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APPROACH REGIONS FOR L^p POTENTIALS WITH RESPECT TO THE SQUARE ROOT OF THE POISSON KERNEL

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Abstract

If one replaces the Poisson kernel of the unit disc by its square root, then normalised Poisson integrals of L^p boundary functions converge along approach regions wider than the ordinary nontangential cones, as proved by Rönning ($1 \leq p < \infty$) and Sjögren ($p = 1$ and $p = \infty$). In this paper we present new and simplified proofs of these results. We also generalise the L^∞ result to higher dimensions.

1. Introduction

The point of this paper is firstly to present a new and simplified proof for two theorems of almost everywhere convergence type. The advantage of the proof, without being precise, is that it reflects that the convergence results are natural consequences of the norm inequalities that characterise the relevant function spaces (Hölder's inequality for L^p), and corresponding norm estimates of the kernel (associated to the normalised square root of the Poisson kernel operator). In the papers by Rönning, [6], and Sjögren, [9], this correspondence is not obvious (even though, of course, present).

$P(z, \beta)$ will denote the Poisson kernel in the unit disc U,

$$P(z, \beta) = \frac{1}{2\pi} \cdot \frac{1 - |z|^2}{|z - e^{i\beta}|^2}$$

where $z \in U$ and $\beta \in \partial U \cong \mathsf{R}/2\pi\mathsf{Z} = \mathsf{T} \cong (-\pi, \pi]$.

It is well known that $P(\,\cdot\,, \beta)$ is the real part of a holomorphic function, and thus that it is harmonic.

Let

$$Pf(z) = \int_{\mathsf{T}} P(z, \beta) f(\beta)\, d\beta,$$

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the Poisson integral (or extension) of $f \in L^1(\mathsf{T})$. Poisson extensions of continuous boundary functions converge unrestrictedly at the boundary, as the following classical result shows:

THEOREM (Schwarz, [7]). *Let $f \in C(\mathsf{T})$. Then $Pf(z) \to f(\theta)$ as $z \to e^{i\theta}$, $z \in U$.*

For less regular boundary functions, unrestricted convergence fails (see the result by Littlewood below). One way to control the approach to the boundary is by means of so called (natural) approach regions. For any function $h : \mathsf{R}_+ \to \mathsf{R}_+$ let

$$\mathscr{A}_h(\theta) = \{z \in U : |\arg z - \theta| \le h(1 - |z|)\}.$$

We refer to $\mathscr{A}_h(\theta)$ as the approach region determined by h at $\theta \in \mathsf{T}$. If $h(t) = \alpha \cdot t$, for some $\alpha > 0$, one refers to $\mathscr{A}_h(\theta)$ as a nontangential cone at $\theta \in \mathsf{T}$. It is natural, but not necessary, to think of h as an increasing function. It should be pointed out that our approach regions certainly have a specific shape. For instance, they are not of Nagel-Stein type.

THEOREM (Fatou, [4]). *Let $f \in L^1(\mathsf{T})$. Then, for a.e. $\theta \in \mathsf{T}$, one has that $Pf(z) \to f(\theta)$ as $z \to e^{i\theta}$ and $z \in \mathscr{A}_h(\theta)$, if $h(t) = O(t)$ as $t \to 0$.*

The theorem of Fatou was proved to be best possible, in the following sense:

THEOREM (Littlewood, [5]). *Let $\gamma_0 \subset U \cup \{1\}$ be a simple closed curve, having a common tangent with the circle at the point 1. Let γ_θ be the rotation of γ_0 by the angle θ. Then there exists a bounded harmonic function f in U with the property that, for a.e. $\theta \in \mathsf{T}$, the limit of f along γ_θ does not exist.*

Littlewood's result has been generalised in several directions. For instance, with the same assumptions as in Littlewood's theorem, Aikawa [1], proves that convergence can be made to fail at *any* point $\theta \in \mathsf{T}$.

For $z = x + iy$ define the hyperbolic Laplacian by

$$L_z = \frac{1}{4}(1 - |z|^2)^2(\partial_x^2 + \partial_y^2).$$

Then the λ-Poisson integral

$$u(z) = P_\lambda f(z) = \int_{\mathsf{T}} P(z, \beta)^{\lambda + 1/2} f(\beta)\, d\beta, \qquad \text{for} \quad \lambda \in \mathsf{C},$$

defines a solution of the equation

$$L_z u = (\lambda^2 - 1/4)u.$$

The case $\lambda = 0$, u is then an eigenfunction at the bottom of the positive spectrum, is particularly interesting. The square root of the Poisson kernel (i.e., $\lambda = 0$) possesses unique properties relative to other powers. In this paper we shall treat convergence questions for normalised Poisson integrals with respect to the square root of the Poisson kernel.

