ON SYMMETRIC WORDS IN THE SYMMETRIC GROUP OF DEGREE THREE

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Abstract

A word $w(x_1, x_2, \ldots, x_n)$ from absolutely free group $F_n$ is called symmetric $n$-word in a group $G$, if the equality $w(g_1, g_2, \ldots, g_n) = w(g_{\sigma 1}, g_{\sigma 2}, \ldots, g_{\sigma n})$ holds for all $g_1, g_2, \ldots, g_n \in G$ and all permutations $\sigma \in S_n$. The set $S^{(n)}(G)$ of all symmetric $n$-words is a subgroup of $F_n$. In this paper the groups of all symmetric 2-words and 3-words for the symmetric group of degree 3 are determined.

0. Introduction

Let $\mathcal{A} = (A, F)$ be an algebra with a family $F$ of fundamental finitary operations on $A$ and let $\mathbf{A}^{(n)}(\mathcal{A})$ denote the set of all $n$-ary polynomials of the algebra $\mathcal{A}$. An element $f \in \mathbf{A}^{(n)}(\mathcal{A})$ is called symmetric if the equality $f(a_{\sigma 1}, a_{\sigma 2}, \ldots, a_{\sigma n}) = f(a_1, a_2, \ldots, a_n)$ holds for all $a_1, a_2, \ldots, a_n \in A$ and all permutations $\sigma \in S_n$. The set of all symmetric polynomials of $\mathcal{A}$ is denoted by $S^{(n)}(\mathcal{A})$. In the case of a group $G$ the set $\mathbf{A}^{(n)}(G)$ consists of all functions $G^n \ni (g_1, g_2, \ldots, g_n) \rightarrow w(g_1, g_2, \ldots, g_n)$, where $w(x_1, x_2, \ldots, x_n)$ is a word of $n$ variables, i.e. an element of the free group $F_n$ on free generators $x_1, x_2, \ldots, x_n$. Let $\mathcal{V}_n(G) = \mathcal{V}_n^c$ be the subgroup of $F_n$ consisting of all words $w$ such that $w(g_1, g_2, \ldots, g_n) = 1$ for all $g_1, g_2, \ldots, g_n \in G$. For a permutation $\sigma \in S_n$ the mapping $x_i \rightarrow x_{\sigma (i)}$, $1 \leq i \leq n$, $\sigma \in S_n$, define an automorphism $\varphi_\sigma$ of $F_n$. Namely we have $\varphi_\sigma (w)(x_1, x_2, \ldots, x_n) = w(x_{\sigma (1)}, x_{\sigma (2)}, \ldots, x_{\sigma (n)})$. Since $\mathcal{V}_n$ is complete characteristic subgroup of $F_n$, the automorphism $\varphi_\sigma$ induces an automorphism $\overline{\varphi}_\sigma$ of the factor group $F_n/\mathcal{V}_n$. Clearly two words $w, v \in F_n$ yield the same element of $\mathbf{A}^{(n)}(G)$ if and only if $w v^{-1} \in \mathcal{V}_n$ and therefore the set $S^{(n)}(G)$ of all $n$-ary symmetric operations in $G$ can be identified with the subgroup of the group $F_n/\mathcal{V}_n$ consisting of all elements which are stable under the action of all automorphisms $\overline{\varphi}_\sigma$, $\sigma \in S_n$. Thus

$$S^{(n)}(G) = \{ w \cdot \mathcal{V}_n \in F_n/\mathcal{V}_n : \overline{\varphi}_\sigma (w \cdot \mathcal{V}_n) = w \cdot \mathcal{V}_n \text{ for all } \sigma \in S_n \}$$

$$= \{ w \in F_n/\mathcal{V}_n : \overline{\varphi}_\sigma (w) \cdot w^{-1} \in \mathcal{V}_n \text{ for all } \sigma \in S_n \}.$$
The question of characterization of symmetric words of \(n\) variables (shortly \(n\)-words) in groups was initiated in [11] and [12]. It has been done for arbitrary nilpotent groups of class \(\leq 3\) and in the case of symmetric 2-words also for dihedral group of order \(2p\). In the papers [2] and [9] 2-words are determined for free metabelian groups and soluble groups of derived length 3. A description of the groups \(S^{(2)}(G)\) and \(S^{(3)}(G)\) for free metabelian and free metabelian, nilpotent group \(G\) is given in [4]. The same for free nilpotent groups of class 4 and 5 has been done in [5], [6] and [7]. Very recently all symmetric \(n\)-words for free metabelian groups are characterized in [8]. Some applications of symmetric words one can find in [13]. In this note all symmetric 2-words for the symmetric group \(S_3\) are listed and using this we determine the group \(S(n)\) of \(S_3\). Unexpectedly enough it turns out that it is isomorphic to the group \((\mathbb{Z}_3)^6\), whereas the group \(S^{(2)}(S_3)\) is non-Abelian. Moreover we prove that all groups \(S^{(n)}(S_3)\) with the exception \(n = 2\) are commutative.

