CONSTRUCTION AND PURE INFINITENESS OF C*-ALGEBRAS ASSOCIATED WITH LAMBDA-GRAPH SYSTEMS

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Abstract
A \( \lambda \)-graph system is a labeled Bratteli diagram with shift transformation. It is a generalization of finite labeled graphs and presents a subshift. In [16] the author has introduced a C*-algebra \( \mathcal{O}_\mathcal{V} \) associated with a \( \lambda \)-graph system \( \mathcal{V} \) by using groupoid method as a generalization of the Cuntz-Krieger algebras. In this paper, we concretely construct the C*-algebra \( \mathcal{O}_\mathcal{V} \) by using both creation operators and projections on a sub Fock Hilbert space associated with \( \mathcal{V} \). We also introduce a new irreducible condition on \( \mathcal{V} \) under which the C*-algebra \( \mathcal{O}_\mathcal{V} \) becomes simple and purely infinite.

0. Introduction
For a finite set \( \Sigma \), a subshift \((\Lambda, \sigma)\) is a topological dynamics defined by a closed shift-invariant subset \( \Lambda \) of the compact set \( \Sigma^\mathbb{Z} \) of all bi-infinite sequences of \( \Sigma \) with shift transformation \( \sigma \) defined by \( \sigma((x_i)_{i \in \mathbb{Z}}) = (x_{i+1})_{i \in \mathbb{Z}} \). The author has introduced the notions of symbolic matrix system and \( \lambda \)-graph system as presentations of subshifts ([15]). They are generalized notions of symbolic matrix and \( \lambda \)-graph (= labeled graph) for sofic subshifts. We henceforth denote by \( \mathbb{Z}_+ \) and by \( \mathbb{N} \) the set of all nonnegative integers and the set of all positive integers respectively. A symbolic matrix system \((\mathcal{M}, I)\) over \( \Sigma \) consists of two sequences of rectangular matrices \((M_{l,l+1}, I_{l,l+1})_{l \in \mathbb{Z}_+}\). The matrices \(M_{l,l+1}\) have their entries in formal sums of \( \Sigma \) and the matrices \(I_{l,l+1}\) have their entries in \( \{0, 1\} \). They satisfy the following commutation relations
\[
I_{l,l+1} M_{l+1,l+2} = M_{l,l+1} I_{l+1,l+2}, \quad l \in \mathbb{Z}_+.
\]
It is required that each row of \( I_{l,l+1} \) has at least one 1 and each column of \( I_{l,l+1} \) has exactly one 1. A \( \lambda \)-graph system \( \mathcal{V} = (V, E, \lambda, \iota) \) over \( \Sigma \) consists of a vertex set \( V = V_0 \cup V_1 \cup V_2 \cup \cdots \), an edge set \( E = E_{0,1} \cup E_{1,2} \cup E_{2,3} \cup \cdots \), a labeling map \( \lambda : E \to \Sigma \) and a surjective map \( \iota (= \iota_{l,l+1}) : V_{l+1} \to V_l \) for each \( l \in \mathbb{Z}_+ \). It naturally arises from a symbolic matrix system \((\mathcal{M}, I)\). The labeled edges from a vertex \( v_i^l \in V_l \) to a vertex \( v_j^{l+1} \in V_{l+1} \) are given by

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the \((i, j)\)-component \(M_{l,l+1}(i, j)\) of \(M_{l,l+1}\). The map \(\iota = \iota_{l,l+1}\) is defined by 
\[
\iota_{l,l+1}(v_{i,j}^{I_{l,l+1}}) = v_i^j \text{ precisely if } I_{l,l+1}(i, j) = 1.
\]
The symbolic matrix systems and the \(\lambda\)-graph systems are the same objects and give rise to subshifts by gathering label sequences appearing in the labeled Bratteli diagrams of the \(\lambda\)-graph systems. Conversely we have a canonical method to construct a symbolic matrix system and a \(\lambda\)-graph system from an arbitrary subshift [15].

In [16], the author has constructed \(C^*\)-algebras from \(\lambda\)-graph systems as groupoid \(C^*\)-algebras by using continuous graphs in the sense of Deaconu (cf. [5], [19]) and studied their structure. Let \(\mathcal{U} = (V, E, \lambda, \iota)\) be a \(\lambda\)-graph system over \(\Sigma\). Let \(\{v_1^I, \ldots, v_m^I\}\) be the vertex set \(V_I\). The \(C^*\)-algebra \(O_\mathcal{U}\) is generated by partial isometries \(S_\alpha\) corresponding to the symbols \(\alpha \in \Sigma\) and projections \(E_{l,i}\) corresponding to the vertices \(v_i^l \in V_l, i = 1, \ldots, m(l), l \in \mathbb{Z}_+\). It is realized as a universal unique \(C^*\)-algebra subject to certain operator relations among \(S_\alpha, \alpha \in \Sigma\) and \(E_{l,i}\), \(i = 1, \ldots, m(l), l \in \mathbb{Z}_+\) encoded by the structure of \(\mathcal{U}\). A condition on \(\mathcal{U}\), called condition (I), has been introduced ([16]). Irreducibility and aperiodicity for \(\mathcal{U}\) have been also defined so that if \(\mathcal{U}\) satisfies condition (I) and is irreducible, the \(C^*\)-algebra \(O_\mathcal{U}\) is shown to be simple. It is also proved that if in particular \(\mathcal{U}\) is aperiodic, \(O_\mathcal{U}\) is simple and purely infinite ([16, Theorem 4.7 and Proposition 4.9]).

In this paper, we will first introduce a new construction of the \(C^*\)-algebras \(O_\mathcal{U}\). We will construct a sub Fock Hilbert space associated with a \(\lambda\) graph system \(\mathcal{U}\) and define creation operators and sequence of projections. We will then show that \(O_\mathcal{U}\) is canonically isomorphic to the quotient \(C^*\)-algebra of the \(C^*\)-algebra generated by the creation operators and the projections by an ideal (Theorem 2.6). This construction is a generalization of a construction of Cuntz-Krieger algebras [3] by [7], [8] and \(C^*\)-algebras associated with subshifts [14]. We will next introduce a new irreducible condition and new condition (I) on \(\mathcal{U}\) such that \(O_\mathcal{U}\) becomes simple and purely infinite (Theorem 3.9). The new conditions are called \(\lambda\)-irreducible condition and \(\lambda\)-condition (I) respectively. In the previously proved result [16, Theorem 4.7 and Proposition 4.9], we needed aperiodicity condition on \(\mathcal{U}\) for \(O_\mathcal{U}\) to be simple and purely infinite. It is well-known that the Cuntz-Krieger algebra \(O_A\) is simple and purely infinite if the matrix \(A\) is irreducible with condition (I). Since the \(C^*\)-algebras \(O_\mathcal{U}\) are a generalization of the Cuntz-Krieger algebras \(O_A\), the aperiodicity condition on \(\mathcal{U}\) is too strong such that \(O_\mathcal{U}\) becomes simple and purely infinite. From this point of view, the \(\lambda\)-irreducible condition with \(\lambda\)-condition (I) on \(\mathcal{U}\) is an exact generalization of the irreducible condition with condition (I) on the nonnegative matrices \(A\).

