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EMBEDDINGS FOR MONOTONE FUNCTIONS AND B_p WEIGHTS, BETWEEN DISCRETE AND CONTINUOUS

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Abstract

We characterize the embedding for general weighted Lebesgue spaces of monotone functions by using the analog embedding for discrete monotone sequences indexed over the integers. We then use these results to obtain the boundedness of the discrete Hardy operator and to study the connections with the Hardy classical operator in the continuous setting.

1. Introduction

Let u and v be two weights in a measure space (X, μ), where X is a partially ordered set. We denote the partial order in X by $x \leq y$ for $x, y \in X$. Recently, there is an ongoing interest in obtaining necessary and sufficient conditions on the weights u and v so that there exists a constant $C > 0$ such that the inequality

$$(1)\qquad \left(\int_X f(x)^q v(x)\, d\mu(x)\right)^{1/q} \leq C\left(\int_X f(x)^p u(x)\, d\mu(x)\right)^{1/p},$$

holds for all decreasing (increasing) functions f in X, for $0 < p, q < \infty$. If we denote by $L^p_{\rm dec}(u)$ the set of positive decreasing functions in the weighted Lebesgue space $L^p(u)$, inequality (1) is equivalent to the embedding $L^p_{\rm dec}(u) \hookrightarrow L^q_{\rm dec}(v)$. If $X = [0, \infty)$ and μ is the Lebesgue measure, the embedding of decreasing functions is closely related to the boundedness of the Hardy operator as well as to the corresponding embeddings for Lorentz spaces. Some results on these questions are in [5], [8], [15] and [17]. If $X = \mathsf{R}^n_+ := \mathsf{R}_+ \times \overset{(n)}{\cdots} \times \mathsf{R}_+$ equipped with the partial order defined in (3) and μ is the Lebesgue measure, further results can be found in [3] and [4]. For the discrete case $X = \mathsf{Z}$ and μ the counting measure, see the paper [9]. For results in a general setting (X, μ) see [6] and [13].

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The paper is organized in two sections that share some ideas and techniques. Section 2 will be devoted to the study of inequality (1) in the general setting of a measure space (X, μ), and in all the range $0 < p, q < \infty$. Our main idea is to begin with the characterization of this inequality for the particular case $X = \mathbb{Z}$ and μ the counting measure (which is done in Theorems 2.1 and 2.4), and to use it to give an answer in the general setting (see Theorem 2.8), by using a discretization technique.

The Hardy operator is defined for any measurable function f on $[0, \infty)$ by

$$Af(x) = \frac{1}{x}\int_0^x f(t)\,dt, \qquad x > 0.$$

In 1990, M. A. Ariño and B. Muckenhoupt ([1]) characterized the weights u for which the boundedness $A : L^p_{\text{dec}}(u) \longrightarrow L^p(u)$ holds. These weights are called the B_p weights. In [6] and [13], the authors studied the boundedness of the discrete Hardy operator defined for positive sequences indexed over $\mathbb{N}$ and showed that the discrete weights for which this holds form, in some sense, a subclass of the classical B_p weights (see Theorem 3.2). In Section 3, we will show that we can reverse this process, that is, we can construct a discrete weight for which the discrete Hardy operator is bounded, for every classical B_p weight (see Lemma 3.3). However, we will give an example to see that this correspondence between discrete and non-discrete weights is not one to one. Finally, we will consider the case of the discrete Hardy operator defined for sequences indexed in $\mathbb{Z}$. The Theorems 3.4 and 3.5 say that, for every B_p (or $B_{p,\infty}$) weight, we can construct a discrete weight for which the boundedness of the discrete Hardy operator holds (or the weak boundedness). As a consequence, we prove that in that case, B_p can be viewed as the set of weights for which the discretized weights belong to the discrete B_p class (see Corollary 3.6).

We will always write $r = pq/(p - q)$ in the case $0 < q < p < \infty$. For a weight $u : X \longrightarrow \mathbb{R}_+$, we denote $U(E) := \int_E u(x)\,d\mu(x)$, where μ is the ambient measure. In the case of $\mathbb{R}_+$ and the Lebesgue measure, for a weight $u : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, we use the notation $U(x) := \int_0^x u(t)\,dt$. We will denote $a_{\mathbb{Z}} := \{a_k : k \in \mathbb{Z}\}$ and $a_{\mathbb{N}} := \{a_k : k \in \mathbb{N}\}$ two sequences of real numbers indexed in $\mathbb{Z}$ and $\mathbb{N}$ respectively. We adopt the convention that $0 \in \mathbb{N}$. A weight $u_{\mathbb{Z}}$ is a positive sequence, and $U_k := \sum_{j=-\infty}^{k} u_j$. We also denote $a_{\mathbb{Z}} \downarrow$ or $a_{\mathbb{N}} \downarrow$ if the sequence is decreasing. Two positive quantities A and B are said to be equivalent ($A \approx B$) if there exists a constant $C > 1$ such that $C^{-1}A \le B \le CA$. If only $B \le CA$, we write $B \lesssim A$. The undetermined cases $0 \cdot \infty$, $\frac{\infty}{\infty}$, $\frac{0}{0}$, will always be taken equal to 0.

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This work is dedicated to the memory of José Antonio Raposo, a brilliant person and mathematician.

2. Embeddings of weighted spaces of monotone functions

We will first characterize the inequality (1) in the case $X = \mathsf{Z}$ and μ the counting measure, for all $0 < p, q < \infty$. We then call a powerful and useful idea to get the characterization of the same inequality in the general case of a measure space (X, μ): we take an appropriate discretization of the set X, and we reduce the problem to the discrete setting of Z, where we have given an answer already. Our first goal is to characterize the inequality

$$\left(\sum_{k\in\mathsf{Z}} a_k^q v_k\right)^{1/q} \leq C\left(\sum_{k\in\mathsf{Z}} a_k^p u_k\right)^{1/p},$$

for all decreasing sequences a_{Z}, where u_{Z} and v_{Z} are positive sequences indexed in Z. For the range $0 < p \leq q < \infty$, the characterization is easy and is proved in the same way as for $X = [0, \infty)$ (see [7] or [17]), and the result is sharp: the supremum is attained in the set of characteristic functions of decreasing sets in Z, that is, χ_B where $B = (\ldots, 1, 1, 1, 0, 0, 0, \ldots)$. The proof is standard and is omitted.