If f and g are positive functions we say that $f \lesssim g$ provided that there exists some positive constant C such that $f(x) \leq Cg(x)$. We write $f \sim g$ if $f \lesssim g$ and $g \lesssim f$.

Let

$$P_0 f(z) = \int_{\mathsf{T}} \sqrt{P(z,\beta)}\, f(\beta)\, d\beta.$$

To get boundary convergence, it is necessary to normalise P_0, since it is readily checked that, for $|z| > 1/2$,

$$P_0 1(z) \sim \sqrt{1 - |z|} \log \frac{1}{1 - |z|},$$

which does not tend to 1, anywhere, as $|z| \to 1$. As mentioned above, Poisson integrals with respect to powers greater than or equal to $1/2$ of the Poisson kernel arise naturally as eigenfunctions to the hyperbolic Laplace operator. When one considers boundary convergence properties of the corresponding normalisations, it is only the square root integral extension that exhibits special properties. Normalisation of higher power integrals behave just like the Poisson integral itself, in the context of boundary convergence.

Denote the normalised operator by $\mathscr{P}_0$, i.e.

$$\mathscr{P}_0 f(z) = \frac{P_0 f(z)}{P_0 1(z)}.$$

DEFINITION 1. If $1 \leq p < \infty$ let

$$S_p = \{h : \mathsf{R}_+ \to \mathsf{R}_+ : h(t) = O(t(\log 1/t)^p) \text{ as } t \to 0\},$$

and let

$$S_\infty = \{h : \mathsf{R}_+ \to \mathsf{R}_+ : h(t) = O(t^{1-\varepsilon}) \text{ for all } \varepsilon > 0 \text{ as } t \to 0\}.$$

Note that $S_p \subset S_\infty$.

Several convergence results for $\mathscr{P}_0$ are known, in different settings. We state a few below:

THEOREM. *Let $f \in C(\mathsf{T})$. Then, for any $\theta \in \mathsf{T}$, one has that $\mathscr{P}_0 f(z) \to f(\theta)$ as $z \to e^{i\theta}$.*

This result follows if one just notes that $\mathscr{P}_0$ is a convolution operator with a kernel which behaves like an approximate identity in T. In the next section we give explicit expressions for the kernel.

THEOREM (Sjögren, [8]). *Let $f \in L^1(\mathsf{T})$. Then, for a.e. $\theta \in \mathsf{T}$, one has that $\mathscr{P}_0 f(z) \to f(\theta)$ as $z \to e^{i\theta}$ and $z \in \mathscr{A}_h(\theta)$, if $h \in S_1$.*

THEOREM (Rönning, [6]). *Let $1 \le p < \infty$ be given and let $f \in L^p(\mathsf{T})$. Then, for a.e. $\theta \in \mathsf{T}$, one has that $\mathscr{P}_0 f(z) \to f(\theta)$ as $z \to e^{i\theta}$ and $z \in \mathscr{A}_h(\theta)$, if $h \in S_p$ (and only if if h is assumed to be monotone).*

The results by Sjögren and Rönning were proved via weak type estimates for the corresponding maximal operators, and approximation with continuous functions.

THEOREM (Sjögren, [9]). *The following conditions are equivalent for any increasing function $h : \mathsf{R}_+ \to \mathsf{R}_+$:*

(i) *For any $f \in L^\infty(\mathsf{T})$ one has for almost all $\theta \in \mathsf{T}$ that*

$$\mathscr{P}_0 f(z) \to f(\theta) \text{ as } z \to e^{i\theta} \text{ and } z \in \mathscr{A}_h(\theta).$$

(ii) $h \in S_\infty$.

In his proof, Sjögren never uses the assumption that h should be increasing. Thus, it remains valid for an even larger class of functions h. The proof of this result differs much from the L^p case, since one has to take into account that the continuous functions are not dense in L^∞. Sjögren instead used a result by Bellow and Jones, [2], "A Banach principle for L^∞". Following the same lines, the author proved the following ($L^{p,\infty}$ denotes weak L^p):

THEOREM (Brundin, [3]). *Let $1 < p < \infty$ be given. Then the following conditions are equivalent for any function $h : \mathsf{R}_+ \to \mathsf{R}_+$:*

(i) *For any $f \in L^{p,\infty}(\mathsf{T})$ one has for almost all $\theta \in \mathsf{T}$ that*

$$\mathscr{P}_0 f(z) \to f(\theta) \text{ as } z \to e^{i\theta} \text{ and } z \in \mathscr{A}_h(\theta).$$

(ii) $\sum_{k=0}^\infty \sigma_k < \infty$, *where* $\sigma_k = \sup_{2^{-2^k} \le s \le 2^{-2^{k-1}}} \frac{h(s)}{s(\log 1/s)^p}$.