The author would like to thank the referee who named the terms \(\lambda\)-irreducible and \(\lambda\)-condition (I) instead of the originally used terms (new) irreducible and (new) condition (I), and for his useful comments.
1. Review of $C^*$-algebras associated with $\lambda$-graph systems

For a $\lambda$-graph system $\mathcal{U} = (V, E, \lambda, i)$ over $\Sigma$, the vertex sets $V_l, l \in \mathbb{Z}_+$ and the edge sets $E_{l,l+1}, l \in \mathbb{Z}_+$ are finite disjoint sets. An edge $e$ in $E_{l,l+1}$ has its source vertex $s(e)$ in $V_l$ and its terminal vertex $t(e)$ in $V_{l+1}$. Every vertex in $V$ has outgoing edges and every vertex in $V_l$ except $V_0$, has incoming edges. The label of an edge $e \in E$ means $\lambda(e) \in \Sigma$. It is then required that there exists an edge in $E_{l,l+1}$ with label $\alpha$ and its terminal is $v \in V_{l+1}$ if and only if there exists an edge in $E_{l-1,l}$ with label $\alpha$ and its terminal is $i(v) \in V_l$. For $u \in V_{l-1}$ and $v \in V_{l+1}$, we put

$$E'(u, v) = \{ e \in E_{l,l+1} \mid t(e) = v, t(s(e)) = u \},$$

$$E_l(u, v) = \{ e \in E_{l-1,l} \mid s(e) = u, t(e) = i(v) \}.$$

Then there exists a bijective correspondence between $E'(u, v)$ and $E_l(u, v)$ that preserves labels for every pair $(u, v) \in V_{l-1} \times V_{l+1}$. This property is called the local property of the $\lambda$-graph system. A finite sequence $(e_1, e_2, \ldots, e_n)$ of edges such that $t(e_i) = s(e_{i+1})$, $i = 1, 2, \ldots, n-1$ is called a path. We put $\Sigma_i = \Sigma$ and define

$$\Lambda_{\mathcal{U}}^+ = \{ (\lambda(e_1), \lambda(e_2), \ldots) \in \prod_{i \in \mathbb{N}} \Sigma_i \mid \alpha_i \in E_{i-1,i}, t(e_i) = s(e_{i+1}), i \in \mathbb{N} \}$$

and

$$\Lambda_{\mathcal{U}} = \{ (\alpha_i)_{i \in \mathbb{Z}} \in \prod_{i \in \mathbb{Z}} \Sigma_i \mid (\alpha_i, \alpha_{i+1}, \ldots) \in \Lambda_{\mathcal{U}}^+, i \in \mathbb{Z} \}.$$

Then $\Lambda_{\mathcal{U}}$ is a subshift over $\Sigma$ called the subshift presented by $\mathcal{U}$. A finite sequence $\mu = (\mu_1, \ldots, \mu_k)$ of $\mu_j \in \Sigma$ that appears in $\Lambda_{\mathcal{U}}$ is called an admissible word of $\mathcal{U}$ of length $|\mu| = k$. Denote by $\Lambda_{\mathcal{U}}^k$ the set of all admissible words of length $k$ of $\mathcal{U}$ and put $\Lambda_{\mathcal{U}}^\alpha = \bigcup_{k=0}^{\infty} \Lambda_{\mathcal{U}}^k$ where $\Lambda_{\mathcal{U}}^0$ denotes the empty word $\emptyset$.

We briefly review the $C^*$-algebra $\mathcal{O}_\mathcal{U}$ associated with $\lambda$-graph system $\mathcal{U}$, that has been originally constructed in [16] to be a groupoid $C^*$-algebra of a groupoid of a continuous graph obtained by $\mathcal{U}$ (cf. [5], [6], [19]).

Let $\mathcal{U} = (V, E, \lambda, i)$ be a left-resolving $\lambda$-graph system over $\Sigma$, that is, for $e, e' \in E$, $\lambda(e) = \lambda(e')$, $t(e) = t(e')$ implies $e = e'$. The vertex set $V_l$ is denoted by $\{v_{1}^l, \ldots, v_{m(l)}^l\}$. Define the transition matrices $A_{l,l+1}, I_{l,l+1}$ of $\mathcal{U}$ by setting for $i = 1, 2, \ldots, m(l), j = 1, 2, \ldots, m(l+1), \alpha \in \Sigma$,

$$A_{l,l+1}(i, \alpha, j) = \begin{cases} 1 & \text{if } s(e) = v_{j}^l, \lambda(e) = \alpha, t(e) = v_{j+1}^l \text{ for some } e \in E_{l,l+1}, \\ 0 & \text{otherwise}, \end{cases}$$

$$I_{l,l+1}(i, j) = \begin{cases} 1 & \text{if } t_{l,l+1}(v_{j+1}^l) = v_{j}^l, \\ 0 & \text{otherwise}. \end{cases}$$
The $C^*$-algebra $\mathcal{O}_U$ is realized as the universal unital $C^*$-algebra generated by partial isometries $S_\alpha$, $\alpha \in \Sigma$ and projections $E_i^l$, $i = 1, 2, \ldots, m(l)$, $l \in \mathbb{Z}_+$ subject to the following operator relations called $(\mathcal{U})$:

\begin{align}
(1.1) & \quad \sum_{\alpha \in \Sigma} S_\alpha S_\alpha^* = 1, \\
(1.2) & \quad \sum_{i=1}^{m(l)} E_i^l = 1, \quad E_i^l = \sum_{j=1}^{m(l+1)} I_{l,j+1}(i, j)E_j^{l+1}, \\
(1.3) & \quad S_\beta S_\beta^* E_i^l = E_i^l S_\beta S_\beta^*, \\
(1.4) & \quad S_\beta^* E_i^l S_\beta = \sum_{j=1}^{m(l+1)} A_{l,j+1}(i, \beta, j)E_j^{l+1},
\end{align}

for $\beta \in \Sigma$, $i = 1, 2, \ldots, m(l)$, $l \in \mathbb{Z}_+$.

For a vertex $v_i^l \in V_l$, we denote by $\Gamma^+(v_i^l)$ the set

$$\Gamma^+(v_i^l) = \{ (\lambda(e_1), \lambda(e_2), \ldots) \in \Lambda^+_U \mid s(e_1) = v_i^l, t(e_j) = s(e_{j+1}), j \in \mathbb{N} \}$$

of all infinite label sequences in $\mathcal{U}$ starting at $v_i^l$. We say that $\mathcal{U}$ satisfies condition (I) if for each $v_i^l \in V$, the set $\Gamma^+(v_i^l)$ contains at least two distinct label sequences.