1. Preliminaries

Let us denote \(e = (1, 2, 3)\), \(1 = (2, 3, 1)\), \(2 = (3, 1, 2)\), \(\varphi = (2, 1, 3)\), \(\varphi 1 = (3, 2, 1)\) and \(\varphi 2 = (1, 3, 2)\). We use standard notation:

\[
x^{-1}yx = y^x, \quad x^{-1}y^{-1}xy = [x, y], \quad x^{\alpha + \beta} = x^\alpha (x^\beta), \quad x^0 = e
\]

for arbitrary group elements \(x, y, z\) and all integers \(\alpha, \beta\). We often shall make use the following simple

**Statements.** (i) *The following relations*

\[
xy = xy[y, x], \quad [y, x]^{-1} = [x, y], \quad [xy, z] = [x, z][x, z, y][y, z],
\]

\[
x, yz] = [x, z][x, y][x, y, z]
\]

*are identities in any group.

(ii) *The following equations*

\[
(1) \quad x^6 = 1, \\
(2) \quad [x, y]^3 = 1, \\
(3) \quad [x^2, [y, z]] = 1, \\
(4) \quad [[x, y], [z, u]] = 1
\]

*are identities in \(S_3\). The group \(S_3\) is metabelian (comp. [10]) and therefore the Jacobi identity \(J(x, y, z) = 1, \text{i.e. the equality} \)

\[
J(x, y, z) = [y, x]^1[z, x]^y^{-1}[z, y]^{1-x} = 1
\]

*is an identity in \(S_3\).
(iii) For arbitrary group G the relation $s(x_1, x_2, \ldots, x_n) \in S^{[n]}(G)$ implies $s(x_1, x_2, \ldots, x_{n-1}, 1) \in S^{(n-1)}(G)$.

(iv) If two words $w, v \in A^{(2)}(S_3)$ are equal on the pairs $(1, e), (e, 1), (1, \varphi), (\varphi, 1), (\varphi, 1\varphi)$ and $(1\varphi, \varphi)$, then $w(x, y) = v(x, y)$ is an identity in $S_3$ ([11, Lemma]).

2. Auxiliary results

We begin with

**Theorem 1.** The group $S^{(2)}(S_3)$ consists of 18 elements

1. $1 \cdot 1 \cdot 1 = 1$
2. $1 \cdot 1v = x^2y^2[y, x]^{1-x}$
3. $1 \cdot 1v^2 = x^4y^4[y, x]^{1-x}$
4. $s1 = [y, x]^{-x}$
5. $s1v = x^2y^2[y, x]^{1-x}$
6. $s1v^2 = x^4y^4[y, x]^{1-x}$
7. $s^21 = [y, x]^{y-x}$
8. $s^21v = x^2y^2$
9. $s^21v^2 = x^4y^4[y, x]^{1-x}$
10. $1t1 = x^3y^3[y, x]^{-x-y}$
11. $1tv = x^5y^4[y, x]^{1-x}$
12. $1tv^2 = x^6y^4[y, x]^{1-x}$
13. $st1 = x^3y^3[y, x]^{y}$
14. $stv = x^5y^4[y, x]^{1-x}$
15. $stv^2 = x^6y^4[y, x]^{1-x}$
16. $s^2t1 = x^3y^3[y, x]^{y}$
17. $s^2tv = x^3y^3[y, x]^{1-x}$
18. $s^2tv^2 = x^5y^4[y, x]^{1-x}$

where

$s = [y, x]^{-x}, \quad t = x^3y^3[y, x]^{-x-y} \quad \text{and} \quad v = x^2y^2[y, x]^{1-x}$.

Thus the group $S^{(2)}(S_3)$ is the direct product of subgroup $gp(s, t) \cong S_3$ and cyclic group $gp(v)$ of order 3.