In this paper we prove the following theorem, with simpler and different methods than those of Rönning and Sjögren.

THEOREM 1.1. *Let $1 \le p \le \infty$ be given and let $h : \mathsf{R}_+ \to \mathsf{R}_+$ be any function. Then the following conditions are equivalent:*

(i) *For any $f \in L^p(\mathsf{T})$ one has, for almost all $\theta \in \mathsf{T}$, that $\mathscr{P}_0 f(z) \to f(\theta)$ as $z \to e^{i\theta}$ and $z \in \mathscr{A}_h(\theta)$.*

(ii) $h \in S_p$.

Obtaining (easily) the result for L^∞ first, we shall use this to treat the L^p case. As in the proofs of Sjögren and Rönning, we decompose the kernel into two parts, one "local" and one "global". The global part is easy. As it turns out here, the local part is also easy. In previous proofs, rather complicated calculations were used to prove that the associated maximal operator is "sufficiently continuous" at 0 (e.g. weak type (p, p) estimates). As it turns out, however, the local part simply does not contribute to convergence and can be treated directly (without estimates of any maximal operator).

One of the advantages of the proof is that the case $p = \infty$ can be easily generalised to higher dimensions, which is done in the section "Higher dimensional results for L^∞". In the paper by Rönning, [6], a certain maximal operator is proved to be of weak type (p, p) (in the L^p case, finite p). If one could prove that it is actually of strong type (p, p) (which is not unreasonable to believe), convergence results for polydiscs would follow easily. The proof in this paper does not rely on hard estimates of maximal operators, but rather on more direct methods. This may suggest that a polydisc result for L^p could be obtained, avoiding maximal operators.

2. The proof of Theorem 1.1

Before turning to the proof we introduce the notation that we shall use.

Let $t = 1 - |z|$ and $z = (1-t)e^{i\theta}$. Then

$$\mathscr{P}_0 f(z) = R_t * f(\theta),$$

where the convolution is taken in T and

$$R_t(\theta) = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{t(2-t)}}{|(1-t)e^{i\theta} - 1|} \frac{1}{P_0 1(1-t)}.$$

Since we are interested only in small values of t, we might as well from now on assume that $t < 1/2$. Then $P_0 1(1-t) \sim \sqrt{t} \log 1/t$, and thus the order of magnitude of R_t is given by

$$R_t(\theta) \sim Q_t(\theta) = \frac{1}{\log 1/t} \cdot \frac{1}{t + |\theta|}.$$

Now, let τ_η denote the translation $\tau_\eta f(\theta) = f(\theta - \eta)$. Then the convergence condition (i) in Theorem 1.1 above means

$$\lim_{\substack{t \to 0 \\ |\eta| < h(t)}} \tau_\eta R_t * f(\theta) = f(\theta).$$

Let

$$R_t(\theta) = R_t^1(\theta) + R_t^2(\theta)$$

where
$$R_t^1(\theta) = R_t(\theta)\chi_{\{|\theta|<2h(t)\}},$$
and let Q_t^1 and Q_t^2 be the corresponding cutoffs of the kernel Q_t.

Define

$$Mf(\theta) = \sup_{\substack{|\eta|<h(t)\\ t<1/2}} \tau_\eta Q_t^2 * |f|(\theta). \tag{1}$$

PROPOSITION 1. *Assume that* $1 \le p \le \infty$ *is given and assume that condition* (ii) *in Theorem 1.1 holds.*

(a) *For a given* $f \in L^p$ *it holds for a.e.* $\theta \in \mathsf{T}$ *that*
$$\lim_{\substack{t\to 0\\ |\eta|<h(t)}} \tau_\eta Q_t^1 * f(\theta) = 0.$$

(b) $Mf \lesssim M_{HL} f$, *where* M_{HL} *denotes the ordinary Hardy-Littlewood maximal operator.*

Let us for the moment postpone the proof and instead see how Proposition 1 is used to prove the implication (ii) $\Rightarrow$ (i) in Theorem 1.1.