**Theorem 1.1** ([16]). Suppose that $\mathcal{U}$ satisfies condition (I). Let $\hat{S}_\alpha$, $\alpha \in \Sigma$ and $\hat{E}_i^l$, $i = 1, 2, \ldots, m(l)$, $l \in \mathbb{Z}_+$ be another family of nonzero partial isometries and nonzero projections satisfying the relations $(\mathcal{U})$. Then the map $S_\alpha \rightarrow \hat{S}_\alpha$, $E_i^l \rightarrow \hat{E}_i^l$ extends to an isomorphism from $\mathcal{O}_U$ onto the $C^*$-algebra $\mathcal{O}_U$ generated by $\hat{S}_\alpha$, $\alpha \in \Sigma$ and $\hat{E}_i^l$, $i = 1, 2, \ldots, m(l)$, $l \in \mathbb{Z}_+$.

Hence the $C^*$-algebra $\mathcal{O}_U$ under the condition that $\mathcal{U}$ satisfies condition (I) is the unique $C^*$-algebra subject to the above relations $(\mathcal{U})$. By the uniqueness of $\mathcal{O}_U$, the correspondence $S_\alpha \rightarrow zS_\alpha$, $E_i^l \rightarrow E_i^l$ for $z \in T = \{ z \in \mathbb{C} \mid |z| = 1 \}$ yields an action $\alpha_U$ of $T$ called the gauge action. Let $\mathcal{F}_k^l$ be the finite dimensional $C^*$-subalgebra of $\mathcal{O}_U$ generated by $\hat{S}_\mu E_i^l S_\mu^*$, $\mu, v \in \Lambda^+_U$, $i = 1, 2, \ldots, m(l)$. Let $\mathcal{F}_k$ be the $C^*$-subalgebra of $\mathcal{O}_U$ generated by the algebras $\mathcal{F}_k^l$, $k \leq l$. It is an AF-algebra realized as the fixed point algebra $\mathcal{O}_U^{\alpha_U}$ of $\mathcal{O}_U$ under $\alpha_U$.

A $\lambda$-graph system $\mathcal{U}$ is said to be **irreducible** if for a vertex $v_i^l \in V_l$ and a sequence $(u^0, u^1, \ldots)$ of vertices $u^n \in V_n$ with $t_{n,n+1}(u^{n+1}) = u^n$, $n \in \mathbb{Z}_+$, there exists a path starting at $v_i^l$ and terminating at $u^{l+N}$ for some $N \in \mathbb{N}$. $\mathcal{U}$ is said to be **aperiodic** if for a vertex $v_i^l \in V_l$ there exists an $N \in \mathbb{N}$ such that there exist paths starting at $v_i^l$ and terminating at all vertices of $V_{l+N}$. These properties for $\lambda$-graph systems are generalizations of the corresponding properties for finite directed graphs.
THEOREM 1.2 ([16], Proposition 4.9). Suppose that a \( \lambda \)-graph system \( \mathcal{L} \) satisfies condition (I). If \( \mathcal{L} \) is irreducible, the \( C^* \)-algebra \( \mathcal{O}_\mathcal{L} \) is simple. If in particular \( \mathcal{L} \) is aperiodic, \( \mathcal{O}_\mathcal{L} \) is simple and purely infinite.

In what follows, we fix a left-resolving \( \lambda \)-graph system \( \mathcal{L} = (V, E, \lambda, \iota) \) over \( \Sigma \).

2. Fock space construction

In this section, we will construct a family of partial isometries and projections satisfying the relations (\( \mathcal{L} \)) in a concrete way. Let \( \Omega_\mathcal{L} \) be the projective limit

\[
\Omega_\mathcal{L} = \left\{ (u^l)_{l \in \mathbb{Z}_+} \in \prod_{l \in \mathbb{Z}_+} V_l \mid u_{l+1}(u^{l+1}) = u^l, l \in \mathbb{Z}_+ \right\}
\]

of the system \( u_{l+1} : V_{l+1} \to V_l, l \in \mathbb{Z}_+ \). We endow \( \Omega_\mathcal{L} \) with the projective limit topology from the discrete topologies on \( V_l, l \in \mathbb{Z}_+ \) so that it is a compact Hausdorff space. An element \( u \) in \( \Omega_\mathcal{L} \) is called a vertex. Let \( E_\mathcal{L} \) be the set of all triplets \((u, \alpha, w) \in \Omega_\mathcal{L} \times \Sigma \times \Omega_\mathcal{L}\) such that there exists \( e_{l+1} \in E_{l+1} \) satisfying \( u^l = s(e_{l+1}), w^{l+1} = t(e_{l+1}) \) and \( \alpha = \lambda(e_{l+1}) \) for each \( l \in \mathbb{Z}_+ \) where \( u = (u^l)_{l \in \mathbb{Z}_+}, w = (w^l)_{l \in \mathbb{Z}_+} \in \Omega_\mathcal{L} \). The set \( E_\mathcal{L} \subset \Omega_\mathcal{L} \times \Sigma \times \Omega_\mathcal{L} \) is a continuous graph in the sense of Deaconu ([14, Proposition 2.1]). For \( w = (w^l)_{l \in \mathbb{Z}_+} \in \Omega_\mathcal{L} \) and \( \alpha \in \Sigma \), the local property of \( \mathcal{L} \) ensures that if there exists \( e_{0,1} \in E_{0,1} \) satisfying \( w^1 = e_{0,1}, \alpha = \lambda(e_{0,1}) \), there uniquely exist \( e_{l+1} \in E_{l+1} \) and \( u = (u^l)_{l \in \mathbb{Z}_+} \in \Omega_\mathcal{L} \) satisfying \( u^l = s(e_{l+1}), w^{l+1} = t(e_{l+1}) \) and \( \alpha = \lambda(e_{l+1}) \) for all \( l \in \mathbb{Z}_+ \). Hence for every \( w \in \Omega_\mathcal{L} \), there exist \( \alpha \in \Sigma \) and \( u \in \Omega_\mathcal{L} \) such that \((u, \alpha, w) \in E_\mathcal{L} \). Let us consider the finite path spaces of the graph \( E_\mathcal{L} \) as follows:

\[
W_\mathcal{L}^0 = \Omega_\mathcal{L},
W_\mathcal{L}^1 = E_\mathcal{L},
W_\mathcal{L}^2 = \{(u_0, \alpha_1, u_1, \alpha_2, u_2) \mid (u_0, \alpha_1, u_1), (u_1, \alpha_2, u_2) \in E_\mathcal{L}\},
\]

\[
\ldots
W_\mathcal{L}^k = \{(u_0, \alpha_1, u_1, \alpha_2, \ldots, \alpha_k, u_k) \mid (u_{i-1}, \alpha_i, u_i) \in E_\mathcal{L}, i = 1, 2, \ldots, k\},
\]

\[
\ldots
\]