**Proof.** It was proved in [11, Theorem 2] that the words $w = xy[y, x]^{-1}$ and $u = x^2y^2$ are symmetric in the group $S_3$ and that the group $S^{(2)}(S_3)$ is generated by this words. Therefore $s, t$ and $v$ are symmetric words in $S_3$. Using the identities from (i) and (ii) one can easily verify the relations $s^3 = 1, t^2 = 1, v^3 = 1, st = t^2s, sv = vs$ and $tv = vt$. Hence the group $gp(s, t)$ is isomorphic to $S_3$, and thus $gp(s, t) \times \{1, v, v^2\} = S^{(2)}(S_3)$.

**Lemma 1.** If for integers $a, b, c$ and $d$ the equality

\begin{equation}
[y, x]^{ax+by+cxy+d} = 1
\end{equation}

holds for all $x, y \in S_3$ then $a \equiv b \equiv c \equiv d \pmod{3}$.

**Proof.** We use (iv). Since the equality (5) holds for the pairs $(1, e), (e, 1)$ and all integers $a, b, c$ and $d$, we can restrict ourself to four pairs $(1, \varphi), (\varphi, 1), (b, 1\varphi)$ and $(1\varphi, \varphi)$. By (iv) the equality (5) is an identity in the group $S_3$ if
and only if the integers $a$, $b$, $c$ and $d$ satisfy the following system of equalities

\[
\begin{align*}
2^a \cdot 1^b \cdot 1^c \cdot 2^d &= e \\
2^a \cdot 1^b \cdot 2^c \cdot 1^d &= e \\
1^a \cdot 1^b \cdot 2^c \cdot 2^d &= e \\
2^a \cdot 2^b \cdot 1^c \cdot 1^d &= e
\end{align*}
\]

Since the mapping $e \rightarrow 0$, $1 \rightarrow 1$, $2 \rightarrow 2$ is an isomorphism of the permutation group $\langle \{e, 1, 2\}; \circ \rangle$ onto the cyclic group $\mathbb{Z}_3$, the last system is equivalent to the homogenous system of equations

\[
\begin{align*}
2a + b + c + 2d &= 0 \\
2a + b + 2c + d &= 0 \\
a + b + 2c + 2d &= 0 \\
2a + 2b + c + d &= 0
\end{align*}
\]

where $+$ is taken modulo 3, of course. Let us observe that vector $(1, 1, 1, 1)$ is a solution of the system. Since the rank of the matrix of the system is 3, the set $\{(1, 1, 1, 1), (2, 2, 2, 2), (0, 0, 0, 0)\}$ consists of all solutions.

**Corollary 1.** The relation

\[ [y, x]^{1+x+y+xy} = 1 \]

is an identity in the group $S_3$.

**Corollary 2.** If for some integers $a$, $b$, $c$ and $d$ the equality

\[ [y, x]^{ax+by+cxy+d} = 1 \]

holds for all $x, y \in S_3$ and at least one from the integers $a$, $b$, $c$ or $d$ is $0$ (mod 3), then all the integers have to be equal $0$ (mod 3).

**Lemma 2.** The equality

\[ [y, x]^{(1-z)(a+bx+cy+dz)}[z, x]^{f(y-1)(x-y)} = 1 \]

holds for all $x, y, z \in S_3$ if and only if $c \equiv a - b - d \equiv f - b \equiv 0$ (mod 3).

**Proof.** If we put $z = x$ into (5) then we get

\[ [y, x]^{(a-b-d)+(b+d-a)x+cy-cxy}. \]
It follows from Lemma 1 that \( c \equiv a - b - d \equiv 0 \pmod{3} \). Because of the equality \([y, x]^{(1-c)(1+c)} = 1\) we can rewrite (5) as
\[
[y, x]^{b(1-c)(x+1)} [z, x]^{f(y-1)(x-y)} = 1.
\]
By putting \( z = y \) and taking into account Lemma 1 we get congruence \( f \equiv b \pmod{3} \).

Conversely, if the congruences hold then the equality (5) is of the form
\[
[y, x]^{b(x+1)} [z, x]^{b(y+1)} = 1,
\]
which is identical with the Jacobi identity \( J(x, y, z) b(x+1) = 1 \).

