2}\}; \circ)$ onto the cyclic group Z_3, the last system is equivalent to the homogenous system of equations

$$\begin{cases} 2a + b + c + 2d = 0 \\ 2a + b + 2c + d = 0 \\ a + b + 2c + 2d = 0 \\ 2a + 2b + c + d = 0 \end{cases}$$

where $+$ is taken modulo 3, of course. Let us observe that vector $(1, 1, 1, 1)$ is a solution of the system. Since the rank of the matrix of the system is 3, the set $\{(1, 1, 1, 1), (2, 2, 2, 2), (0, 0, 0, 0)\}$ consists of all solutions.

Corollary 1. *The relation*

$$[y, x]^{1+x+y+xy} = 1$$

is an identity in the group S_3.

Corollary 2. *If for some integers a, b, c and d the equality*

$$[y, x]^{ax+by+cxy+d} = 1$$

holds for all $x, y \in S_3$ and at least one from the integers a, b, c or d is 0 (mod 3)*, then all the integers have to be equal* 0 (mod 3).

Lemma 2. *The equality*

$$[y, x]^{(1-z)(a+bx+cy+dz)}[z, x]^{f(y-1)(x-y)} = 1 \tag{6}$$

holds for all $x, y, z \in S_3$ if and only if $c \equiv a - b - d \equiv f - b \equiv 0$ (mod 3).

Proof. If we put $z = x$ into (5) then we get

$$[y, x]^{(a-b-d)+(b+d-a)x+cy-cxy}.$$

It follows from Lemma 1 that $c \equiv a - b - d \equiv 0 \pmod 3$. Because of the equality $[y, x]^{(1-z)(1+z)} = 1$ we can rewrite (5) as

$$[y, x]^{b(1-z)(x+1)}[z, x]^{f(y-1)(x-y)} = 1.$$

By putting $z = y$ and taking into account Lemma 1 we get congruence $f \equiv b \pmod 3$.

Conversely, if the congruences hold then the equality (5) is of the form

$$[y, x]^{b(x+1)(1-z)}[z, x]^{b(x+1)(y-1)} = 1,$$

which is identical with the Jacobi identity $J(x, y, z)^{b(x+1)} = 1$.

3. Symmetric 3-words

Let w be a word of 3 variables in general form

$$w = x^{\alpha_1} y^{\beta_1} z^{\gamma_1} x^{\alpha_2} y^{\beta_2} z^{\gamma_2} \cdots x^{\alpha_n} y^{\beta_n} z^{\gamma_n}, \tag{7}$$

where $\alpha_i, \beta_i, \gamma_i \in \mathbb{Z}$, $i = 1, 2, \ldots, n$ and $n \in \mathbb{N}$. In view of the identity (1), we may assume that all integers α_i, β_i and γ_i belong to the set $\{0, 1, 2, 3, 4, 5\}$. Using the identities from (i) it is possible to remove each x of the word (7) at the first place. One obtains a word of the form $x^{\alpha_1} uv \ldots$, where $u, v, \ldots$ are variables y, z or commutators of the form $[y, x]^{x^i}$ and $[z, x]^{x^j}$ for some $i, j = 1, 2, \ldots$. Since squares of elements of the group S_3 commutes with commutators (comp. (3)), one can assume i and j equal 0 or 1. Now we remove all $y's$ at the second place and apply (ii). We get a word $x^{\alpha} y^{\beta} u'v' \cdots$, where $u', v', \ldots$ are words of the form z, $[y, x]^{x^i y^j}$, $[z, x]^{x^k y^l}$ or $[z, x]^{y^m}$ for some $i, j, k, l, m \in \{0, 1\}$. Clearly, the same collecting process can be made with the last variable z. This together with Corollary 2 gives

Lemma 3. *Any word of variables x, y and z in the group S_3 is equivalent modulo $\mathcal{V}_3(S_3)$ to the following word*

$$\begin{aligned} w(x, y, z) = x^a y^b z^c &\cdot [y, x]^{\alpha_0+\alpha_1 x+\alpha_2 y+\alpha_3 z+\alpha_{13} xz+\alpha_{23} yz} \\ &\cdot [z, x]^{\beta_0+\beta_1 x+\beta_2 y+\beta_3 z+\beta_{12} xy+\beta_{23} yz} \\ &\cdot [z, y]^{\gamma_0+\gamma_1 x+\gamma_2 y+\gamma_3 z+\gamma_{12} xy+\gamma_{13} xz} \end{aligned} \tag{8}$$

for some elements $a, b, c \in \{0, 1, 2, 3, 4, 5\}$ and $\alpha_0, \ldots, \gamma_{13} \in \{0, 1, 2\}$.