THEOREM 2.1. *If* $0 < p \leq q < \infty$, *then*

$$\sup_{0\leq a_{\mathsf{Z}}\downarrow} \frac{\left(\sum_{k\in\mathsf{Z}} a_k^q v_k\right)^{1/q}}{\left(\sum_{k\in\mathsf{Z}} a_k^p u_k\right)^{1/p}} = \sup_{k\in\mathsf{Z}} \frac{V_k^{1/q}}{U_k^{1/p}}.$$

In 1990, E. Sawyer characterized the inequality (1) in the case $X = [0, \infty)$, μ the Lebesgue measure, and $0 < q < p < \infty$. The next theorem states his result.

THEOREM 2.2 ([15]). *If* $0 < q < p < \infty$, *then:*

$$\sup_{0\leq f\downarrow} \frac{\left(\int_0^\infty f(x)^q v(x)\,dx\right)^{1/q}}{\left(\int_0^\infty f(x)^p u(x)\,dx\right)^{1/p}} \approx \left(\int_0^\infty V(x)^{r/p} U(x)^{-r/p} v(x)\,dx\right)^{1/r}$$
$$\approx \left(\int_0^\infty V(x)^{r/q} U(x)^{-r/q} u(x)\,dx\right)^{1/r} + \frac{V(\infty)^{1/q}}{U(\infty)^{1/p}},$$

where $U(\infty) = \int_0^\infty u(x)\,dx$, *and analogously for* $V(\infty)$.

In the range $0 < q < p < \infty$, the case of decreasing sequences in Z does not seem to be known in the literature, except for the result of H. P. Heinig and A. Kufner ([9]), where extra hypotheses on the weights are needed. To solve the problem in its full generality, we consider the next proposition. We see that the required discrete embedding is equivalent to a continuous embedding.

PROPOSITION 2.3. *If* $0 < q < p < \infty$, *then*

$$A = \sup_{0\leq a_{\mathsf{Z}}\downarrow} \frac{\left(\sum_{k\in\mathsf{Z}} a_k^q v_k\right)^{1/q}}{\left(\sum_{k\in\mathsf{Z}} a_k^p u_k\right)^{1/p}} \approx \sup_{0\leq f\downarrow} \frac{\left(\int_0^\infty f(x)^q \widetilde{v}(x)\,dx\right)^{1/q}}{\left(\int_0^\infty f(x)^p \widetilde{u}(x)\,dx\right)^{1/p}} = B,$$

where $\widetilde{v}(x) = \sum_{k\in\mathsf{Z}} \frac{v_k}{2^k} \chi_{[2^k,2^{k+1})}(x)$ *and* $\widetilde{u}(x) = \sum_{k\in\mathsf{Z}} \frac{u_k}{2^k} \chi_{[2^k,2^{k+1})}(x)$. *Moreover,* $A \leq B \leq \left(\frac{2^{r/q}-1}{r/q}\right)^{q/r} A$.

PROOF. For every positive decreasing sequence a_{Z}, we consider the function

$$f(x) = \sum_{k\in\mathsf{Z}} a_k \chi_{[2^k,2^{k+1})}(x),$$

and that easily gives $A \leq B$. Now, for every positive decreasing function f, we define the positive and decreasing sequence

$$a_k = \left(\int_{2^k}^{2^{k+1}} f(x)^p \frac{dx}{x}\right)^{1/p},$$

and we get the case $B \lesssim A$ applying Hölder's inequality.

We now use this proposition and Theorem 2.2 to characterize the embedding in the desired range.

THEOREM 2.4. *If* $0 < q < p < \infty$, *then*

$$\sup_{0\leq a_{\mathsf{Z}}\downarrow} \frac{\left(\sum_{k\in\mathsf{Z}} a_k^q v_k\right)^{1/q}}{\left(\sum_{k\in\mathsf{Z}} a_k^p u_k\right)^{1/p}} \approx \left(\int_0^1 \left[\sum_{k\in\mathsf{Z}} \left(\frac{V_{k-1} + v_k t}{U_{k-1} + u_k t}\right)^{r/p} v_k\right] dt\right)^{1/r}$$

$$\approx \left(\int_0^1 \left[\sum_{k\in\mathsf{Z}} \left(\frac{V_{k-1} + v_k t}{U_{k-1} + u_k t}\right)^{r/q} u_k\right] dt\right)^{1/r} + \frac{V_\infty^{1/q}}{U_\infty^{1/p}},$$

where $U_\infty = \sum_{k\in\mathsf{Z}} u_k$, *and analogously for* V_∞.

PROOF. By the previous proposition and Theorem 2.2, we know that

$$\sup_{0\le a_{\mathsf{Z}}\downarrow} \frac{\left(\sum_{k\in\mathsf{Z}} a_k^q v_k\right)^{1/q}}{\left(\sum_{k\in\mathsf{Z}} a_k^p u_k\right)^{1/p}} \approx \left(\int_0^\infty \widetilde{V}(x)^{r/p}\widetilde{U}(x)^{-r/p}\widetilde{v}(x)\,dx\right)^{1/r}$$

$$\approx \left(\int_0^\infty \widetilde{V}(x)^{r/q}\widetilde{U}(x)^{-r/q}\widetilde{u}(x)\,dx\right)^{1/r} + \frac{\widetilde{V}(\infty)^{1/q}}{\widetilde{U}(\infty)^{1/p}},$$