3. Symmetric 3-words

Let \( w \) be a word of 3 variables in general form
\[
w = x^{a_1} y^{\beta_1} z^{\gamma_1} x^{a_2} y^{\beta_2} z^{\gamma_2} \cdots x^{a_n} y^{\beta_n} z^{\gamma_n},
\]
where \( a_i, \beta_i, \gamma_i \in \mathbb{Z}, i = 1, 2, \ldots, n \) and \( n \in \mathbb{N} \). In view of the identity (1), we may assume that all integers \( a_i, \beta_i \) and \( \gamma_i \) belong to the set \( \{0, 1, 2, 3, 4, 5\} \).

Using the identities from (i) it is possible to remove each \( x \) of the word (7) at the first place. One obtains a word of the form \( x^i u v \cdots \), where \( u, v, \ldots \) are variables \( y, z \) or commutators of the form \([y, x]^i x^j y^k z^l\) for some \( i, j = 1, 2, \ldots \). Since squares of elements of the group \( S_3 \) commutes with commutators (comp. (3)), one can assume \( i \) and \( j \) equal 0 or 1. Now we remove all \( y \)'s at the second place and apply (ii). We get a word \( x^i y^j u' v' \cdots \), where \( u', v', \ldots \) are words of the form \([y, x]^i x^j y^k z^l\) or \([z, x]^m\) for some \( i, j, k, l, m \in \{0, 1\} \). Clearly, the same collecting process can be made with the last variable \( z \). This together with Corollary 2 gives

**Lemma 3.** Any word of variables \( x, y \) and \( z \) in the group \( S_3 \) is equivalent modulo \( V_3(S_3) \) to the following word
\[
w(x, y, z) = x^a y^b z^c \cdot [y, x]^{a_0 + \alpha_1 x + \alpha_2 y + \alpha_3 z + \alpha_4 x z + \alpha_5 y z}
\]
\[
\cdot [z, x]^{b_0 + \beta_1 x + \beta_2 y + \beta_3 z + \beta_4 x y + \beta_5 x z}
\]
\[
\cdot [z, y]^{c_0 + \gamma_1 x + \gamma_2 y + \gamma_3 z + \gamma_4 x y + \gamma_5 x z}
\]
for some elements \( a, b, c \in \{0, 1, 2, 3, 4, 5\} \) and \( \alpha_0, \ldots, \gamma_{13} \in \{0, 1, 2\} \).

From now on we write \( w = u \) instead of \( w \equiv u \pmod{V_3(S_3)} \) and we prefer to write \(-1\) than 2, when 2 is an exponent of a commutator. Now we are able to determine all words of three variables in the group \( S_3 \). First we show
that the image of each symmetric 3-word in $S_3$ treated as a function $S_3^3 \rightarrow S_3$ is contained in the commutator subgroup of $S_3$. More precisely we have

**Theorem 2.** Let $s(x, y, z)$ be a symmetric 3-word of the form (8) in the group $S_3$. Then $a = b = c = 2i$ for $i = 0, 1$ or 2.

**Proof.** Clearly $s(x, 1, 1) = s(1, x, 1) = s(x, 1, 1)$ and therefore we have $a \equiv b \equiv c \pmod{6}$.

Let us suppose that $a = 1$. We have

\[
s(x, y, 1) = xy[y, x]^{(\alpha_0 + \alpha_1 + \alpha_2 + \gamma_0 + \gamma_1 + \gamma_2 + \gamma_3)}
\]

\[
s(x, 1, y) = xy[y, x]^{(\beta_0 + \beta_1 + \beta_2 + \gamma_0 + \gamma_1 + \gamma_2 + \gamma_3)}
\]

\[
s(1, x, y) = xy[y, x]^{(\gamma_0 + \gamma_1 + \gamma_2 + \gamma_3)}
\]

By (iii) $s(x, y, 1)$ is a symmetric 2-word in $S_3$. It follows from Theorem 1 that $s(x, y, 1)$ equals

\[
xy[y, x]^{-1} \text{ or } xy[y, x]^{-1-x+y} \text{ or } xy[y, x]^{-1+x-y}.
\]

Without loss of generality we can assume that

\[
s(x, y, 1) = xy[y, x]^{-1}.
\]

Indeed, it is easy to verify the equalities $w(x, y, z) = w(y, x, z) = w(x, z, y)$, where

\[
w(x, y, z) = [y, x]^{(y-x)[z, x]^{(z-x)[z, y]^{(z-y)x}}}.
\]

which means that $w$ is a symmetric 3-word in the group $S_3$ such that $w(x, y, 1) = [y, x]^{y-x}$. Therefore if $s(x, y, 1)$ does not equal $xy[y, x]^{-1}$, then we can consider $s \cdot w$ or $s \cdot w^2$ instead of $s$.