From now on we write $w = u$ instead of $w \equiv u \pmod{\mathcal{V}_3(S_3)}$ and we prefer to write -1 than 2, when 2 is an exponent of a commutator. Now we are able to determine all words of three variables in the group S_3. First we show that the image of each symmetric 3-word in S_3 treated as a function $S_3^3 \longrightarrow S_3$ is contained in the commutator subgroup of S_3. More precisely we have

THEOREM 2. *Let $s(x, y, z)$ be a symmetric 3-word of the form (8) in the group S_3. Then $a = b = c = 2i$ for $i = 0$, 1 or 2.*

PROOF. Clearly $s(x, 1, 1) = s(1, x, 1) = s(x, 1, 1)$ and therefore we have $a \equiv b \equiv c \pmod 6$.

Let us suppose that $a = 1$. We have

$$\begin{aligned} s(x, y, 1) &= xy[y, x]^{(\alpha_0+\alpha_3)+(\alpha_1+\alpha_{13})x+(\alpha_2+\alpha_{23})y}, \\ s(x, 1, y) &= xy[y, x]^{(\beta_0+\beta_2)+(\beta_1+\beta_{12})x+(\beta_3+\beta_{23})y}, \\ s(1, x, y) &= xy[y, x]^{(\gamma_0+\gamma_1)+(\gamma_2+\gamma_{12})x+(\gamma_3+\gamma_{13})y}. \end{aligned}$$

By (iii) $s(x, y, 1)$ is a symmetric 2-word in S_3. It follows from Theorem 1 that $s(x, y, 1)$ equals

$$xy[y, x]^{-1} \qquad \text{or} \qquad xy[y, x]^{-1-x+y} \qquad \text{or} \qquad xy[y, x]^{-1+x-y}.$$

Without loss of generality we can assume that

$$s(x, y, 1) = xy[y, x]^{-1}.$$

Indeed, it is easy to verify the equalities $w(x, y, z) = w(y, x, z) = w(x, z, y)$, where

$$w(x, y, z) = [y, x]^{(y-x)z}[z, x]^{(z-x)y}[z, y]^{(z-y)x},$$

which means that w is a symmetric 3-word in the group S_3 such that $w(x, y, 1) = [y, x]^{y-x}$. Therefore if $s(x, y, 1)$ does not equal $xy[y, x]^{-1}$, then we can consider $s \cdot w$ or $s \cdot w^2$ instead of s.

By Corollary 2 we get the congruences

$$\begin{aligned} (9) \quad \alpha_0 + \alpha_3 + 1 &\equiv \beta_0 + \beta_2 + 1 \equiv \gamma_0 + \gamma_1 + 1 \equiv \alpha_1 + \alpha_{13} \equiv \beta_1 + \beta_{12} \\ &\equiv \gamma_2 + \gamma_{12} \equiv \alpha_2 + \alpha_{23} \equiv \beta_3 + \beta_{23} \equiv \gamma_3 + \gamma_{13} \equiv 0 \pmod 3. \end{aligned}$$

Therefore we can rewrite the word s as

$$\begin{aligned} s(x, y, z) = xyz &\cdot [y, x]^{(1-z)(\alpha_0+\alpha_1 x+\alpha_2 y)-z} \\ &\cdot [z, x]^{(1-y)(\beta_0+\beta_1 x+\beta_3 z)-y} \\ &\cdot [z, y]^{(1-x)(\gamma_0+\gamma_2 y+\gamma_3 z)-x}. \end{aligned}$$