where $\widetilde{v}(x) = \sum_{k\in\mathsf{Z}} \frac{v_k}{2^k}\chi_{[2^k,2^{k+1})}(x)$ and $\widetilde{u}(x) = \sum_{k\in\mathsf{Z}} \frac{u_k}{2^k}\chi_{[2^k,2^{k+1})}(x)$. First, observe that if $2^k \le x < 2^{k+1}$ then $\widetilde{V}(x) = V_{k-1} + v_k\frac{x-2^k}{2^k}$, and the same for $\widetilde{U}$. Now, splitting the integral into dyadic intervals, we get:

$$\begin{aligned}
&\int_0^\infty \widetilde{V}(x)^{r/p}\widetilde{U}(x)^{-r/p}\widetilde{v}(x)\,dx \\
&\quad = \sum_{k\in\mathsf{Z}} \frac{v_k}{2^k}\int_{2^k}^{2^{k+1}} \left(V_{k-1} + v_k\frac{x-2^k}{2^k}\right)^{r/p}\left(U_{k-1} + u_k\frac{x-2^k}{2^k}\right)^{-r/p} dx \\
&\quad = \int_0^1 \left(\sum_{k\in\mathsf{Z}} (V_{k-1}+v_k t)^{r/p}(U_{k-1}+u_k t)^{-r/p} v_k\right) dt,
\end{aligned}$$

where the last equality follows by the change of variable $t = \frac{x-2^k}{2^k}$ in each integral. The other equivalence is analogous.

We now deal with the problem in the general setting of a measure space (X,μ), where X is a partially ordered set. Our results are based on a discretization technique which shows that the embedding (1) is equivalent to a collection of embeddings of sequences in Z. This technique was pointed out by E. Sawyer in [14], and has been also used in [2], [4], [10] and [11]. In our case, the results are given in terms of covering sequences of decreasing sets in X. A positive function $f : X \longrightarrow [0,\infty)$ is said to be decreasing if $f(x) \le f(y)$ whenever $x \ge y$. We will denote $f\downarrow$ whenever f is a decreasing function, and $D\downarrow$ for a decreasing set $D \subset X$, that is, a set D for which χ_D is a decreasing function. In what follows, we will assume that every decreasing set is μ-measurable.

DEFINITION 2.5. A collection of sets $\{D_k : k \in \mathsf{Z}\}$ is a *covering family of decreasing sets* for the set X if:

- D_k is decreasing for all $k \in \mathsf{Z}$.

- $D_k \subset D_{k+1}$ for all $k \in \mathbb{Z}$.
- $\bigcup_{k\in\mathbb{Z}} D_k = X$.

The set of all covering families of decreasing sets in X is denoted by $\mathscr{D}(X)$ or simply $\mathscr{D}$ if there is no possible confusion. For a fixed family $\{D_k : k \in \mathbb{Z}\}$ in $\mathscr{D}(X)$, we denote $\Delta_k = D_{k+1} \setminus D_k$.

We now present a lemma that can be found in a slightly different version in [10] for the case $X = [0, \infty)$ and in [2] for $X = \mathbb{R}^n_+$. In our case, we do not require any additional condition neither on the covering family of decreasing sets nor in the modular functions. Recall that a modular function P is a positive and increasing function $P : [0, \infty) \longrightarrow [0, \infty)$ such that $P(0) = 0$ and $P(\infty) = \infty$. We will use the subsequent corollary for our purpose.

LEMMA 2.6. *Let (X, μ) be a measure space, where X is a partially ordered set. Let Q and P be two modular functions. Let A be the infimum of the constants $C > 0$ such that the inequality*

$$Q^{-1}\left(\int_X Q(f(x))v(x)\,d\mu(x)\right) \le P^{-1}\left(\int_X P(Cf(x))u(x)\,d\mu(x)\right),$$

holds for all $0 \le f \downarrow$, and let B be the infimum of the constants $C > 0$ such that the inequality

$$Q^{-1}\left(\sum_{k\in\mathbb{Z}} Q(\delta_k)\int_{\Delta_k} v(x)\,d\mu(x)\right) \le P^{-1}\left(\sum_{k\in\mathbb{Z}} P(C\delta_k)\int_{\Delta_k} u(x)\,d\mu(x)\right),$$

holds for all $0 \le \delta_{\mathbb{Z}} \downarrow$ and for all $\{D_k\} \subset \mathscr{D}$. Then $A = B$.

PROOF. For every positive decreasing sequence $\delta_{\mathbb{Z}}$ and every family $\{D_k : k \in \mathbb{Z}\}$, if we consider the decreasing function

$$f(x) = \sum_{k\in\mathbb{Z}} \delta_k \chi_{\Delta_k}(x),$$

we easily get that $A \le B$. Now, take a positive decreasing function f such that

$$Q^{-1}\left(\int_X Q(f(x))v(x)\,d\mu(x)\right) \le P^{-1}\left(\int_X P((A+\varepsilon)f(x))u(x)\,d\mu(x)\right),$$

for a fixed $\varepsilon > 0$. Take $c > 1$ and $\Delta_k = \left\{x \in X : c^{-k-1} < f(x) \le c^{-k}\right\}$, for all $k \in \mathbb{Z}$. Now, by using that P, Q, P^{-1} and Q^{-1} are increasing functions and our hypothesis, we have

$$
\begin{aligned}
Q^{-1}\Big(\sum_k Q(c^{-k-1})\int_{\Delta_k} v(x)\,d\mu(x)\Big)
&\le Q^{-1}\Big(\int_X Q(f(x))v(x)\,d\mu(x)\Big)\\
&\le P^{-1}\Big(\sum_k \int_{\Delta_k} P((A+\varepsilon)f(x))u(x)\,d\mu(x)\Big)\\
&\le P^{-1}\Big(\sum_k P((A+\varepsilon)c^{-k})\int_{\Delta_k} u(x)\,d\mu(x)\Big),
\end{aligned}
$$

and this is $B \le c(A+\varepsilon)$. Letting $\varepsilon \to 0^+$ and $c \to 1^+$, we obtain $B \le A$.