By Corollary 2 we get the congruences

\[
(9) \quad \alpha_0 + \alpha_3 + 1 \equiv \beta_0 + \beta_2 + 1 \equiv \gamma_0 + \gamma_1 + 1 \equiv \alpha_1 + \alpha_{13} \equiv \beta_1 + \beta_{12}
\]

\[
\equiv \gamma_2 + \gamma_{12} \equiv \alpha_2 + \alpha_{23} \equiv \beta_3 + \beta_{23} \equiv \gamma_3 + \gamma_{13} \equiv 0 \pmod{3}.
\]

Therefore we can rewrite the word $s$ as

\[
s(x, y, z) = xyz \cdot [y, x]^{(1-z)[x, y]^{(\alpha_0 + \alpha_1 + \alpha_2 + \beta_0 + \beta_1 + \beta_{12})}}
\]

\[
\cdot [z, x]^{(1-y)[x, z]^{(\beta_0 + \beta_1 + \beta_{12})}}
\]

\[
\cdot [z, y]^{(1-x)[y, z]^{(\gamma_0 + \gamma_1 + \gamma_{13})}}.
\]
After eliminating $\beta_s(x,y,z)$ the condition this, by Lemma 2, implies the congruences

\[
(s(x,y,z))^{1-\gamma}(\gamma_0 + \gamma_2 y + \gamma_3 z) - y
\]

\[
(s(x,z,y))^{1-x}(\beta_0 + \beta_1 y + \beta_2 z) - x
\]

and similarly

\[
(s(x,y,z))^{1-z}(\beta_0 + \beta_1 x + \beta_2 y) - z
\]

\[
(s(y,x,z))^{1-y}(\gamma_0 + \gamma_2 x + \gamma_3 z) - y
\]

\[
(s(x,z,y))^{1-x}(\gamma_0 - \gamma_2 y - \gamma_3 z) + x
\]

The condition $s(x,y,z) = s(y,x,z)$ implies

\[
[y,x]^{1-\gamma}(\gamma_0 + \gamma_2 y + \gamma_3 z) - y
\]

\[
[z,x]^{(\beta_0 - \gamma_0)} + (\beta_1 - \gamma_2) x + (\beta_3 - \gamma_3) z
\]

\[
[z,y]^{1-x}(\gamma_0 - \gamma_3 y - \gamma_3 z) + x
\]

which in the case $z = x$ gives

\[
[y,x]^{1-x}(\gamma_0 + \gamma_2 y + \gamma_3 z) = 1
\]

This, by Lemma 2, implies the congruences

\[
\beta_1 \equiv \gamma_2 - \alpha_1 - \alpha_2 \pmod{3}
\]

\[-\alpha_0 + \beta_0 - \gamma_0 \equiv \alpha_1 + \alpha_2 + \beta_3 - \gamma_3 \pmod{3}
\]

Similarly the equality $s(x,y,z) = s(x,z,y)$ gives

\[
[y,x]^{1-\gamma}(\gamma_0 + \gamma_2 y + \gamma_3 z) - y
\]

\[
[z,x]^{(\beta_0 - \gamma_0)} + (\beta_1 - \gamma_2) x + (\beta_3 - \gamma_3) z
\]

\[
[z,y]^{1-x}(\gamma_0 - \gamma_3 y - \gamma_3 z) + x
\]

which in the case $y = x$ together with Lemma 2 yields the congruences

\[
\beta_3 \equiv \alpha_2 - \gamma_2 - \gamma_3 \pmod{3}
\]

\[-\alpha_0 + \beta_0 - \gamma_0 - 1 \equiv \beta_1 - \alpha_1 + \gamma_2 + \gamma_3 \pmod{3}
\]

After eliminating $\beta_1$ and $\beta_3$ from above system of four congruences we see that it has no solution.