PROOF OF THEOREM 1.1, (ii) $\Rightarrow$ (i). By Proposition 1, part (a), it suffices to prove that, for almost all $\theta \in \mathsf{T}$, one has

$$\lim_{\substack{t\to 0\\ |\eta|<h(t)}} \tau_\eta R_t^2 * f(\theta) = f(\theta). \tag{2}$$

Note that, if $f \in C(\mathsf{T})$, then
$$\lim_{\substack{t\to 0\\ |\eta|<h(t)}} \tau_\eta R_t * f(\theta) = f(\theta).$$

This fact, together with Proposition 1, part (a), and $C(\mathsf{T}) \subset L^p(\mathsf{T})$ gives that (2) must hold for $f \in C(\mathsf{T})$. Hence, to establish (2) for any $f \in L^p$, it suffices to prove that the corresponding maximal operator is of weak type (1, 1). But since it is dominated by M, which in turn is dominated by M_{HL} by Proposition 1, part (b), we are done.

We now proceed with the proof of Proposition 1. The proof of implication (i) $\Rightarrow$ (ii) in Theorem 1.1 can be found in the end of this section.

PROOF OF PROPOSITION 1. We start by proving part (b). Since $|\eta| < h(t)$, we have that
$$\tau_\eta Q_t^2(\theta) = \frac{1}{\log 1/t} \cdot \frac{1}{t + |\theta - \eta|}\chi_{\{|\theta-\eta|>2h(t)\}} \lesssim \frac{1}{\log 1/t} \cdot \frac{1}{t + |\theta|},$$

which is a decreasing function of θ, whose integral in T is uniformly bounded in t. It is well known that convolution with such a function is controlled by the Hardy-Littlewood maximal operator. Part (b) is thus established.

We proceed now with the proof of part (a), in the case $p = \infty$.

Let $\varepsilon > 0$ be given. We have

$$\begin{aligned}\tau_\eta Q_t^1 * |f|(\theta) &= \frac{1}{\log 1/t}\int_{|\varphi|<2h(t)} \frac{|f(\theta-\eta-\varphi)|}{t+|\varphi|}\,d\varphi \\ &\le \frac{\|f\|_\infty}{\log 1/t}\int_{|\varphi|<2h(t)} \frac{d\varphi}{t+|\varphi|} \lesssim \frac{\|f\|_\infty}{\log 1/t}\log\left(h(t)/t\right).\end{aligned}$$

By condition (ii) in Theorem 1.1, we have that $h(t) \le Ct^{1-\varepsilon}$, and we get

$$\limsup_{\substack{t\to 0\\ |\eta|<h(t)}} \tau_\eta Q_t^1 * |f|(\theta) \lesssim \varepsilon\|f\|_\infty,$$

as desired.

Now, assume that $1 \le p < \infty$ and that $q = p/(p-1)$ (where $q = \infty$ if $p = 1$). Assume also that $f \ge 0$, without loss of generality.

Note, first of all, that

$$\|Q_t\|_q \le C_q \frac{1}{t^{1/p}\log 1/t} \tag{3}$$

Write $f = f_- + f_R$, where $f_- = f\chi_{\{f\le R\}} \in L^\infty$, and where $R > 0$ is arbitrary. By (3) and by assumption we have, for $t \in (0, 1/2)$ and $\theta \in \mathsf{T}$, that

$$\begin{aligned}\tau_\eta Q_t^1 * f_R(\theta) &= \int_{|\varphi|<2h(t)} Q_t(\varphi) f_R(\theta-\varphi-\eta) \\ &\lesssim \frac{1}{t^{1/p}\log 1/t}\cdot\left(\int_{|\varphi+\eta-\theta|\le 2h(t)} f_R(\varphi)^p\,d\varphi\right)^{1/p} \\ &\lesssim \frac{1}{t^{1/p}\log 1/t}\cdot\left(\int_{|\varphi-\theta|\le 3h(t)} f_R(\varphi)^p\,d\varphi\right)^{1/p} \\ &\lesssim \left(\frac{h(t)}{t(\log 1/t)^p}\cdot\frac{1}{6h(t)}\int_{|\varphi-\theta|\le 3h(t)} f_R(\varphi)^p\,d\varphi\right)^{1/p} \\ &\lesssim \left(\frac{1}{6h(t)}\int_{|\varphi-\theta|\le 3h(t)} f_R(\varphi)^p\,d\varphi\right)^{1/p}.\end{aligned}$$

For a.e. $\theta \in \mathsf{T}$ (Lebesgue points of f_R^p) we have (using Proposition 1, part (a) for L^∞) that

$$\begin{aligned}\limsup_{\substack{t\to 0\\ |\eta|<h(t)}} \tau_\eta Q_t^1 * f(\theta) &\leq \limsup_{\substack{t\to 0\\ |\eta|<h(t)}} \tau_\eta Q_t^1 * f_-(\theta) + \limsup_{\substack{t\to 0\\ |\eta|<h(t)}} \tau_\eta Q_t^1 * f_R(\theta)\\ &\leq 0 + C\cdot f_R(\theta).\end{aligned}$$

By choosing R sufficiently large, we can make $f_R(\theta) = 0$ on a set with measure arbitrarily close to 2π , so part (a) of Proposition 1 is now established also for $1 \leq p < \infty$.