COROLLARY 2.7. *If $P(t) = t^p$ and $Q(t) = t^q$, then the previous lemma reads as*

$$
\sup_{0\le f\downarrow} \frac{\Big(\int_X f(x)^q v(x)\,d\mu(x)\Big)^{1/q}}{\Big(\int_X f(x)^p u(x)\,d\mu(x)\Big)^{1/p}} = \sup_{\{D_k\}\subset\mathscr{D}}\ \sup_{0\le\delta_{\mathbb{Z}}\downarrow} \frac{\Big(\sum_{k\in\mathbb{Z}} \delta_k^q v_k\Big)^{1/q}}{\Big(\sum_{k\in\mathbb{Z}} \delta_k^p u_k\Big)^{1/p}}, \tag{2}
$$

where we are using the notation $u_k := \int_{\Delta_k} u(x)\,d\mu(x) = U(\Delta_k)$, and the same for v.

We are ready to prove our main result of this section.

THEOREM 2.8. *Let (X,μ) be a measure space, where X is a partially ordered set. We have:*

(a) *If $0 < p \le q < \infty$, then*

$$
\sup_{0\le f\downarrow} \frac{\Big(\int_X f(x)^q v(x)\,d\mu(x)\Big)^{1/q}}{\Big(\int_X f(x)^p u(x)\,d\mu(x)\Big)^{1/p}} = \sup_{D\downarrow} \frac{V(D)^{1/q}}{U(D)^{1/p}}.
$$

(b) *If $0 < q < p < \infty$, then the following conditions are equivalent:*

(i) *There exists $C > 0$ such that*

$$
\Big(\int_X f(x)^q v(x)\,d\mu(x)\Big)^{1/q} \le C\Big(\int_X f(x)^p u(x)\,d\mu(x)\Big)^{1/p},
$$

for all positive decreasing functions f.

(ii) *There exists $C > 0$ such that*

$$\left(\int_0^1 \left[\sum_{k\in\mathbb{Z}} \left(\frac{V(D_k)+V(\Delta_k)t}{U(D_k)+U(\Delta_k)t}\right)^{r/p} V(\Delta_k)\right] dt\right)^{1/r} \leq C,$$

for all $\{D_k\} \subset \mathscr{D}$.

(iii) *There exists $C > 0$ such that*

$$\left(\int_0^1 \left[\sum_{k\in\mathbb{Z}} \left(\frac{V(D_k)+V(\Delta_k)t}{U(D_k)+U(\Delta_k)t}\right)^{r/q} U(\Delta_k)\right] dt\right)^{1/r} + \frac{V(X)^{1/q}}{U(X)^{1/p}} \leq C,$$

for all $\{D_k\} \subset \mathscr{D}$.

Additionally, if X is connected, these conditions are equivalent to

(iv) *There exists a constant $C \geq 0$ such that*

$$\left(\int_0^\infty U(D_{f,t})^{-r/p}\, d\big[-V(D_{f,t})^{r/q}\big]\right)^{1/r} \leq C,$$

for all $0 \leq f \downarrow$, where $D_{f,t} = \{x : f(x) > t\}$.

(v) *There exists a constant $C \geq 0$ such that*

$$\left(\sum_{k\in\mathbb{Z}} V(\Delta_k)^{r/q}\, U(D_{k+1})^{-r/p}\right)^{1/r} \leq C,$$

for all $\{D_k\} \subset \mathscr{D}$.

Proof. In the case (a), we apply Theorem 2.1 and (2) to characterize the embedding for sequences in $\mathbb{Z}$, and thus we get

$$A := \sup_{0\leq f\downarrow} \frac{\left(\int_X f(x)^q v(x)\, d\mu(x)\right)^{1/q}}{\left(\int_X f(x)^p u(x)\, d\mu(x)\right)^{1/p}} = \sup_{\{D_k\}\subset\mathscr{D}} \sup_{k\in\mathbb{Z}} \frac{V_k^{1/q}}{U_k^{1/p}} = \sup_{\{D_k\}\subset\mathscr{D}} \sup_{k\in\mathbb{Z}} \frac{V(D_k)^{1/q}}{U(D_k)^{1/p}},$$

observing that $U_k = U(D_{k+1})$, and similarly for V_k, and the right hand side of this equality is trivially equal to $\sup_{D\downarrow} \frac{V(D)^{1/q}}{U(D)^{1/p}}$. For the case (b), (2) and

Theorem 2.4 give

$$A \approx \sup_{\{D_k\}\subset\mathcal{D}} \left(\int_0^1 \left[\sum_{k\in\mathbb{Z}} \left(\frac{V_{k-1}+v_k t}{U_{k-1}+u_k t}\right)^{r/p} v_k\right] dt\right)^{1/r}$$
$$\approx \sup_{\{D_k\}\subset\mathcal{D}} \left(\left(\int_0^1 \left[\sum_{k\in\mathbb{Z}} \left(\frac{V_{k-1}+v_k t}{U_{k-1}+u_k t}\right)^{r/q} u_k\right] dt\right)^{1/r} + \frac{V_\infty^{1/q}}{U_\infty^{1/p}}\right),$$

and we finally get the characterizations (ii) and (iii) if we observe that $U_{k-1} = U(D_k)$ and $V_{k-1} = V(D_k)$. Let us see that (iii) implies (iv). Given a function $0 \le f \downarrow$, we assume that $U(D_{f,t})$ is continuous and strictly decreasing in t. The general case follows by a standard limiting argument. We define a positive decreasing sequence $t_{\mathbb{Z}}$ as follows: fix $t_0 = 1$ and

$$t_k = \sup\left\{t : U(D_{f,t}) = 2U(D_{f,t_{k-1}})\right\} \qquad \text{if} \quad k \ge 1,$$
$$t_k = \inf\left\{t : 2U(D_{f,t}) = U(D_{f,t_{k+1}})\right\} \qquad \text{if} \quad k \le -1.$$