We assign to a finite path \( \eta \in W_\mathcal{L}^k \) the vector \( e_\eta \). For each \( k \in \mathbb{Z}_+ \), let \( \mathcal{H}_\mathcal{L}^k \) be the Hilbert space spanned by the complete orthonormal basis \( \{e_\eta \mid \eta \in W_\mathcal{L}^k\} \). The Hilbert space \( \mathcal{H}_\mathcal{L} \) is defined by their direct sums

\[
\mathcal{H}_\mathcal{L} = \bigoplus_{k=0}^{\infty} \mathcal{H}_\mathcal{L}^k.
\]
We define creation operators $T_\beta$ for $\beta \in \Sigma$ and projections $P_i^l$ for $v_i^l \in V$ on $\mathcal{H}$ by setting

$$T_\beta e(u_0, \alpha_1, u_1, \alpha_2, \ldots, \alpha_k, u_k) = \begin{cases} e(u_{-1}, \beta, u_0, \alpha_1, u_1, \alpha_2, \ldots, \alpha_k, u_k) & \text{if there exists } u_{-1} \in \Omega \text{ such that } (u_{-1}, \beta, u_0) \in E, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$P_i^l e(u_0, \alpha_1, u_1, \alpha_2, \ldots, \alpha_k, u_k) = \begin{cases} e(u_0, \alpha_1, u_1, \alpha_2, \ldots, \alpha_k, u_k) & \text{if } u_0^l = v_i^l, \text{ where } u_0 = (u_0^l)_{l \in \mathbb{Z}} \in \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the vertex $u_{-1} \in \Omega$ satisfying $(u_{-1}, \beta, u_0) \in E$ is unique for $\beta$ and $u_0$ if it exists, because $\mathcal{L}$ is left-resolving. It is direct to see that

$$T_\beta^* e(u_0, \alpha_1, u_1, \alpha_2, \ldots, \alpha_k, u_k) = \begin{cases} e(u_1, \alpha_2, \ldots, \alpha_k, u_k) & \text{if } k \geq 1 \text{ and } \alpha_1 = \beta, \\ 0 & \text{otherwise.} \end{cases}$$

**Lemma 2.1.** For $\beta \in \Sigma$

(i) $T_\beta T_\beta^*$ is the projection onto the subspace spanned by the vectors $e_\eta$ such that $\eta = (u_0, \alpha_1, u_1, \alpha_2, \ldots, \alpha_k, u_k) \in W_\mathcal{L}, \alpha_1 = \beta, k \in \mathbb{N}$,

(ii) $T_\beta^* T_\beta$ is the projection onto the subspace spanned by the vectors $e_\xi$ such that $\xi = (u_0, \alpha_1, u_1, \alpha_2, \ldots, \alpha_k, u_k) \in W_\mathcal{L}$, $k \in \mathbb{Z}_+$, $(u_{-1}, \beta, u_0) \in E$ for some $u_{-1} \in \Omega$.

Let $P_0$ denote the projection on $\mathcal{H}$ onto the subspace $\mathcal{H}^0$. It is immediate to see that $P_0 T_\beta = 0$ for $\beta \in \Sigma$ and $P_0 P_i^l = P_i^l P_0$ for $v_i^l \in V$. We then have

**Lemma 2.2.**

(2.1) $\sum_{\alpha \in \Sigma} T_\alpha^* T_\alpha + P_0 = 1$,

(2.2) $\sum_{i=1}^{m(l)} P_i^l = 1, \quad P_i^l = \sum_{j=1}^{m(l+1)} I_{l+1}(i, j) P_j^{l+1}$,

(2.3) $T_\beta T_\beta^* P_i^l = P_i^l T_\beta T_\beta^*$,

(2.4) $T_\beta^* P_i^l T_\beta = \sum_{j=1}^{m(l+1)} A_{l+1}(i, \beta, j) P_j^{l+1}$,

for $\beta \in \Sigma$, $i = 1, 2, \ldots, m(l)$, $l \in \mathbb{Z}_+$. 
Proof. We will show the relation (2.4). Other relations are direct. For \( \beta \in \Sigma, v^j_i \in V, (u_0, \alpha_1, u_1, \alpha_2, \ldots, \alpha_k, u_k) \in W^k_{\beta}, \) it follows that

\[
T^*_\beta P^j_i T^*_\alpha e_{(u_0, \alpha_1, u_1, \alpha_2, \ldots, \alpha_k, u_k)} = \begin{cases} 
   e_{(u_0, \alpha_1, u_1, \alpha_2, \ldots, \alpha_k, u_k)} & \text{if } (u_{-1}, \beta, u_0) \in E_{v_j_i} \text{ for some } u_{-1} \in \Omega_{v_j_i} \\
   0 & \text{otherwise},
\end{cases}
\]

if \( s(e) = v^j_i, t(e) = u^i_{l+1}, \lambda(e) = \beta \) for some \( e \in E_{l,l+1}, \)

\[
= \begin{cases} 
   0 & \text{otherwise},
\end{cases}
\]

\[
= \sum_{j=1}^{m(l+1)} A_{l,l+1}(i, \beta, j) P^{l+1}_j e_{(u_0, \alpha_1, u_1, \alpha_2, \ldots, \alpha_k, u_k)}.
\]

Hence the relation (2.4) holds.

For a word \( v = \alpha_1 \cdots \alpha_k \in \Lambda^*_\beta, \) we set \( T_v = T_{\alpha_1} \cdots T_{\alpha_k}. \)

Lemma 2.3. Every polynomial of \( T_\alpha, P^j_i, \alpha \in \Sigma, i = 1, 2, \ldots, m(l), l \in \mathbb{Z}_+, \)

is a finite linear combination of elements of the form \( T_\mu P^j_i T^*_v \) for \( \mu, v \in \Lambda^*_\beta, \)

\( i = 1, 2, \ldots, m(l), l \in \mathbb{Z}_+. \)

Proof. It follows that by (2.3) and (2.4)

\[
P^j_i T^*_\alpha = T^*_\alpha T^*_\mu P^j_i T^*_\alpha = \sum_{j=1}^{m(l+1)} A_{l,l+1}(i, \alpha, j) T^*_\alpha P^{l+1}_j
\]

and hence

\[
T^*_\alpha P^j_i = \sum_{j=1}^{m(l+1)} A_{l,l+1}(i, \alpha, j) P^{l+1}_j T^*_\alpha.
\]

The assertion is immediately seen by these equations.