**Theorem 3.** Let $s(x,y,z)$ be a symmetric 3-word in the group $S_3$, then

\[
s = u_i \cdot s_j^1 \cdot s_k^1 \cdot s_j^m \cdot w^n \quad \text{for some} \quad i, j, k, l, m \in \{0, 1, -1\}.
\]
where

\[ u(x, y, z) = x^2 y^2 z^2, \]
\[ s_0(x, y, z) = [y, x]^{x-1}[z, x]^{y-1}, \]
\[ s_1(x, y, z) = [y, x]^{(1-y)z} [z, x]^{1-y} [z, y]^{(1-y)(y-z)}, \]
\[ s_2(x, y, z) = [y, x]^{(1-z)x} [z, x]^{1-y} [z, y]^{(1-x)(y-z)}, \]
\[ s_3(x, y, z) = [z, x]^{(1-y)(y+z)} [z, y]^{1-y} [y, z]^{(y-x)}, \]
\[ w(x, y, z) = [y, x]^{(y-x)z} [z, x]^{(z-x)y} [z, y]^{(z-y)x}. \]

Presentation (12) is unique and therefore \( S^3(3) \) is Abelian group isomorphic to \((\mathbb{Z}_3)^6\).

Proof. Let \( s(x, y, z) \) be a symmetric 3-word in \( S_3 \) of the form (8) with \( a = b = c = 0 \). We have

\[ s(x, y, 1) = [y, x]^{(\alpha_0 + \alpha_3) + (\alpha_1 + \alpha_{13})x + (\alpha_2 + \alpha_{23})y}, \]
\[ s(x, 1, y) = [y, x]^{(\beta_0 + \beta_2) + (\beta_1 + \beta_{12})x + (\beta_3 + \beta_{23})y}, \]
\[ s(1, x, y) = [y, x]^{(\gamma_0 + \gamma_3) + (\gamma_2 + \gamma_{12})x + (\gamma_1 + \gamma_{23})y}. \]

By Theorem 1 every symmetric word \( s(x, y, 1) \) of two variables equals

either 1 or \([y, x]^{y-x}\) or else \([y, x]^{x-y}\).

Let us consider first case \( s(x, y, 1) = 1 \).

By Corollary 2 we have the following congruences

\[ \alpha_0 + \alpha_3 \equiv \beta_0 + \beta_2 \equiv \gamma_0 + \gamma_1 \equiv \alpha_1 + \alpha_{13} \equiv \beta_1 + \beta_{12} \equiv \gamma_2 + \gamma_{12} \equiv \alpha_2 + \alpha_{23} \equiv \beta_3 + \beta_{23} \equiv \gamma_3 + \gamma_{13} \equiv 0 \pmod{3}, \]

which enables us to rewrite the word \( s \) in the form

\[ s(x, y, z) = [y, x]^{(1-z)(\alpha_0 + \alpha_3 x + \alpha_2 y)} \cdot [z, x]^{(1-y)(\beta_0 + \beta_2 x + \beta_3 y)} \cdot [z, y]^{(1-x)(\gamma_0 + \gamma_3 y + \gamma_2 z)}. \]

It is well known the transpositions \((1, 2)\) and \((2, 3)\) generate the symmetric group \( S_3 \) of degree 3 and therefore \( s(x, y, z) \) is symmetric if and only if two equalities

\[ s(y, x, z)^{-1} \cdot s(x, y, z) = 1, \quad s(x, z, y)^{-1} \cdot s(x, y, z) = 1 \]
Thus hold for all elements $x$, $y$, $z$ from $S_3$. We check

$$s(x, y, z) = [y, x]^{1-z}(-\alpha_0-\alpha_2 x+\alpha_1 y) \cdot [z, x]^{(1-y)(\gamma_0+\gamma_2 x+\gamma_2 y)} \cdot [z, y]^{(1-x)(\beta_0+\beta_1 x+\beta_2 z)}$$

and similarly

$$s(x, z, y) = [y, x]^{(1-z)(\beta_0+\beta_1 x+\beta_2 y)} \cdot [z, x]^{(1-y)(\alpha_0+\alpha_2 x+\alpha_2 z)} \cdot [z, y]^{(1-x)(\gamma_0+\gamma_1 y+\gamma_2 z)}.$$

Hence we get

$$f(x, y, z) = s(y, x, z)^{-1} \cdot s(x, y, z)$$

\begin{equation}
= [y, x]^{1-z}(-\alpha_0+(\alpha_1+\alpha_2)x+(\alpha_1+\alpha_2)y) \cdot [z, x]^{(y-1)(\gamma_0-\beta_0)+(\gamma_2-\beta_1)x+(\gamma_2-\beta_1)z) \cdot [z, y]^{(1-x)(\gamma_0-\beta_0)+(\gamma_1-\beta_1)y+(\gamma_1-\beta_1)z)]
\end{equation}

and also

$$g(x, y, z) = s(x, z, y)^{-1} \cdot s(x, y, z)$$

\begin{equation}
= [y, x]^{(1-z)(\alpha_0-\beta_0)+(x_1+\alpha_1)y} \cdot [z, x]^{(y-1)(\gamma_0-\beta_0)+(\gamma_1-\beta_1)x+(\gamma_1-\beta_1)z) \cdot [z, y]^{(1-x)(\gamma_0-\beta_0)+(\gamma_2-\beta_1)y+(\gamma_2-\beta_1)z)]
\end{equation}