We denote $D_k = D_{f,t_k}$ and we observe that this is a decreasing set for all $k \in \mathbb{Z}$. Using that $U(D_k) + U(\Delta_k)t \le U(D_{k+1})$, if $0 \le t \le 1$ we have:

$$\int_0^1 \sum_{k\in\mathbb{Z}} \left[\frac{V(D_k)+V(\Delta_k)t}{U(D_k)+U(\Delta_k)t}\right]^{r/p} V(\Delta_k)\, dt$$
$$\ge \sum_{k\in\mathbb{Z}} V(\Delta_k) U(D_{k+1})^{-r/p} \int_0^1 (V(D_k)+V(\Delta_k)t)^{r/p}\, dt$$
$$= 2^{-r/p}(q/r) \sum_k U(D_k)^{-r/p}(V(D_{k+1})^{r/q} - V(D_k)^{r/q}).$$

The last equality follows from the definition of the sequence $t_{\mathbb{Z}}$. It is now enough to see that the expression in (iv) is smaller than this last quantity, and this is done using the fact that $D_k \subset D_{f,t} \subset D_{k+1}$:

$$\int_0^\infty U(D_{f,t})^{-r/p}\, d\big[-V(D_{f,t})^{r/q}\big] = \sum_{k\in\mathbb{Z}} \int_{t_{k+1}}^{t_k} U(D_{f,t})^{-r/p}\, d\big[-V(D_{f,t})^{r/q}\big]$$
$$\le \sum_{k\in\mathbb{Z}} U(D_k)^{-r/p}(V(D_{k+1})^{r/q} - V(D_k)^{r/q}).$$

Let us see that (iv) implies (v). For a fixed family $\{D_k : k \in \mathbb{Z}\}$, define a decreasing function $f(x) = \sum_{k\in\mathbb{Z}} 2^{-k}\chi_{\Delta_k}(x)$. Then $D_k = \left\{x : f(x) > 2^{-k}\right\}$ and $D_{f,t} = D_{k+1}$ if $2^{-k-1} < t \le 2^{-k}$ and thus, splitting the integral into dyadic intervals, we have:

$$\int_0^\infty U(D_{f,t})^{-r/p}\, d\big[-V(D_{f,t})^{r/q}\big] = \sum_{k\in\mathbb{Z}} U(D_{k+1})^{-r/p} V(D_{k+1})^{r/p} V(\Delta_k)$$
$$\geq \sum_{k\in\mathbb{Z}} U(D_{k+1})^{-r/p}\, V(\Delta_k)^{r/q}.$$

Finally we prove that (v) implies (i). For a fixed decreasing function f, let $t_{\mathbb{Z}}$ be a decreasing sequence constructed in the same way as in the implication (ii) $\Rightarrow$ (iii). Also denote $D_k = D_{f,t_k}$ and $\Delta_k = D_{k+1}\backslash D_k = \{x : t_{k+1} < t \leq t_k\}$. Then, applying Hölder's inequality, we have:

$$\left(\int_X f(x)^q v(x)\, dx\right)^{1/q}$$
$$= \left(\sum_{k\in\mathbb{Z}} \int_{\Delta_k} f(x)^q v(x)\, dx\right)^{1/q} \leq \left(\sum_{k\in\mathbb{Z}} t_k^q V(\Delta_k)\right)^{1/q}$$
$$\leq \left(\sum_{k\in\mathbb{Z}} t_k^p U(D_{k+1})\right)^{1/p} \left(\sum_{k\in\mathbb{Z}} V(\Delta_k)^{r/q} U(D_{k+1})^{-r/p}\right)^{1/r}.$$

By construction, we have $U(\Delta_k) = \frac{1}{2}U(D_{k+1})$, and thus $U(\Delta_{k-1}) = \frac{1}{4}U(D_{k+1})$. Now, the hypothesis and this equality give:

$$\left(\int_X f(x)^q v(x)\, dx\right)^{1/q} \leq C4^{1/p}\left(\sum_{k\in\mathbb{Z}} t_k^p U(\Delta_{k-1})\right)^{1/p}$$
$$\leq C4^{1/p}\left(\int_X f(x)^p u(x)\, dx\right)^{1/p}.$$

The above theorem gives a simpler proof of the embedding characterization of the inequality (1) for $0 < q < p < \infty$ given in [4] for the particular case $X = \mathbb{R}^n_+$ equipped with the partial order defined by

$$(a_1, a_2, \ldots, a_n) \leq (b_1, b_2, \ldots, b_n) \tag{3}$$

if and only if $a_i \leq b_i$ for $i = 1, \ldots, n$, and μ be the Lebesgue measure. We observe that our notation is slightly different from the one used in that paper.

REMARK 2.9. All the results in this section can be reformulated for increasing functions with mild changes.

3. B_p weights and the discrete Hardy operator

Recall that B_p is the set of weights u for which the Hardy operator is bounded from $L^p_{\text{dec}}(u)$ to $L^p(u)$. Analogously, $B_{p,\infty}$ is the set of weights u for which the Hardy operator is bounded from $L^p_{\text{dec}}(u)$ to $L^{p,\infty}(u)$. We present a new characterization for a weight to be in the B_p class in terms of the boundedness of the discrete Hardy operator defined for sequences indexed in Z. Further characterizations of these classes are known, and we collect some of them in the next theorem, that will be used later.

THEOREM 3.1 ([1], [8], [16]).
(a) *For* $0 < p < \infty$, *the following conditions are equivalent:*
(i) $u \in B_p$.
(ii) *There exists a constant* $C > 0$ *such that*

$$\int_r^\infty \frac{u(x)}{x^p}\,dx \le C\frac{1}{r^p}\int_0^r u(x)\,dx, \quad \forall r > 0. \tag{4}$$

(iii) *There exists a constant* $C > 0$ *such that*

$$\int_0^r \frac{1}{U(x)^{1/p}}\,dx \le C\frac{r}{U(r)^{1/p}}, \quad \forall r > 0. \tag{5}$$

(b) *For* $0 < p \le 1$, *the following conditions are equivalent:*
(i) $u \in B_{p,\infty}$.
(ii) *There exists a constant* $C > 0$ *such that*

$$\frac{U(r)^{1/p}}{r} \le C\frac{U(s)^{1/p}}{s}, \qquad \forall\, 0 < s < r. \tag{6}$$