Let \( \mathcal{T}_\beta \) be the \( C^* \)-algebra on \( \mathfrak{H}_\beta \) generated by \( T_\alpha, P^j_i, P_0, \alpha \in \Sigma, i = 1, 2, \ldots, m(l), l \in \mathbb{Z}_+ \) and \( \mathcal{I} \) the closed two-sided ideal of \( \mathcal{T}_\beta \) generated by \( P_0. \)

Lemma 2.4. \( \mathcal{I} \) is the closure of the algebra of all finite linear combinations of elements of the form \( T_\mu P^j_i T^*_v \) for \( \mu, v \in \Lambda^*_\beta, i = 1, 2, \ldots, m(l), l \in \mathbb{Z}_+. \)

Proof. Since \( P_0 T_\beta = 0, \) one sees \( T_\mu P^j_i T^*_v P_0 = P_0 T^*_\mu T^*_v = 0. \) As the algebra \( \mathcal{T}_\beta \) is generated by elements of the form \( T_\mu P^j_i T^*_v \) and \( P_0, \) by
using the relation $P_0 P_i^l = P_i^l P_0$, $\mathcal{T}_\lambda$ is the closure of the algebra of all linear combinations of elements of the forms $T_\mu P_i^l P_0 T_\nu^*$ and $T_\mu P_i^l T_\nu^*$. Since $\mathcal{I} = \mathcal{T}_\lambda P_0 \mathcal{T}_\lambda$, one concludes that $\mathcal{I}$ is the closure of the algebra of all finite linear combinations of elements of the form $T_\mu P_i^l P_0 T_\nu^*$.

**Lemma 2.5.** $T_\beta, P_i^l \notin \mathcal{I}$.

**Proof.** Suppose $T_\beta \in \mathcal{I}$. By Lemma 2.4, there exists a finite linear combination $X = \sum_{\mu,v,l,i} c_{\mu,v,i,l} T_\mu P_i^l P_0 T_\nu^*$ of $T_\mu P_i^l P_0 T_\nu^*$, $\mu, v \in \Lambda^*_\nu, i = 1, 2, \ldots, m(l), l \in \mathbb{Z}_+$. Let $K$ denote the maximum length of the words $v$ that appear in the element $\sum_{\mu,v,l,i} c_{\mu,v,i,l} T_\mu P_i^l P_0 T_\nu^*$. Take a finite path $\xi = (u_0, \alpha_1, u_1, \alpha_2, \ldots, \alpha_{K+1}, u_{K+1}) \in W^K_{\nu_{K+1}}$ such that there exists a vertex $u_{-1} \in \Omega_\nu$ satisfying $(u_{-1}, \beta, u_0) \in E_\nu$. We have $X e_\xi = 0$ and $T_\beta e_\xi = e_{(u_{-1}, \beta, u_0, \ldots, u_{K+1})}$ so that
\[
( X - T_\beta ) e_\xi = \| e_{(u_{-1}, \beta, u_0, \ldots, u_{K+1})} \| = 1,
\] a contradiction.

Suppose next $P_i^l \in \mathcal{I}$. There exists similarly an element $Y = \sum_{\mu,v,l,i} c_{\mu,v,i,l} T_\mu P_i^l P_0 T_\nu^*$ such that $\| Y - P_i^l \| < \frac{1}{2}$. Take a finite path $\eta = (u_0, \alpha_1, u_1, \alpha_2, \ldots, \alpha_{K+1}, u_{K+1}) \in W^K_{\nu_{K+1}}$ such that $u_0 = u_i^l$, where $u_0 = (u_i^l)_{l \in \mathbb{Z}_+} \in \Omega_\nu$ so that $Y e_\eta = 0$ and $P_i^l e_\eta = e_\eta$ a contradiction.

**Definition.** Let $\widehat{\mathcal{T}}_\lambda$ be the quotient $C^*$-algebra $\mathcal{T}_\lambda / \mathcal{I}$ of $\mathcal{T}_\lambda$ by the ideal $\mathcal{I}$, and the operators $\widehat{S}_\alpha$ and $\widehat{E}_i^l$ the quotient images of $S_\alpha$ and $P_i^l$ in $\widehat{\mathcal{T}}_\lambda$ respectively.

By Lemma 2.5, the elements $\widehat{S}_\alpha$ and $\widehat{E}_i^l$ are not zeros for each $\alpha \in \Sigma$ and $v_i^l \in V$, and satisfy the relations $\mathcal{L}$ by Lemma 2.2. Thus by Theorem 1.1 we obtain

**Theorem 2.6.** Suppose that $\mathcal{L}$ satisfies condition (I). Then the $C^*$-algebra $\widehat{\mathcal{T}}_\lambda$ is canonically isomorphic to the $C^*$-algebra $\mathcal{O}_\nu$ associated with $\lambda$-graph system $\mathcal{L}$.

Define a unitary representation $U$ of the circle group $T$ on the Hilbert space $\mathcal{H}_\lambda$ by $U z e_\eta = z^e_\eta$ for $\eta \in W^K_{\nu_{K+1}}$. It is easy to see that the automorphisms $Ad(U z)$, $z \in T$ on the algebra of all bounded linear operators on $\mathcal{H}_\lambda$ leave invariant globally both the algebras $\mathcal{T}_\lambda$ and $\mathcal{I}$. They give rise to an action on the $C^*$-algebra $\widehat{\mathcal{T}}_\lambda$ that is the gauge action $\alpha_\nu$ on $\mathcal{O}_\nu$.

This construction of the $C^*$-algebra $\widehat{\mathcal{T}}_\lambda$ is inspired by the construction of the $C^*$-algebras of Hilbert $C^*$-bimodules by [18] and [10] (cf. [9]). Our construction can work for the construction of the $C^*$-algebras of general continuous graphs of Deaconu [5].
3. \(\lambda\)-irreducibility and pure infiniteness

As in Section 1, it has been proved in [16] that if \(\mathcal{U}\) is aperiodic, the \(C^*\)-algebra \(\mathcal{O}_{\mathcal{U}}\) becomes simple and purely infinite. The aperiodic condition on \(\mathcal{U}\) however is too strong such that the algebra \(\mathcal{O}_{\mathcal{U}}\) is simple and purely infinite. In fact, the Cuntz-Krieger algebra \(\mathcal{O}_A\) is simple and purely infinite if the matrix \(A\) is irreducible with condition (I). In this section, we introduce a new irreducible condition along with a new condition (I) on \(\mathcal{U}\) under which the \(C^*\)-algebra \(\mathcal{O}_{\mathcal{U}}\) is simple and purely infinite. The new conditions are called \(\lambda\)-irreducible condition and \(\lambda\)-condition (I) respectively. They are exact generalization of the corresponding conditions on a finite square matrix \(A\) with entries in \(\{0, 1\}\).

**Definition.** A \(\lambda\)-graph system \(\mathcal{U}\) is \(\lambda\)-irreducible if for an ordered pair of vertices \(v_i, v_i^j \in V\), there exists a number \(L_l(i, j) \in N\) such that for a vertex \(v_i^L_l(i,j) \in V_{l+L_l(i,j)}\) with \(t_{l(i,j)}(v_i^L_l(i,j)) = v_i^j\), there exists a path \(\gamma\) in \(\mathcal{U}\) such that 

\[
\gamma = v_i^j, \quad t(\gamma) = v_i^L_l(i,j),
\]

where \(t_{l(i,j)}\) means the \(L_l(i, j)\)-times compositions of \(t\), and \(s(\gamma)\), \(t(\gamma)\) denote the source vertex, the terminal vertex of \(\gamma\) respectively. It is obvious that if \(\mathcal{U}\) is \(\lambda\)-irreducible, then it is irreducible in the sense of Section 1. Let \(\mathcal{G}\) be a finite directed graph and \(\mathcal{G}_\mathcal{U}\) the associated \(\lambda\)-graph system defined in [16, Section 7]. It is then immediate that \(\mathcal{G}\) is irreducible if and only if \(\mathcal{G}_\mathcal{U}\) is \(\lambda\)-irreducible.