Proof of the implication (i) $\Rightarrow$ (ii). We assume here that $1 < p < \infty$, since the results for $p = 1$ and $p = \infty$ are already established by Sjögren[1]. Assume that condition (ii) in Theorem 1.1 is false. We show that this implies that (i) is false also.

Assume that

$$\limsup_{t\to 0} \frac{h(t)}{t(\log 1/t)^p} = \infty, \tag{4}$$

Pick any decreasing sequence $\{t_i\}_1^\infty$, converging to 0, such that

$$1 \leq \frac{h(t_i)}{t_i(\log 1/t_i)^p} \uparrow \infty, \tag{5}$$

as $i \to \infty$. Let

$$f_i(\varphi) = t_i^{1/(p-1)} \log 1/t_i \cdot \left(\frac{1}{t_i + |\varphi|}\right)^{1/(p-1)} \cdot \chi_{\{|\varphi|<h(t_i)\}},$$

Now,

$$\begin{aligned}\|f_i\|_p^p &\lesssim t_i^{p/(p-1)}(\log 1/t_i)^p \int_0^{h(t_i)} \left(\frac{1}{t_i+\varphi}\right)^{p/(p-1)} d\varphi\\ &\lesssim t_i^{p/(p-1)}(\log 1/t_i)^p t_i^{1-p/(p-1)} = t_i(\log 1/t_i)^p,\end{aligned}$$

where the constant depends only on p. It follows that

$$\frac{h(t_i)}{\|f_i\|_p^p} \geq C(p)\cdot \frac{h(t_i)}{t_i(\log 1/t_i)^p}.$$

[1] In section "Higher dimensional results for L^∞", we give a proof of the case $p = \infty$ in two dimensions, which is actually just a trivial extension of Sjögrens proof.

By (5) the right hand side tends to ∞ as $i \to \infty$. Thus, by standard techniques, we can pick a subsequence of $\{t_i\}$, with possible repetitions, for simplicity denoted $\{t_i\}$ also, such that

$$\sum_1^\infty h(t_i) = \infty, \qquad \text{and} \tag{6}$$

$$\sum_1^\infty \|f_i\|_p^p < \infty. \tag{7}$$

Let $A_1 = h(t_1)$, and for $n \geq 2$ let $A_n = h(t_n) + \sum_{j=1}^{n-1} 2h(t_j)$. By (6) one has that $\lim_{n\to\infty} A_n = \infty$.

Define (on T) $F_j(\varphi) = \tau_{A_j} f_j(\varphi)$, and let

$$F^{(N)}(\varphi) = \sup_{j \geq N} F_j(\varphi).$$

It is clear by construction that any given $\varphi \in \mathsf{T}$ lies in the support of infinitely many F_j:s.

Since $[F^{(N)}(\varphi)]^p = \sup_{j\geq N}[F_j(\varphi)]^p \leq \sum_{j\geq N}[F_j(\varphi)]^p$, it follows that

$$\|F^{(N)}\|_p^p \leq \sum_{j=N}^\infty \|F_i\|_p^p = \sum_{j=N}^\infty \|f_i\|_p^p \to 0$$

as $N \to \infty$, by (7). Thus, in particular, $F^{(N)} \in L^p$ for any $N \geq 1$.

For $\theta \in \mathsf{T}$ and a given $\xi_0 > 0$ we can, by construction, find $j \in \mathsf{N}$ so that $\theta \in \operatorname{supp}(F_j)$ and so that $t_j \in (0, \xi_0)$. We can then choose η, with $|\eta| < h(t_j)$, so that $\theta - \eta \equiv A_j \mod 2\pi$. It follows that

$$\limsup_{t\to 0,\ |\eta|<h(t)} \mathscr{P}_0 F^{(N)}((1-t)e^{i(\theta-\eta)}) \geq \limsup_{j\to\infty} \mathscr{P}_0 F_j((1-t_j)e^{iA_j}).$$