For the cases $X = \mathsf{N}$ or $X = \mathsf{Z}$, we recall that

$$\ell^p(u_X) = \left\{a_X : \left(\sum_{k\in X}|a_k|^p\,u_k\right)^{1/p} < \infty\right\},$$

and

$$\ell^{p,\infty}(u_X) = \left\{a_X : \|a_X\|_{\ell^{p,\infty}_{\text{dec}}(u_X)} = \sup_{t>0} t\left(\sum_{k\in E_t} u_k\right)^p < \infty\right\},$$

where $E_t = \{k \in X : |a_k| > t\}$. Then, $\ell^p_{\text{dec}}(u_X)$ and $\ell^{p,\infty}_{\text{dec}}(u_X)$ are the sets of positive decreasing sequences in $\ell^p(u_X)$ and $\ell^{p,\infty}(u_X)$, respectively. If a_X is a positive decreasing sequence, it easily can be shown that

$$\|a_X\|_{\ell^{p,\infty}_{\text{dec}}(u_X)} = \sup_{k\in X} U_k^{1/p} a_k. \tag{7}$$

Analogously, in the case of a positive decreasing function f defined in R_+, we also can write

$$\|f\|_{L^{p,\infty}_{\mathrm{dec}}(u)} = \sup_{t>0} U(t)^{1/p} f(t). \tag{8}$$

In [6] and [13], the authors studied a discrete Hardy operator defined for sequences indexed in N, namely

$$A_{\mathsf{N}} f(n) = \frac{1}{n+1} \sum_{j=0}^{n} f_j, \qquad n = 0, 1, 2, \dots,$$

where $(f_n)_{n\in\mathsf{N}} \subset \mathsf{R}$. In particular, they proved the following result:

THEOREM 3.2 ([6], [13]). *For $1 < p < \infty$, the following conditions are equivalent for a weight u_{N}:*

(i) $A_{\mathsf{N}} : \ell^p_{\mathrm{dec}}(u_{\mathsf{N}}) \longrightarrow \ell^{p,\infty}(u_{\mathsf{N}})$.

(ii) $A_{\mathsf{N}} : \ell^{p,\infty}_{\mathrm{dec}}(u_{\mathsf{N}}) \longrightarrow \ell^{p,\infty}(u_{\mathsf{N}})$.

(iii) $A_{\mathsf{N}} : \ell^p_{\mathrm{dec}}(u_{\mathsf{N}}) \longrightarrow \ell^p(u_{\mathsf{N}})$.

(iv) $\widetilde{u}(x) = \sum_{n=0}^{\infty} u_n \, \chi_{[n,n+1)}(x) \in B_p$.

(v) $\sum_{k=0}^{n} \frac{1}{U_k^{1/p}} \le C \frac{n+1}{U_n^{1/p}}$, *for all $n \ge 0$, where* $U_k = \sum_{j=0}^{k} u_j$.

We see that the boundedness $A_{\mathsf{N}} : \ell^p_{\mathrm{dec}}(u_{\mathsf{N}}) \longrightarrow \ell^p(u_{\mathsf{N}})$ for a discrete weight u_{N} is equivalent to $\widetilde{u} \in B_p$ for an extended weight $\widetilde{u} : \mathsf{R}_+ \longrightarrow \mathsf{R}_+$. Can this process be reversed in some sense? That is, if $u \in B_p$, is there a discrete weight u_{N} related to u such that A_{N} is bounded from $\ell^p_{\mathrm{dec}}(u_{\mathsf{N}})$ to $\ell^p(u_{\mathsf{N}})$? The answer is affirmative, as the next lemma state (the proof is trivial and it is left to the reader). We denote $B_p(\mathsf{N})$ the class of discrete weights u_{N} such that $A_{\mathsf{N}} : \ell^p_{\mathrm{dec}}(u_{\mathsf{N}}) \longrightarrow \ell^p(u_{\mathsf{N}})$ is bounded.

LEMMA 3.3. *If $u : \mathsf{R}_+ \longrightarrow \mathsf{R}_+$ is a B_p weight, then u_{N} defined by $u_n = \int_n^{n+1} u(x)\,dx$, is a $B_p(\mathsf{N})$ weight.*

Theorem 3.2 says that $B_p(\mathsf{N})$ can be viewed as a subset of the B_p weights that are constant at each interval $[n, n+1)$, and Lemma 3.3 says that every B_p weight which is constant at each interval $[n, n+1)$ can be viewed as a $B_p(\mathsf{N})$ weight. In other words,

$$B_p(\mathsf{N}) \equiv \left\{ u \in B_p : u(x) = c_n \; \forall x \in [n, n+1), \text{ for some positive } c_{\mathsf{N}} \right\}.$$

Now, another question arises. Can we characterize B_p as the class of weights such that the boundedness of A_{N} holds for the discretized weights? That is, can we characterize B_p in terms of $B_p(\mathsf{N})$? Now the answer is negative. There are weights which are not in B_p but their discretized ones are in $B_p(\mathsf{N})$. For example, take the weight

$$u(x) = \begin{cases} 0, & 0 < x < 1/2 \\ 2, & 1/2 \leq x < 1 \\ 1, & x \geq 1, \end{cases}$$

which is not a B_p weight because it equals zero in a neighborhood of 0 (contradicting condition (4)). But, it is easily seen that $\widetilde{u} \equiv 1$, and this is a B_p weight for all $1 < p < \infty$, and therefore A_{N} is bounded from $\ell^p_{\mathrm{dec}}(u_{\mathsf{N}})$ to $\ell^p(u_{\mathsf{N}})$, by Theorem 3.2, that is, u_{N} is a $B_p(\mathsf{N})$ weight.