The following lemma is direct from the local property of \(\lambda\)-graph system.

**Lemma 3.1.** Suppose that a \(\lambda\)-graph system \(\mathcal{U}\) is \(\lambda\)-irreducible. For a vertex \(v_i^j \in V\), let \(L\) be the number \(L_l(i, i) \in N\) such that for a vertex \(v_i^{L+L_l(i,i)} \in V_{l+L_l(i,i)}\) with \(t_{L_l(i,i)}(v_i^{L+L_l(i,i)}) = v_i^j\), there exists a path \(\gamma\) in \(\mathcal{U}\) such that 

\[
s(\gamma) = v_i^j, \quad t(\gamma) = v_i^{L+L_l(i,i)},
\]

We will introduce \(\lambda\)-condition (I).

**Definition.** A \(\lambda\)-graph system \(\mathcal{U}\) is said to satisfy \(\lambda\)-condition (I) if for a vertex \(v_i^j \in V\) there exist two distinct paths \(\gamma_1, \gamma_2\) in \(\mathcal{U}\) such that

\[
s(\gamma_1) = s(\gamma_2) = v_i^j, \quad t(\gamma_1) = t(\gamma_2), \quad \lambda(\gamma_1) \neq \lambda(\gamma_2).
\]

It is obvious that if \(\mathcal{U}\) satisfies \(\lambda\)-condition (I), it satisfies condition (I) in the sense of Section 1. One immediately sees that the adjacency matrix of a finite
directed graph $G$ satisfies condition (I) in the sense of Cuntz-Krieger [3] if and only if $G$ satisfies $\lambda$-condition (I).

Let $A_{l,l+1}, I_{l,l+1}$ be the transition matrices of $\mathcal{U}$ as in Section 1. Define the matrices $A_{l,l+k}, I_{l,l+k}$ for $k \in \mathbb{N}$ by setting for $i = 1, 2, \ldots, m(l), j = 1, 2, \ldots, m(l+k), \mu \in \Lambda^k_\mathcal{U}$,

$$A_{l,l+k}(i, \mu, j) = \begin{cases} 
1 & \text{if } s(\gamma) = v^l_i, \lambda(\gamma) = \mu, t(\gamma) = v^{l+k}_j \\
& \text{for some path } \gamma \text{ in } \mathcal{U}, \\
0 & \text{otherwise,}
\end{cases}$$

$$I_{l,l+k}(i, j) = \begin{cases} 
1 & \text{if } t^k(v^{l+k}_j) = v^l_i, \\
0 & \text{otherwise,}
\end{cases}$$

where $\lambda(\gamma) = \lambda(\gamma_1) \cdots \lambda(\gamma_k)$ for $\gamma = (\gamma_1, \ldots, \gamma_k), \gamma_i \in E, 1 \leq i \leq k$.

**Lemma 3.2.** Suppose that $\mathcal{U}$ is $\lambda$-irreducible and satisfies $\lambda$-condition (I). For a vertex $v^l_i \in V_l$, let $L$ be the number as in Lemma 3.1. Then one of the following two conditions holds:

1. There exist a word $\eta \in \Lambda^L_\mathcal{U}$ and a vertex $v^{l+L}_h \in V_{l+L}$ such that $A_{l,l+L}(i, \eta, j) = 1, I_{l,l+L}(i, j) = 0$.

2. There exists $k \in \mathbb{N}$ such that $I_{l,l+kL}(i, h) = 1$ implies $A_{l,l+kL}(i, \mu, h) = 1$ for some $\mu \in \Lambda^k_\mathcal{U}$, and there exists $h \in \{1, \ldots, m(l+L)\}$ such that $\sum_{\mu \in \Lambda^k_\mathcal{U}} A_{l,l+kL}(i, \mu, h) \geq 2$.

**Proof.** Suppose that the condition (1) does not hold. As $\mathcal{U}$ is $\lambda$-irreducible, it satisfies the assumption of Lemma 3.1(ii). By the $\lambda$-condition (I), we may take a number $k \in \mathbb{N}$ and a vertex $v^{l+kL}_h \in V_{l+kL}$ and two distinct paths $\gamma_1, \gamma_2$ in $\mathcal{U}$ such that

$$s(\gamma_1) = s(\gamma_2) = v^l_i, \quad t(\gamma_1) = t(\gamma_2) = v^{l+kL}_h, \quad \lambda(\gamma_1) \neq \lambda(\gamma_2).$$

Hence we have $A_{l,l+kL}(i, \gamma_1, h) = A_{l,l+kL}(i, \gamma_2, h) = 1$ so that

$$\sum_{\mu \in \Lambda^k_\mathcal{U}} A_{l,l+kL}(i, \mu, h) \geq 2$$

and the condition (2) holds.

**Proposition 3.3.** Assume that $\mathcal{U}$ is $\lambda$-irreducible and satisfies $\lambda$-condition (I). For the projection $E_i^l$ in the $C^*$-algebra $\mathcal{O}_\mathcal{U}$ corresponding to the vertex $v^l_i \in V_l$, there exists a number $L \in \mathbb{N}$ such that for every vertex $v^{l+L}_h \in V_{l+L}$
with \( t^L(v_h^{i+L}) = v_i^j \), there exists an admissible word \( \mu(h) \) in \( \Lambda^L_\mathcal{L} \) such that

\[
S_{\mu(h)} E_{h}^{i+L} S_{\mu(h)}^* \neq 0 \quad \text{and} \quad \sum_{h=1}^{m(l+L)} I_{i,j+L}(i, h) S_{\mu(h)} E_{h}^{i+L} S_{\mu(h)}^* < E_i^j,
\]

**Proof.** For \( v_i^j \in V_l \), let \( L \) be the number as in Lemma 3.1. One of the two conditions (1) and (2) in the preceding lemma holds. Suppose that (1) holds. As \( \mathcal{L} \) is \( \lambda \)-irreducible, for a vertex \( v_h^{i+L} \in V_{l+L} \) with \( t^L(v_h^{i+L}) = v_i^j \), there exists an admissible word \( \mu(h) \) in \( \Lambda^L_\mathcal{L} \) such that \( S_{\mu(h)} E_{h}^{i+L} S_{\mu(h)}^* \neq 0 \) and

\[
\sum_{h=1}^{m(l+L)} I_{i,j+L}(i, h) S_{\mu(h)} E_{h}^{i+L} S_{\mu(h)}^* < E_i^j.
\]