Thus $s$ is symmetric if and only if the equalities $f(x, y, z) = g(x, y, z) = 1$ hold for all $x$, $y$, $z \in S_3$. Applying Jacobi identity

$$(y, x)^{1-z}[z, x]^{y-1}[z, y]^{1-x}(\gamma_0-\beta_0)+(\gamma_2-\beta_1)y+(\gamma_2-\beta_1)z) = 1$$

to the equality (14) and

$$(y, x)^{1-z}[z, x]^{y-1}[z, y]^{1-x}(\alpha_0-\beta_0)+(\alpha_1-\beta_1)x+(\alpha_2-\beta_2)z) = 1$$

to the equality (15) we see that $s(x, y, z)$ is symmetric 3-word if and only if the following two equalities

$$1 = f(x, y, z) = [y, x]^{1-z}((\beta_0-\alpha_0-\gamma_0)+\alpha_1 x+(\alpha_2+\alpha_2+\beta_1 y)+(\beta_2-\gamma_1)z) \cdot [z, x]^{(\gamma_2-\beta_1)(y-1)(x-y)},$$

$$1 = g(x, y, z) = [z, y]^{(1-x)((\beta_0-\alpha_0-\gamma_0)+\beta_1 x+(\gamma_2+\gamma_2+\beta_2 y)+(\gamma_2+\gamma_2)z) \cdot [z, x]^{(\alpha_2-\beta_2)(y-1)(z-y)}.$$
holds for all $x, y, z \in S_3$. By Lemma 2 this is equivalent to the following system of congruences

$$\begin{align*}
\alpha_1 + \alpha_2 + \beta_1 - \gamma_2 &\equiv 0 \pmod{3}, \\
-\alpha_0 + \beta_0 - \gamma_0 &\equiv \alpha_1 + \alpha_2 + \beta_3 - \gamma_3 \pmod{3}, \\
\beta_3 - \gamma_3 &\equiv \gamma_2 - \beta_1 \pmod{3}, \\
-\alpha_2 + \beta_3 + \gamma_2 + \gamma_3 &\equiv 0 \pmod{3}, \\
-\alpha_0 + \beta_0 - \gamma_0 &\equiv \beta_1 - \alpha_1 + \gamma_2 + \gamma_3 \pmod{3}.
\end{align*}$$

Choosing $\alpha_0, \gamma_0, \alpha_1, \alpha_2$ and $\beta_1$ as parameters we obtain the following solution of the system

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>$\alpha_0$</td>
<td>$\alpha_1$</td>
<td>$\alpha_2$</td>
<td>$-\alpha_0$</td>
</tr>
<tr>
<td>$\beta$</td>
<td>$\alpha_0 + \gamma_0 - \alpha_1 - \alpha_2$</td>
<td>$\beta_1$</td>
<td>$\alpha_0 - \gamma_0 + \alpha_1 + \alpha_2$</td>
<td>$\beta_1 - \alpha_2$</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>$\gamma_0$</td>
<td>$-\gamma_0$</td>
<td>$\alpha_1 + \alpha_2 + \beta_1$</td>
<td>$-\alpha_1 + \alpha_2 + \beta_1$</td>
</tr>
</tbody>
</table>