Now using (i) and (ii) we get

$$\begin{aligned} s(y,x,z) = xyz[y,x]^z \cdot [y,x]^{(1-z)(-\alpha_0-\alpha_2 x-\alpha_1 y)+z} \\ \cdot [z,x]^{(1-y)(\gamma_0+\gamma_2 x+\gamma_3 z)-y} \\ \cdot [z,y]^{(1-x)(\beta_0+\beta_1 y+\beta_3 z)-x} \end{aligned}$$

and similarly

$$\begin{aligned} s(x,z,y) = xyz[z,y] \cdot [y,x]^{(1-z)(\beta_0+\beta_1 x+\beta_3 y)-z} \\ \cdot [z,x]^{(1-y)(\alpha_0+\alpha_1 x+\alpha_2 z)-y} \\ \cdot [z,y]^{(1-x)(-\gamma_0-\gamma_3 y-\gamma_2 z)+x}. \end{aligned}$$

The condition $s(x,y,z) = s(y,x,z)$ implies

$$\begin{aligned} (10)\quad [y,x]^{(1-z)\{(-\alpha_0+(\alpha_1+\alpha_2)x+(\alpha_1+\alpha_2)y\}} \\ \cdot [z,x]^{(1-y)\{(\beta_0-\gamma_0)+(\beta_1-\gamma_2)x+(\beta_3-\gamma_3)z\}} \\ \cdot [z,y]^{(1-x)\{(\gamma_0-\beta_0)+(\gamma_2-\beta_1)y+(\gamma_3-\beta_3)z\}} = 1, \end{aligned}$$

which in the case $z = x$ gives

$$[y,x]^{(1-x)\{(-\alpha_0+\beta_0-\gamma_0)+(\alpha_1+\alpha_2+\beta_3-\gamma_3)x+(\alpha_1+\alpha_2+\beta_1-\gamma_2)y\}} = 1,$$

This, by Lemma 2, implies the congruences

$$\begin{aligned} \beta_1 \equiv \gamma_2 - \alpha_1 - \alpha_2 \pmod 3 \\ -\alpha_0 + \beta_0 - \gamma_0 \equiv \alpha_1 + \alpha_2 + \beta_3 - \gamma_3 \pmod 3 \end{aligned}$$

Similarly the equality $s(x,y,z) = s(x,z,y)$ gives

$$\begin{aligned} (11)\quad [y,x]^{(1-z)\{(\alpha_0-\beta_0)+(\alpha_1-\beta_1)x+(\alpha_2-\beta_3)y\}} \\ \cdot [z,x]^{(1-y)\{(\beta_0-\alpha_0)+(\beta_1-\alpha_1)x+(\beta_3-\alpha_2)z\}} \\ \cdot [z,y]^{(1-x)\{(-\gamma_0-1+(\gamma_2+\gamma_3)y+(\gamma_2+\gamma_3)z\}} = 1, \end{aligned}$$

which in the case $y = x$ together with Lemma 2 yields the congruences

$$\begin{aligned} \beta_3 \equiv \alpha_2 - \gamma_2 - \gamma_3 \pmod 3 \\ -\alpha_0 + \beta_0 - \gamma_0 - 1 \equiv \beta_1 - \alpha_1 + \gamma_2 + \gamma_3 \pmod 3. \end{aligned}$$

After eliminating β_1 and β_3 from above system of four congruences we see that it has no solution.

THEOREM 3. *Let $s(x,y,z)$ be a symmetric 3-word in the group S_3, then*

$$(12)\quad s = u^i \cdot s_0^j \cdot s_1^k \cdot s_2^l \cdot s_3^m \cdot w^n \qquad \textit{for some} \quad i,j,k,l,m,n \in \{0,1,-1\},$$

where

$$u(x,y,z) = x^2y^2z^2,$$
$$s_0(x,y,z) = [y,x]^{z-1}[z,x]^{y-1},$$
$$s_1(x,y,z) = [y,x]^{(1-z)x}[z,x]^{1-y}[z,y]^{(1-x)(y-z)},$$
$$s_2(x,y,z) = [y,x]^{(1-z)y}[z,x]^{1-y}[z,y]^{(1-x)(y+z)}$$
$$s_3(x,y,z) = [z,x]^{(1-y)(x+z)}[z,y]^{(1-y)(y+z)},$$
$$w(x,y,z) = [y,x]^{(y-x)z}[z,x]^{(z-x)y}[z,y]^{(z-y)x}.$$

Presentation (12) *is unique and therefore* $\mathbf{S}^{(3)}(S_3)$ *is Abelian group isomorphic to* $(\mathsf{Z}_3)^6$.