In order to get complete results, we work with discrete weights indexed in Z rather than in N. The Hardy operator, defined for sequences f_{Z}, is defined by

$$A_{\mathsf{Z}} f(k) = \frac{1}{2^{k+1}} \sum_{j=-\infty}^{k} 2^j f_j, \qquad k \in \mathsf{Z}.$$

The function $A_{\mathsf{Z}} f$ is decreasing if f_{Z} is decreasing. A weight u_{Z} is in the $B_p(\mathsf{Z})$ ($B_{p,\infty}(\mathsf{Z})$) class if and only if $A_{\mathsf{Z}} : \ell^p_{\mathrm{dec}}(u_{\mathsf{Z}}) \longrightarrow \ell^p(u_{\mathsf{Z}})$ ($A_{\mathsf{Z}} : \ell^p_{\mathrm{dec}}(u_{\mathsf{Z}}) \longrightarrow \ell^{p,\infty}(u_{\mathsf{Z}})$).

THEOREM 3.4. *For a weight* $u : \mathsf{R}_+ \longrightarrow \mathsf{R}_+$, *denote* $u_k = \int_{2^k}^{2^{k+1}} u(x)\,dx$ *and* $\widetilde{u}(x) = \sum_{k\in\mathsf{Z}} \frac{u_k}{2^k} \chi_{[2^k,2^{k+1})}(x)$. *Then, if* $0 < p < \infty$, *the following conditions are equivalent:*

(i) $u \in B_p$.

(ii) $u_{\mathsf{Z}} \in B_p(\mathsf{Z})$.

(iii) $$\sum_{j=k+1}^{\infty} \frac{u_j}{(2^{j+1})^p} \leq C \frac{U_k}{(2^{k+1})^p}, \quad \forall k \in \mathsf{Z}.$$

(iv) $\widetilde{u} \in B_p$.

Additionally if $1 < p < \infty$, *these conditions are equivalent to*

(v) $u_{\mathsf{Z}} \in B_{p,\infty}(\mathsf{Z})$.

PROOF. Suppose that (i) holds. For a positive decreasing sequence f_{Z}, consider the decreasing function $\widetilde{f}(x) = \sum_{k\in\mathsf{Z}} f_k \chi_{[2^k,2^{k+1})}(x)$. Using that $\int_0^{2^{n+1}} \widetilde{f}(x)\,dx = \sum_{k\leq n} 2^k f_k$, and that $A\widetilde{f}$ is decreasing we easily have:

$$\sum_{k\in\mathsf{Z}} A_{\mathsf{Z}} f(k)^p u_k \leq \sum_{k\in\mathsf{Z}} \int_{2^k}^{2^{k+1}} \left(\frac{1}{x} \int_0^x \widetilde{f}(y)\,dy \right)^p u(x)\,dx \leq C \sum_{k\in\mathsf{Z}} f_k^p\, u_k,$$

and this is condition (ii). To see that (ii) implies (iii), it is enough to consider the boundedness of the operator on the functions $f(l) = \chi_{\{j:j\le k\}}(l)$, for all $k \in \mathsf{Z}$. Let us see that (iii) implies condition (4) for the weight $\widetilde{u}$, which is (iv). For $r > 0$, take $2^k \le r < 2^{k+1}$ and use that $\int_{2^j}^{2^{j+1}} \widetilde{u}(x)\,dx = u_j$:

$$\int_r^\infty \frac{\widetilde{u}(x)}{x^p}\,dx \le \sum_{j=k}^\infty \frac{u_j}{2^{jp}} \le 2^p C \frac{U_{k-1}}{2^{kp}} \le 4^p C \frac{1}{r}\int_0^r \widetilde{u}(x)\,dx.$$

That (iv) implies (i) is easy, if we use the characterization (4) of the B_p weights, and observe that for all k

$$\int_{2^k}^{2^{k+1}} \frac{u(x)}{x^p}\,dx \approx \int_{2^k}^{2^{k+1}} \frac{\widetilde{u}(x)}{x^p}\,dx,$$

and $\int_0^{2^k} u(x)\,dx = \int_0^{2^k} \widetilde{u}(x)\,dx$. Trivially, (ii) implies (v). Finally, let us see that (v) implies (iv). The boundedness of A_{Z} is equivalent, by (7), to

$$\frac{U_n^{1/p}}{2^{n+1}} \sup_{0\le(f_k)\downarrow} \frac{\sum_{j\le n} 2^j f_j}{\left(\sum_{k\in\mathsf{Z}} f_k^p\, u_k\right)^{1/p}} \le C.$$

We use Proposition 2.3 with weights $v_k = 2^k$, if $k \le n$, and 0 otherwise, and u_k to obtain that the boundedness of A_{Z} is equivalent to

$$\frac{U_n^{1/p}}{2^{n+1}} \sup_{0\le f\downarrow} \frac{\int_0^{2^{n+1}} f}{\left(\int_0^\infty f(x)^p \widetilde{u}(x)\,dx\right)^{1/p}} \le C,$$

and using the fact that $\widetilde{U}(2^{n+1}) = U_n$, this is

$$\widetilde{U}(2^{n+1})^{1/p} Af(2^{n+1}) \le C\left(\int_0^\infty f(x)^p \widetilde{u}(x)\,dx\right)^{1/p},$$

for all positive decreasing f. We claim that this condition also holds for all $t > 0$ instead of 2^{n+1}. Observe that the hypothesis on A_{Z} implies the necessity of condition (9) (simply by taking $f = \chi_{\{j:j\le k\}}$ for every $k \in \mathsf{Z}$), and therefore $\widetilde{U}(2^{n+1})^{1/p} = U_n^{1/p} \le C U_{n-1}^{1/p} = C\widetilde{U}(2^n)^{1/p}$. Then, if $2^n < t \le 2^{n+1}$ we have

$$\widetilde{U}(t)^{1/p} Af(t) \le C\widetilde{U}(2^n)^{1/p} Af(2^n) \le C\left(\int_0^\infty f(x)^p \widetilde{u}(x)\,dx\right)^{1/p}.$$

This last condition is equivalent to $A : L^p_{\text{dec}}(\widetilde{u}) \to L^{p,\infty}(\widetilde{u})$ by (8), and this is equivalent to $\widetilde{u} \in B_p$, since $B_p = B_{p,\infty}$ if $1 < p < \infty$ (see [12]).