Thus the word $s$ can be written as

$$s(x, y, z) = [y, x]^{1-x}(\alpha_0 + \alpha_1 x + \alpha_2 y) \cdot [z, x]^{1-y}((\alpha_0 + \gamma_0 - \alpha_1 - \alpha_2) + \beta_1 x + (\beta_1 - \alpha_2) z) \cdot [z, y]^{1-z}((\alpha_0 + \alpha_1 + \alpha_2 + \beta_1) + (\gamma_0 - \alpha_1 + \gamma_2) (y - z)),$$

or

$$s = s_0^{\alpha_0} \cdot s_4^{\gamma_0} \cdot s_1^{\alpha_1} \cdot s_2^{\alpha_2} \cdot s_3^\beta,$$

where

$$\begin{align*}
s_0(x, y, z) &= [y, x]^{1-x} [z, x]^{1-y}, \\
s_4(x, y, z) &= [z, x]^{1-y} [z, y]^{1-z}, \\
s_1(x, y, z) &= [y, x]^{1-x} [z, x]^{1-y} [z, y]^{1-x} (y - z), \\
s_2(x, y, z) &= [y, x]^{1-x} [z, x]^{1-y} [z, y]^{1-x} (y + z), \\
s_3(x, y, z) &= [z, x]^{1-y} (x + z) [z, y]^{1-x} (y + z).
\end{align*}$$

Above we have made use of (i) and (ii). Observe that

$$s_0(x, y, z) \cdot s_4(x, y, z) = J(x, y, z)^{-1} = 1,$$

which yields

$$s = s_0^j \cdot s_1^k \cdot s_2^l \cdot s_3^m.$$
for some \( j, k, l, m \in \{0, 1, -1\} \). We claim that the presentation (16) of the word \( s \) is unique. Indeed, if for some \( j, k, l, m \in \{0, 1, -1\} \) the equality

\[
(17) \quad s_0(x, y, z)^j s_1(x, y, z)^k s_2(x, y, z)^l s_3(x, y, z)^m = 1
\]

hold for all \( x, y, z \in S_3 \), then, in the case \( z = x \), we get

\[
[y, x]^{(j+k+l+m)+(j-k-l-m)x-(k+m)y+(k+m)xy)} = 1,
\]

which by Lemma 1 implies \( l - j \equiv k + m \equiv 0 \pmod{3} \). If we put \( z = y \) into (16), we get

\[
[y, x]^{(j+k+m)+(j+k+m)x+(j-k-m)y-(k+m)xy} = 1.
\]

By Corollary 2 we have \( j = k = l = m = 0 \), as required.

As we have mentioned earlier \( u = x^2 y^2 z^2 \) is symmetric 3-word in \( S_3 \) and

\[
w(x, y, z) = [y, x]^{(y-x)z} [z, x]^{(z-x)y} [z, y]^{(z-y)x}
\]

is a symmetric 3-word with \( w(x, y, 1) = [y, x]^{y-x} \). Let \( s(x, y, z) \) be arbitrary symmetric 3-word in \( S_3 \) such that \( s(x, y, 1) = (x^2 y^2)[y, x]^{n(y-x)} \) for some \( i, n \in \{0, 1, -1\} \). Then the following product \( (u^{-i} s w^{-n}) \) is a symmetric 3-word with \( (u^{-i} s w^{-n})(x, y, 1) = 1 \) and therefore, in view of what we have just established, \( u^{-i} s w^{-n} = s_0^j s_1^k s_2^l s_3^m \) for some \( j, k, l, m, n \), which completes the proof.

**Theorem 4.** For all \( n \neq 2 \) the groups \( S^{(n)}(S_3) \) of \( n \)-symmetric words of the group \( S_3 \) are commutative.

**Proof.** Let \( s(x_1, x_2, \ldots, x_n) \) be symmetric \( n \)-word, \( n \geq 3 \), in \( S_3 \). Using the same arguments as in the proof of Theorem 1 it is possible to present the word \( s \) as \( x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} c \), where \( c \) is a product of commutators of the form

\[
[x_{i_1}, x_{i_2}]^P
\]

for \( P \) being a polynomial in variables \( x_1, x_2, \ldots, x_n \). Since \( s(x, 1, 1, \ldots, 1) = s(1, x, 1, \ldots, 1) = \cdots = s(1, 1, \ldots, 1, x) \), we have the equality \( a_1 \equiv a_2 \equiv \cdots \equiv a_n \equiv a \pmod{6} \). In view of statement (iii) \( s(x, y, z, 1, \ldots, 1) = x^a y^a z^a c' \) is a symmetric 3-word in \( S_3 \). By Theorem 2, the number \( a \) has to be even, which together with (2) of (ii) finishes the proof.

**Remark.** Every symmetric \( n \)-word in a group \( G \) is symmetric in any group from the variety \( \text{var}(G) \) of groups generated by \( G \) and therefore the results of the paper are valid not only for the group \( S_3 \) but also for all groups from \( \text{var}(S_3) = HSP(S_3) \).
REFERENCES