Now \( A_{i,j+L}(i, \eta, j) = 1 \) so that \( S_{\eta} E_{h}^{i+L} S_{\eta}^* \neq 0 \). By (1.1), (1.3) and (1.4), the equality

\[
(3.1) \quad \sum_{h=1}^{m(l+L)} \sum_{\eta \in \Lambda^L_\mathcal{L}} A_{i,j+L}(i, \eta, h) S_{\mu(h)} E_{h}^{i+L} S_{\eta}^* = E_i^j,
\]

holds so that

\[
\sum_{h=1}^{m(l+L)} I_{i,j+L}(i, h) S_{\mu(h)} E_{h}^{i+L} S_{\mu(h)}^* < E_i^j.
\]

We next assume that the condition (2) holds. There exists \( k \in \mathbb{N} \) such that \( I_{i,j+kL}(i, h) = 1 \) implies \( A_{i,j+kL}(i, \mu, h) = 1 \) for some \( \mu \in \Lambda^L_\mathcal{L} \), and there exists \( h = 1, \ldots, m(l+L) \) such that \( \sum_{\mu \in \Lambda^L_\mathcal{L}} A_{i,j+kL}(i, \mu, h) \geq 2 \). By (3.1) we obtain

\[
\sum_{h=1}^{m(l+kL)} I_{i,j+kL}(i, h) S_{\mu(h)} E_{h}^{i+kL} S_{\mu(h)}^* < E_i^j.
\]

Take \( L \) as \( kL \) so that we get the desired assertion.

Let \( \Lambda_{h}^{i+n,n} \) be the number of paths \( \gamma \) in \( \mathcal{L} \) starting at a vertex in \( V_l \) and terminating at \( v_h^{i+n} \). As \( \mathcal{L} \) is left-resolving, it is the number of admissible words
μ in $Λ_Ω$ of length $n$ such that $S_μ E^{l+n}_h S_μ^*$ $≠$ 0. It satisfies the equality

$$N^{l+n}_h E^{l+n}_h = \left( \sum_{μ} S_μ S_μ^* \right) E^{l+n}_h.$$ 

By the local property of $λ$-graph system, we have $N^{l+n}_h = N^{l+n}_h$ if $t^a (v^{l+n}_k) = t^a (v^{l+n}_k)$. For a vertex $v^{l+n}_i ∈ V_{l+n}$, define a projection $P^{l+n}_h$ by setting

$$P^{l+n}_h = \frac{1}{N^{l+n}_h} \sum_{μ, v ∈ Λ_Ω^*} S_μ E^{l+n}_h S_μ^*.$$ 

**Lemma 3.4.** Take $μ ∈ Λ_Ω^n$ satisfying $S_μ E^{l+n}_h S_μ^* ≠ 0$. Then there exists a partial isometry $U^{l+n}_{h,μ}$ in $O_{H_5222}$ such that

$$U^{l+n}_{h,μ} U^{l+n}_{h,μ}^* = \sum_{v ∈ Λ_Ω^*} S_v E^{l+n}_h S_v^*,$$

$$U^{l+n}_{h,μ} P^{l+n}_h U^{l+n}_{h,μ}^* = S_μ E^{l+n}_h S_μ^*.$$ 

**Proof.** The elements $S_ξ E^{l+n}_h S_η^*$, $ξ, η ∈ Λ_Ω^n$ form a matrix units of the $C^*$-subalgebra of $O_{H_5222}$ generated by $S_ξ E^{l+n}_h S_η^*$, $ξ, η ∈ Λ_Ω^n$ that is isomorphic to the full matrix algebra of size $N^{l+n}_h$. As $P^{l+n}_h$ is a projection of rank one in the subalgebra, one can find a desired partial isometry by elementary linear algebra.

The following lemma is straightforward.

**Lemma 3.5.** Put $V_L = \frac{1}{\sqrt{N^{l+n}_h}} \sum_{μ} S_μ E^{l+n}_h$. Then we have

$$V^*_L V_L = 1, \quad V_L E^{l+n}_h V^*_L = \sum_{h=1}^{m(l+n)} I_{i,l+n}(i, h) P^{l+n,L}_h.$$ 

**Proposition 3.6.** Assume that $Ω$ is $λ$-irreducible and satisfies $λ$-condition (I). Then the projection $E^i_h$ for $v^i_i ∈ V$ is an infinite projection in $O_Ω$.

**Proof.** Suppose that the number $m(l)$ of the vertex set $V_l$ is one for all $l ∈ Z_+$. Then we have $E^i_l = 1$. Since $Ω$ satisfies $λ$-condition (I), the alphabet $Σ$ is not singleton. Now $1 = \sum_{α ∈ Σ} S_α S_α^*$ and $A_{l,l+1}(i, α, j) = 1$ for all $i, α, j$. Hence we see by the relations ($Ω$),

$$S_α S_α^* = \sum_{α ∈ Σ} S_α S_α^* = 1.$$
This implies that the unit 1 is an infinite projection. In this case, the C*-algebra \( O \) is isomorphic to the Cuntz algebra \( \mathcal{O}_{|\Sigma|} \) of order \(|\Sigma|\) the number of \( \Sigma \).

Suppose next that there exists \( l_0 \in \mathbb{Z}_+ \) such that \( m(l_0) \geq 2 \). Hence \( m(l) \geq 2 \) for \( l \geq l_0 \). For a projection \( E'_l \) with \( l \geq l_0 \), by Proposition 3.3 for \( h = 1, \ldots, m(l + L) \) with \( I_{l,l+L}(i,h) = 1 \), there exists an admissible word \( \mu(h) \) in \( \Lambda^L \) such that

\[
S_{\mu(h)}E'_hE'_{h}'S_{\mu(h)}^* \neq 0 \quad \text{and} \quad \sum_{h=1}^{m(l+L)} I_{l,l+L}(i,h)S_{\mu(h)}E'_hE'_{h}'S_{\mu(h)}^* < E'_l.
\]

Let \( V_L \) be the isometry as in Lemma 3.5 and \( U_{h,k}^{l+L} \) the partial isometry as in Lemma 3.4. Then we set \( W_i^l = (\sum_{h=1}^{m(l+L)} U_{h,k}^{l+L})V_L \). As \( U_{h,k}^{l+L}P_{h}^{l+L} = 0 \) for \( k \neq h \), it follows that \( W_i^l \) is an infinite projection.

**Lemma 3.7.** Assume that \( \mathcal{O} \) is \( \lambda \)-irreducible and satisfies \( \lambda \)-condition (I). Then for the projection \( E'_l \in \mathcal{O} \) for \( v'_l \in V \), there exists an element \( U \in \mathcal{O} \) such that \( UU^* = 1 \) and \( UE'_lU^* = 1 \).