We have

$$\begin{aligned}
&\mathscr{P}_0 F_j((1-t_j)e^{iA_j}) \\
&\quad \geq \frac{C}{\log 1/t_j} \int_{|\varphi|<h(t_j)} \frac{F_j(A_j-\varphi)}{t_j+|\varphi|}\, d\varphi = \frac{C}{\log 1/t_j} \int_{|\varphi|<h(t_j)} \frac{f_j(\varphi)}{t_j+|\varphi|}\, d\varphi \\
&\quad = 2Ct_j^{1/(p-1)} \int_0^{h(t_j)} \left(\frac{1}{t_j+\varphi}\right)^{1+1/(p-1)} d\varphi \geq C_p'' > 0.
\end{aligned}$$

To sum up, we have shown that for any $\theta \in \mathsf{T}$ one has

$$\limsup_{t\to 0,\ |\eta|<h(t)} \mathscr{P}_0 F^{(N)}((1-t)e^{i(\theta-\eta)}) \geq C_p'' > 0.$$

Take N so large so that the measure of $\{F^{(N)} > C''_p/2\}$ is small, and a.e. convergence to $F^{(N)}$ is disproved.

3. Higher dimensional results for L^∞

In this section we prove results for the polydisc U^n, with bounded boundary functions. To simplify, we give the notation and proof for $n = 2$. The generalisation to arbitrary n is clear.

We define the Poisson integral of $f \in L^1(\mathsf{T}^2)$ to be

$$Pf(z_1, z_2) = \int_{\mathsf{T}^2} P(z_1, z_2, \beta_1, \beta_2) f(\beta_1, \beta_2)\, d\beta_1\, d\beta_2,$$

where

$$P(z_1, z_2, \beta_1, \beta_2) = P(z_1, \beta_1) P(z_2, \beta_2).$$

For any functions $h_i : \mathsf{R}_+ \to \mathsf{R}_+$, $i = 1, 2$, let

$$\mathscr{A}_{h_1,h_2}(\theta_1, \theta_2) = \{(z_1, z_2) \in U^2 : |\arg z_i - \theta_i| \le h_i(1 - |z_i|),\ i = 1, 2\}. \tag{8}$$

We refer to $\mathscr{A}_{h_1,h_2}(\theta_1, \theta_2)$ as the approach region determined by h_1, h_2 at $(\theta_1, \theta_2) \in \mathsf{T}^2$.

Let

$$P_0 f(z_1, z_2) = \int_{\mathsf{T}^2} \sqrt{P(z_1, z_2, \beta_1, \beta_2)}\, f(\beta_1, \beta_2)\, d\beta_1\, d\beta_2,$$

and denote the normalised operator by $\mathscr{P}_0$, i.e.

$$\mathscr{P}_0 f(z_1, z_2) = \frac{P_0 f(z_1, z_2)}{P_0 1(z_1, z_2)}.$$

We shall prove the following theorem:

THEOREM 3.1. *The following conditions are equivalent for any functions $h_i : \mathsf{R}_+ \to \mathsf{R}_+$, $i = 1, \ldots, n$:*

(i) *For any $f \in L^\infty(\mathsf{T}^n)$ one has for almost all $(\theta_1, \ldots, \theta_n) \in \mathsf{T}^n$ that*

$$\mathscr{P}_0 f(z_1, \ldots, z_n) \to f(\theta_1, \ldots, \theta_n)$$

as $(z_1, \ldots, z_n) \to (\theta_1, \ldots, \theta_n)$ and $(z_1, \ldots, z_n) \in \mathscr{A}_{h_1,\ldots,h_n}(\theta_1, \ldots, \theta_n)$.

(ii) *$h_i \in S_\infty$, $i = 1, \ldots, n$. (For S_∞, see Definition 1.)*

4. The proof of Theorem 3.1

We may assume, without loss of generality, that $\lim_{t\to 0} h_j(t)/t = \infty$, $j = 1, 2$.

We shall begin by proving the implication (ii) $\Rightarrow$ (i) in Theorem 3.1.

Let $t_j = 1 - |z_j|$ and $z_j = (1 - t_j)e^{i\theta_j}$, $j = 1, 2$. Then

$$\mathscr{P}_0 f(z_1, z_2) = R_{t_1,t_2} * f(\theta_1, \theta_2),$$

where the convolution is taken in T^2 and

$$R_{t_1,t_2}(\theta_1, \theta_2) = \prod_{j=1}^{2} \frac{1}{\sqrt{2\pi}} \frac{\sqrt{t_j(2 - t_j)}}{|(1 - t_j)e^{i\theta_j} - 1|} \frac{1}{P_0^{(1)}1(1 - t_j)},$$

$P_0^{(1)}$ denoting the square root operator in *one* variable.