In the following theorem, we complete the results by considering the case of the $B_{p,\infty}$ weights in the range $0 < p \le 1$.

THEOREM 3.5. *For a weight $u : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, denote $u_k = \int_{2^k}^{2^{k+1}} u(x)\,dx$ and $\widetilde{u}(x) = \sum_{k\in\mathbb{Z}} \frac{u_k}{2^k} \chi_{[2^k,2^{k+1})}(x)$. Then, if $0 < p \le 1$, the following conditions are equivalent:*

(i) $u \in B_{p,\infty}$.

(ii) $u_{\mathbb{Z}} \in B_{p,\infty}(\mathbb{Z})$.

(iii) $\dfrac{U_n^{1/p}}{2^{n+1}} \le C \dfrac{U_k^{1/p}}{2^{k+1}}, \ \ \forall k \le n.$

(iv) $\widetilde{u} \in B_{p,\infty}$.

PROOF. Let us see that condition (ii) holds if and only if (iii) holds. The boundedness of $A_{\mathbb{Z}}$ is equivalent, by (7), to

$$\frac{U_n^{1/p}}{2^{n+1}} \sup_{0\le(f_k)\downarrow} \frac{\displaystyle\sum_{j\le n} 2^j f_j}{\left(\displaystyle\sum_{k\in\mathbb{Z}} f_k^p u_k\right)^{1/p}} \le C,$$

for all decreasing sequences $f_{\mathbb{Z}}$. Using Theorem 2.1 with $v_k = 2^k$, if $k \le n$, and 0 otherwise, and $q = 1$, the last expression is equivalent to

$$\frac{U_n^{1/p}}{2^{n+1}} \frac{2^{k+1}}{U_k^{1/p}} \le C, \qquad \forall\, k \le n, \tag{9}$$

which is (iii). Let us see that (iii) holds if and only if (iv) holds. Using that $\widetilde{U}(2^{n+1}) = U_n$, it is not difficult to see that (9) is equivalent to

$$\frac{\widetilde{U}(r)^{1/p}}{r} \le C \frac{\widetilde{U}(s)^{1/p}}{s}, \qquad \forall\, 0 < s < r,$$

and this condition is actually equivalent to (iv) by (6). Finally, let us see that (i) is equivalent to (iii). As before, using that $U(2^{n+1}) = U_n$, it is easy to see that condition (9) is equivalent to

$$\frac{U(r)^{1/p}}{r} \le C \frac{U(s)^{1/p}}{s}, \qquad \forall\, 0 < s < r,$$

which is in fact equivalent to (i) by (6).

Corollary 3.6. *If* $0 < p < \infty$ *and* $u_k = \int_{2^k}^{2^{k+1}} u(x)\,dx$ *for a weight u in* R_+*, we have*

$$B_p = \{u \ge 0 : u_{\mathsf{Z}} \in B_p(\mathsf{Z})\} \qquad \textit{and} \qquad B_{p,\infty} = \{u \ge 0 : u_{\mathsf{Z}} \in B_{p,\infty}(\mathsf{Z})\} .$$

Our last result is the next theorem.

Theorem 3.7. *For* $0 < p < \infty$*, we have*

$$A_{\mathsf{Z}} : \ell^{p,\infty}_{\text{dec}}(u_{\mathsf{Z}}) \longrightarrow \ell^{p,\infty}(u_{\mathsf{Z}})$$

if and only if

$$\sum_{j \le k} \frac{2^{j+1}}{U_j^{1/p}} \le C \frac{2^{k+1}}{U_k^{1/p}}, \qquad \forall\, k \in \mathsf{Z}.$$

Proof. The boundedness of A_{Z} is equivalent to

$$\|A_{\mathsf{Z}} f\|_{\ell^{p,\infty}(u_{\mathsf{Z}})} \le C\, \|f\|_{\ell^{p,\infty}(u_{\mathsf{Z}})} ,$$

for all decreasing f_{Z}. We observe that the sequence $f_k = U_k^{-1/p}$ is decreasing, and that $\|f\|_{\ell^{p,\infty}(u_{\mathsf{Z}})} = 1$, by using (7), and therefore the boundedness of A_{Z} implies

$$\|A_{\mathsf{Z}} f\|_{\ell^{p,\infty}(u_{\mathsf{Z}})} \le C. \tag{10}$$

On the other hand, we observe that for every decreasing $f_{\mathsf{Z}} \in \ell^{p,\infty}(u_{\mathsf{Z}})$, we have that $f_k \le C U_k^{-1/p}$ for all $k \in \mathsf{Z}$, and this implies that $A_{\mathsf{Z}} f(k) \le A_{\mathsf{Z}}(U^{-1/p})(k)$ for all $k \in \mathsf{Z}$, and thus, (10) is also sufficient for the boundedness of A_{Z}. Now, if we write condition (10) by using (7), we find the desired condition.

By considering Theorems 3.4 and 3.7, we can state the following result:

Corollary 3.8. *If* $1 < p < \infty$*, the following conditions are equivalent for a weight* u_{Z}*:*

(i) $\widetilde{u}(x) = \sum_{k \in \mathsf{Z}} \frac{u_k}{2^k} \chi_{[2^k, 2^{k+1})}(x) \in B_p$.

(ii) $\sum_{j \le k} \frac{2^{j+1}}{U_j^{1/p}} \le C \frac{2^{k+1}}{U_k^{1/p}}, \quad \forall\, k \in \mathsf{Z}$.

(iii) $u_{\mathsf{Z}} \in B_{p,\infty}(\mathsf{Z})$.

(iv) $A_{\mathsf{Z}} : \ell^{p,\infty}_{\text{dec}}(u_{\mathsf{Z}}) \longrightarrow \ell^{p,\infty}(u_{\mathsf{Z}})$.

(v) $u_{\mathsf{Z}} \in B_p(\mathsf{Z})$.

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