**Proof.** Assume that \( \mathcal{O} \) is \( \lambda \)-irreducible and satisfies \( \lambda \)-condition (I), so that \( \mathcal{O} \) is irreducible and satisfies condition (I). Hence \( \mathcal{O} \) is simple. By [4, Lemma V.5.4] with Proposition 3.6, the unit 1 of \( \mathcal{O} \) is equivalent to a subprojection of \( E'_l \). Take an element \( U \in \mathcal{O} \) such that \( UU^* = 1 \) and \( U^*U \leq E'_l \). This implies \( UE'_lU^* = 1 \).

**Theorem 3.8.** If \( \mathcal{O} \) is \( \lambda \)-irreducible and satisfies \( \lambda \)-condition (I), for any nonzero \( X \in \mathcal{O} \) there exist \( A,B \in \mathcal{O} \) such that \( AXB = 1 \).

**Proof.** Let \( E : \mathcal{O} \rightarrow \mathcal{F} \) be the canonical conditional expectation given by

\[
E(X) = \int_T (\alpha_t)(X) dt, \quad X \in \mathcal{O}.
\]

Since \( E \) is faithful, we may assume that \( \|E(X^*X)\| = 1 \). Let \( P_\mathcal{O} \) be the *-algebra generated algebraically by the generators \( S_\alpha, E'_l, \alpha \in \Sigma, v'_l \in V \). For
any $0 < \epsilon < \frac{1}{4}$, we may find $0 \leq Y \in \mathcal{P}_k$ such that $\| X^* X - Y \| < \frac{\epsilon}{2}$ so that $\| E(Y) \| > 1 - \frac{\epsilon}{2}$. As in the discussion in [16, Section 3], the element $Y$ is expressed as

$$Y = \sum_{|\nu| \geq 1} Y_{-\nu} s^*_\nu Y_0 + \sum_{|\mu| \geq 1} S_\mu Y_\mu$$

for some $Y_{-\nu}, Y_0, Y_\mu \in \mathcal{F}_0 \cap \mathcal{P}_k$.

Take $k \leq l$ such that $Y_{-\nu}, Y_0, Y_\mu \in \mathcal{F}_l$ for all $\mu, \nu$ in the above expression. Now $\mathcal{U}$ satisfies condition (I). By [16, Lemma 3.1 and Lemma 4.2] there exists a projection $Q_k^l$ in the diagonal algebra of $\mathcal{F}_l$ for $k \leq l$ satisfying the following properties

1. $Q_k^l$ commutes with $\mathcal{F}_l$.
2. The map $X \in \mathcal{F}_l \rightarrow Q_k^l X Q_k^l \in Q_k^l \mathcal{F}_l Q_k^l$ is an isomorphism.
3. $Q_k^l S_\mu Q_k^l = Q_k^l S^*_\mu Q_k^l = 0$ for $1 \leq |\mu|, |\nu| \leq k$.

As $E(Y) = Y_0$, it follows that by (1) and (3),

$$Q_k^l Y Q_k^l = \sum_{|\nu| \geq 1} Y_{-\nu} s^*_\nu Q_k^l + Q_k^l Y_0 Q_k^l + \sum_{|\mu| \geq 1} Q_k^l S_\mu Q_k^l Y_\mu = Q_k^l E(Y) Q_k^l.$$

Since $Q_k^l E(Y) Q_k^l \in \mathcal{U}$, there exists $0 \leq Z \in \mathcal{F}_l$ for some $k' \leq l$ such that $\| Q_k^l E(Y) Q_k^l - Z \| < \frac{\epsilon}{2}$. By (2), we note $\| Q_k^l E(Y) Q_k^l \| = \| E(Y) \|$ so that

$$\| Z \| \geq \| E(Y) \| - \frac{\epsilon}{2} > 1 - \epsilon$$

and

$$\| Z \| < \| Q_k^l E(Y) Q_k^l \| + \frac{\epsilon}{2} = \| E(Y) \| + \frac{\epsilon}{2} \leq \| E(X^* X) \| + \frac{\epsilon}{2} + \frac{\epsilon}{2} < 1 + \epsilon.$$

As the algebra $\mathcal{F}_l$ is finite dimensional, we have spectral decomposition $Z = \sum_{j=1}^r \lambda_j R_j$ of $Z$ for some real numbers $\lambda_j \geq 0$ and minimal projections $R_j \in \mathcal{F}_l$. Since $1 - \epsilon < \| Z \| < 1 + \epsilon$, we may find $i_0$ such that $1 - \epsilon < \lambda_{i_0} < 1 + \epsilon$, and may assume that $R_{i_0} = S_{\mu_0} E_{i_0}^l S^*_{\mu_0}$ for some $|\mu_0| = k'$ and $v_{i_0}^l \in V_l$. By Lemma 3.7, there exists $U \in \mathcal{O}_l$ such that $UU^* = 1, U E_{i_0}^l U^* = 1$. Put $A = U S^*_{\mu_0} R_{i_0} Q_k^l$. It follows that

$$\| AX^* A^* - 1 \| \leq \| AX^* A^* - AY A^* \| + \| AYA^* - US_{\mu_0} R_{i_0} Z R_{i_0} S_{\mu_0} U^* \| + \| US_{\mu_0} R_{i_0} Z R_{i_0} S_{\mu_0} U^* - 1 \|.$$
One then sees
\[
\|AX^*XA^* - AYA^*\| \leq \|X^*X - Y\| < \frac{\epsilon}{2},
\]
\[
2\|AY^* - US^*_\mu_0 R_0 Z R_0 S_{\mu_0} U^*\| = \|US^*_\mu_0 R_0 (Q^k_1 Y Q^k_1 - Z) R_0 S_{\mu_0} U^*\| < \frac{\epsilon}{2},
\]
\[
US^*_\mu_0 R_0 Z R_0 S_{\mu_0} U^* = \lambda_i_0 U E^i_0 U^* = \lambda_i_0.
\]

Thus we obtain
\[
\|AX^*XA^* - 1\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} + |\lambda_i_0 - 1| < 2\epsilon < \frac{1}{4}.
\]

Hence $AX^*XA^*$ is invertible so that we have an element $C \in \mathcal{O}_\mathcal{U}$ such that $AX^*XA^*C = 1$.

Therefore we conclude by [4, Theorem V.5.5]

**Theorem 3.9.** If $\mathcal{U}$ is $\lambda$-irreducible and satisfies $\lambda$-condition (I), then the $C^*$-algebra $\mathcal{O}_\mathcal{U}$ is simple and purely infinite.

Let $A$ be a finite square matrix with entries in $\{0, 1\}$ and $G_A$ its corresponding directed graph. By considering the associated $\lambda$-graph system $\mathcal{U}_{GA}$, we have the following well-known result:

**Corollary 3.10 ([1], [2], [3]).** If $A$ satisfies condition (I) in the sense of Cuntz-Krieger [3] and is irreducible, the Cuntz-Krieger algebra $\mathcal{O}_A$ is simple and purely infinite.

In [12], [16, Theorem 7.7] and [17], examples of $\lambda$-graph systems that are $\lambda$-irreducible and satisfy $\lambda$-condition (I) are presented and the $K$-groups for the associated $C^*$-algebras are computed.

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