As before, we are interested only in small values of t_j, so we assume from now on that $t_j < 1/2$, $j = 1, 2$. Then $P_0^{(1)}1(1 - t) \sim \sqrt{t}\log 1/t$, and thus the order of magnitude of R_{t_1,t_2} is given by

$$R_{t_1,t_2}(\theta_1, \theta_2) \sim Q_{t_1,t_2}(\theta_1, \theta_2) = \prod_{j=1}^{2} \frac{1}{\log 1/t_j} \cdot \frac{1}{t_j + |\theta_j|}.$$

Now, let τ_{η_1,η_2} denote the translation $\tau_{\eta_1,\eta_2} f(\theta_1, \theta_2) = f(\theta_1 - \eta_1, \theta_2 - \eta_2)$. Then the convergence condition (i) in Theorem 3.1 above means

$$\lim_{\substack{t_1,t_2\to 0\\ |\eta_j|<h_j(t_j),\ j=1,2}} \tau_{\eta_1,\eta_2} R_{t_1,t_2} * f(\theta_1, \theta_2) = f(\theta_1, \theta_2).$$

We are now ready to prove Theorem 3.1.

Proof. Assume that condition (ii) holds. We prove that it implies (i).

If we let

$$R_{t_1,t_2}(\theta_1, \theta_2) = R^1_{t_1,t_2}(\theta_1, \theta_2) + R^2_{t_1,t_2}(\theta_1, \theta_2)$$

where

$$R^2_t(\theta_1, \theta_2) = R_{t_1,t_2}(\theta_1, \theta_2)\chi_{\{|\theta_j|\geq 2h_j(t_j),\ j=1,2\}}(\theta_1, \theta_2),$$

we claim that

$$\lim_{\substack{t_1,t_2\to 0\\ |\eta_j|<h_j(t_j),\ j=1,2}} \tau_{\eta_1,\eta_2} R^1_{t_1,t_2} * f(\theta_1, \theta_2) = 0 \tag{9}$$

and, for almost all $(\theta_1, \theta_2) \in \mathsf{T}^2$,

$$\lim_{\substack{t_1,t_2\to 0\\ |\eta_j|<h_j(t_j),\ j=1,2}} \tau_{\eta_1,\eta_2} R^2_{t_1,t_2} * f(\theta_1, \theta_2) = f(\theta_1, \theta_2). \tag{10}$$

To prove (9), it suffices to prove that

$$\limsup_{\substack{t_1,t_2\to 0\\ |\eta_j|<h_j(t_j),\ j=1,2}} \tau_{\eta_1,\eta_2} Q^1_{t_1,t_2} * f(\theta_1,\theta_2) = 0,$$

where $Q^1_{t_1,t_2}$ corresponds to Q_{t_1,t_2} as $R^1_{t_1,t_2}$ corresponds to R_{t_1,t_2}. Note that $Q^1_{t_1,t_2}$ is supported in a set where $|\varphi_j| < 2h_j(t_j)$ for $j = 1$ or $j = 2$. Assume, without loss of generality, that $|\varphi_1| < 2h_1(t_1)$ and observe that we then have

$$Q^1_{t_1,t_2}(\varphi_1,\varphi_2) \le \chi_{\{|\varphi_1|<2h_1(t_1)\}}(\varphi_1,\varphi_2) \prod_{j=1}^{2} \frac{1}{\log 1/t_1} \cdot \frac{1}{t_j + |\varphi_j|}.$$

It follows that

$$\begin{aligned}
&\tau_{\eta_1,\eta_2} Q^1_{t_1,t_2} * |f|(\theta_1,\theta_2)\\
&\quad \le \|f\|_\infty \int_{\mathsf{T}^2} Q^1_{t_1,t_2}(\varphi_1,\varphi_2)\, d\varphi_1\, d\varphi_2\\
&\quad = \frac{\|f\|_\infty}{(\log 1/t_1)(\log 1/t_2)} \cdot \int_{|\varphi_1|<2h_1(t_1)} \frac{d\varphi_1}{t_1+|\varphi_1|} \cdot \int_{\mathsf{T}} \frac{d\varphi_2}{t_2+|\varphi_2|}\\
&\quad \lesssim \frac{\|f\|_\infty}{\log 1/t_1} \log\,(h_1(t_1)/t_1).
\end{aligned}$$

Let $\varepsilon > 0$ be given. By condition (ii) in Theorem 3.1, we have that $h_1(t_1) \le Ct_1^{1-\varepsilon}$. Thus,

$$\limsup_{\substack{t_1,t_2\to 0\\ |\eta_j|<h_j(t_j),\ j=1,2}} \tau_{\eta_1,\eta_2} Q^1_{t_1,t_2} * f(\theta_1,\theta_2) \lesssim \varepsilon \|f\|_\infty,$$

and (9) follows.