Proof. Let $s(x,y,z)$ be a symmetric 3-word in S_3 of the form (8) with $a=b=c=0$. We have

$$s(x,y,1) = [y,x]^{(\alpha_0+\alpha_3)+(\alpha_1+\alpha_{13})x+(\alpha_2+\alpha_{23})y},$$
$$s(x,1,y) = [y,x]^{(\beta_0+\beta_2)+(\beta_1+\beta_{12})x+(\beta_3+\beta_{23})y},$$
$$s(1,x,y) = [y,x]^{(\gamma_0+\gamma_1)+(\gamma_2+\gamma_{12})x+(\gamma_3+\gamma_{13})y}.$$

By Theorem 1 every symmetric word $s(x,y,1)$ of two variables equals

$$\text{either} \quad 1 \quad \text{or} \quad [y,x]^{y-x} \quad \text{or else} \quad [y,x]^{x-y}.$$

Let us consider first case $s(x,y,1)=1$.

By Corollary 2 we have the following congruences

$$\begin{aligned}(13)\quad \alpha_0+\alpha_3 &\equiv \beta_0+\beta_2 \equiv \gamma_0+\gamma_1 \equiv \alpha_1+\alpha_{13} \equiv \beta_1+\beta_{12} \\ &\equiv \gamma_2+\gamma_{12} \equiv \alpha_2+\alpha_{23} \equiv \beta_3+\beta_{23} \equiv \gamma_3+\gamma_{13} \equiv 0 \pmod 3,\end{aligned}$$

which enables us to rewrite the word s in the form

$$\begin{aligned}s(x,y,z) = [y,x]&^{(1-z)(\alpha_0+\alpha_1x+\alpha_2y)} \\ &\cdot [z,x]^{(1-y)(\beta_0+\beta_1x+\beta_3z)} \\ &\cdot [z,y]^{(1-x)(\gamma_0+\gamma_2y+\gamma_3z)}.\end{aligned}$$

It is well known the transpositions $(1,2)$ and $(2,3)$ generate the symmetric group S_3 of degree 3 and therefore $s(x,y,z)$ is symmetric if and only if two equalities

$$s(y,x,z)^{-1}\cdot s(x,y,z)=1, \qquad s(x,z,y)^{-1}\cdot s(x,y,z)=1$$

hold for all elements x, y, z from S_3. We check

$$s(y,x,z) = [y,x]^{(1-z)(-\alpha_0-\alpha_2 x-\alpha_1 y)} \cdot [z,x]^{(1-y)(\gamma_0+\gamma_2 x+\gamma_3 z)} \cdot [z,y]^{(1-x)(\beta_0+\beta_1 y+\beta_3 z)}$$

and similarly

$$s(x,z,y) = [y,x]^{(1-z)(\beta_0+\beta_1 x+\beta_3 y)} \cdot [z,x]^{(1-y)(\alpha_0+\alpha_1 x+\alpha_2 z)} \cdot [z,y]^{(1-x)(-\gamma_0-\gamma_3 y-\gamma_2 z)}.$$

Hence we get

$$\begin{aligned} f(x,y,z) &= s(y,x,z)^{-1} \cdot s(x,y,z) \\ &= [y,x]^{(1-z)\{-\alpha_0+(\alpha_1+\alpha_2)x+(\alpha_1+\alpha_2)y\}} \\ &\quad \cdot [z,x]^{(y-1)\{(\gamma_0-\beta_0)+(\gamma_2-\beta_1)x+(\gamma_3-\beta_3)z\}} \\ &\quad \cdot [z,y]^{(1-x)\{(\gamma_0-\beta_0)+(\gamma_2-\beta_1)y+(\gamma_3-\beta_3)z\}} \end{aligned} \tag{14}$$

and also

$$\begin{aligned} g(x,y,z) &= s(x,z,y)^{-1}(s(x,y,z) \\ &= [y,x]^{(1-z)\{(\alpha_0-\beta_0)+(\alpha_1-\beta_1)x+(\alpha_2-\beta_3)y\}} \\ &\quad \cdot [z,x]^{(y-1)\{(\alpha_0-\beta_0)+(\alpha_1-\beta_1)x+(\alpha_2-\beta_3)z\}} \\ &\quad \cdot [z,y]^{(1-x)\{-\gamma_0+(\gamma_2+\gamma_3)y+(\gamma_2+\gamma_3)z\}} \end{aligned} \tag{15}$$

Thus s is symmetric if and only if the equalities $f(x,y,z) = g(x,y,z) = 1$ hold for all $x, y, z \in S_3$. Applying Jacobi identity

$$([y,x]^{1-z}[z,x]^{y-1}[z,y]^{1-x})^{\{(\gamma_0-\beta_0)+(\gamma_2-\beta_1)y+(\gamma_3-\beta_3)z\}} = 1$$

to the equality (14) and

$$([y,x]^{1-z}[z,x]^{y-1}[z,y]^{1-x})^{\{(\alpha_0-\beta_0)+(\alpha_1-\beta_1)x+(\alpha_2-\beta_3)y\}} = 1$$

to the equality (15) we see that $s(x,y,z)$ is symmetric 3-word if and only if the following two equalities

$$1 = f(x,y,z) = [y,x]^{(1-z)\{(\beta_0-\alpha_0-\gamma_0)+(\alpha_1+\alpha_2)x+(\alpha_1+\alpha_2+\beta_1-\gamma_2)y+(\beta_3-\gamma_3)z\}} \cdot [z,x]^{(\gamma_2-\beta_1)(y-1)(x-y)},$$

$$1 = g(x,y,z) = [z,y]^{(1-x)\{(\beta_0-\alpha_0-\gamma_0)+(\beta_1-\alpha_1)x+(\gamma_2+\gamma_3+\beta_3-\alpha_2)y+(\gamma_2+\gamma_3)z\}} \cdot [z,x]^{(\alpha_2-\beta_3)(y-1)(z-y)}$$

holds for all $x, y, z \in S_3$. By Lemma 2 this is equivalent to the following system of congruences

$$\alpha_1 + \alpha_2 + \beta_1 - \gamma_2 \equiv 0 \pmod 3,$$
$$-\alpha_0 + \beta_0 - \gamma_0 \equiv \alpha_1 + \alpha_2 + \beta_3 - \gamma_3 \pmod 3,$$
$$\beta_3 - \gamma_3 \equiv \gamma_2 - \beta_1 \pmod 3,$$
$$-\alpha_2 + \beta_3 + \gamma_2 + \gamma_3 \equiv 0 \pmod 3,$$
$$-\alpha_0 + \beta_0 - \gamma_0 \equiv \beta_1 - \alpha_1 + \gamma_2 + \gamma_3 \pmod 3.$$

Choosing $\alpha_0, \gamma_0, \alpha_1, \alpha_2$ and β_1 as parameters we obtain the following solution of the system

	0	1	2	3
α	α_0	α_1	α_2	$-\alpha_0$
β	$\alpha_0 + \gamma_0 - \alpha_1 - \alpha_2$	β_1	$\alpha_0 - \gamma_0 + \alpha_1 + \alpha_2$	$\beta_1 - \alpha_2$
γ	γ_0	$-\gamma_0$	$\alpha_1 + \alpha_2 + \beta_1$	$-\alpha_1 + \alpha_2 + \beta_1$

Thus the word s can be written as

$$s(x, y, z) = [y, x]^{(1-z)(\alpha_0+\alpha_1 x+\alpha_2 y)} \cdot [z, x]^{(1-y)((\alpha_0+\gamma_0-\alpha_1-\alpha_2)+\beta_1 x+(\beta_1-\alpha_2)z} \cdot [z, y]^{(1-x)(\gamma_0+(\alpha_1+\alpha_2+\beta_1)(y-z))},$$

or

$$s = s_0^{\alpha_0} \cdot s_4^{\gamma_0} \cdot s_1^{\alpha_1} s_2^{\alpha_2} \cdot s_3^{\beta_1},$$

where

$$s_0(x, y, z) = [y, x]^{z-1}[z, x]^{y-1},$$
$$s_4(x, y, z) = [z, x]^{y-1}[z, y]^{x-1},$$
$$s_1(x, y, z) = [y, x]^{(1-z)x}[z, x]^{(1-y)}[z, y]^{(1-x)(y-z)},$$
$$s_2(x, y, z) = [y, x]^{(1-z)y}[z, x]^{1-y}[z, y]^{(1-x)(y+z)},$$
$$s_3(x, y, z) = [z, x]^{(1-y)(x+z)}[z, y]^{(1-x)(y+z)}.$$

Above we have made use of (i) and (ii). Observe that

$$s_0(x, y, z) \cdot s_4(x, y, z) = J(x, y, z)^{-1} = 1,$$

which yields

$$s = s_0^j \cdot s_1^k \cdot s_2^l \cdot s_3^m \tag{16}$$

for some $j, k, l, m \in \{0, 1, -1\}$. We claim that the presentation (16) of the word s is unique. Indeed, if for some $j, k, l, m \in \{0, 1, -1\}$ the equality

$$s_0(x, y, z)^j s_1(x, y, z)^k s_2(x, y, z)^l s_3(x, y, z)^m = 1 \tag{17}$$

hold for all $x, y, z \in S_3$, then, in the case $z = x$, we get

$$[y, x]^{\{(-j+k+l+m)+(j-k-l-m)x-(k+m)y+(k+m)xy)\}} = 1,$$

which by Lemma 1 implies $l - j \equiv k + m \equiv 0 \pmod 3$. If we put $z = y$ into (16), we get

$$[y, x]^{(j+k-m)+(j+k+m)x+(j-k+m)y-(k+m)xy} = 1.$$

By Corollary 2 we have $j = k = l = m = 0$, as required.

As we have mentioned earlier $u = x^2y^2z^2$ is symmetric 3-word in S_3 and

$$w(x, y, z) = [y, x]^{(y-x)z}[z, x]^{(z-x)y}[z, y]^{(z-y)x}$$

is a symmetric 3-word with $w(x, y, 1) = [y, x]^{y-x}$. Let $s(x, y, z)$ be arbitrary symmetric 3-word in S_3 such that $s(x, y, 1) = (x^2y^2)^i[y, x]^{n(y-x)}$ for some $i, n = 0, 1, -1$. Then the following product $(u^{-i}sw^{-n})$ is a symmetric 3-word with $(u^{-i}sw^{-n})(x, y, 1) = 1$ and therefore, in view of what we have just established, $u^{-i}sw^{-n} = s_0^j s_1^k s_2^l s_3^m$ for some j, k, l, m, n, which completes the proof.

THEOREM 4. *For all $n \neq 2$ the groups $\mathbf{S}^{(n)}(S_3)$ of n-symmetric words of the group S_3 are commutative.*

PROOF. Let $s(x_1, x_2, \ldots, x_n)$ be symmetric n-word, $n \geq 3$, in S_3. Using the same arguments as in the proof of Theorem 1 it is possible to present the word s as $x_1^{a_1}x_2^{a_2}, \ldots, x_n^{a_n}c$, where c is a product of commutators of the form

$$[x_{i_1}, x_{i_2}]^P$$

for P being a polynomial in variables $x_1, x_2, \ldots, x_n$. Since $s(x, 1, 1, \ldots, 1) = s(1, x, 1, \ldots, 1) = \cdots = s(1, 1, \ldots, 1, x)$, we have the equality $a_1 \equiv a_2 \equiv \cdots \equiv a_n \equiv a \pmod 6$. In view of statement (iii) $s(x, y, z, 1, \ldots, 1) = x^a y^a z^a c'$ is a symmetric 3-word in S_3. By Theorem 2, the number a has to be even, which together with (2) of (ii) finishes the proof.

REMARK. Every symmetric n-word in a group G is symmetric in any group from the variety $\mathrm{var}(G)$ of groups generated by G and therefore the results of the paper are valid not only for the group S_3 but also for all groups from $\mathrm{var}(S_3) = HSP(S_3)$.

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