To prove (10), it now suffices to prove that the maximal operator M, defined by

$$Mf(\theta) = \limsup_{\substack{t_1,t_2\to 0\\ |\eta_j|<h_j(t_j),\ j=1,2}} \tau_{\eta_1,\eta_2} Q^2_{t_1,t_2} * |f|(\theta_1,\theta_2),$$

is dominated by a strong type (p, p) operator, for some $p \ge 1$. Then convergence follows by standard arguments, since the continuous functions, for which unrestricted convergence holds for $R^2_{t_1,t_2}$, form a dense subset of L^p.

Since $|\eta_j| < h_j(t_j)$, $j = 1, 2$, we have that

$$\tau_{\eta_1,\eta_2} Q^2_{t_1,t_2}(\theta_1, \theta_2) = \prod_{j=1}^{2} \frac{1}{\log 1/t_j} \cdot \frac{1}{t_j + |\theta_j - \eta_j|} \chi_{\{|\theta_j-\eta_j|\geq 2h_j(t_j)\}}$$
$$\lesssim \prod_{j=1}^{2} \frac{1}{\log 1/t_j} \cdot \frac{1}{t_j + |\theta_j|}.$$

Each factor in the above product is a decreasing function of $|\theta_j|$ whose integral in T is bounded uniformly in t_j. Convolution (in one variable) with such a function is dominated by the Hardy-Littlewood maximal operator, as is well known.Since, for example, $L^\infty \subset L^2$ and since the Hardy-Littlewood maximal operator is of strong type (2, 2), we have that

$$\tau_{\eta_1,\eta_2} Q^2_{t_1,t_2} * |f|(\theta_1, \theta_2) \leq \frac{1}{\log 1/t_2} \cdot \int_{\mathsf{T}} \frac{1}{t_2 + |\varphi_2|} M^{(1)}_{HL} f(\theta_1, \theta_2 - \varphi_2)\, d\varphi_2,$$
$$\leq M^{(2)}_{HL} M^{(1)}_{HL} f(\theta_1, \theta_2)$$

where $M^{(j)}_{HL}$ denotes the ordinary (one-dimensional) Hardy-Littlewood maximal operator in variable j. But, since $M^{(2)}_{HL} M^{(1)}_{HL}$ is of strong type (2, 2) (weak type is sufficient), we are done.

It remains to prove that (i) implies (ii). The method is similar to that of Sjögren. Assume that (ii) is false. Without loss of generality, we may assume that there exists $\varepsilon > 0$ and a sequence $s_k \to 0$, such that $h_1(s_k)/s_k^{1-\varepsilon} \to \infty$. We may also assume that

$$\sum_{k=1}^{\infty} \frac{s_k^{1-\varepsilon}}{h_1(s_k)} < \infty.$$

Let $E_k \subset \mathsf{T}$ be the union of at most $C/h_1(s_k)$ intervals of length $s_k^{1-\varepsilon}$, chosen such that the distance from E_k to any point in T is at most $h_1(s_k)$. If $\theta_1 \in \partial E_k$, it is clear that

$$\mathscr{P}_0 \chi_{E_k\times \mathsf{T}} \left((1 - s_k)e^{i\theta_1}, (1 - t)e^{i\theta_2}\right)$$
$$\geq \frac{C}{(\log 1/s_k)(\log 1/t)} \cdot \int_0^{s_k^{1-\varepsilon}} \frac{d\varphi_1}{s_k + \varphi_1} \cdot \int_{\mathsf{T}} \frac{d\varphi_2}{t + |\varphi_2|} \geq C\varepsilon.$$

Thus, for any $(\theta_1, \theta_2) \in \mathsf{T}^2$ we have

$$\sup_{|\eta_j|<h_j(t_j),\ j=1,2} \mathscr{P}_0 \chi_{E_k\times \mathsf{T}} \left((1 - s_k)e^{i(\theta_1-\eta_1)}, (1 - t)e^{i(\theta_2-\eta_2)}\right) \geq C\varepsilon.$$

Now, since $|E_k| \lesssim s_k^{1-\varepsilon}/h_1(s_k)$, we can choose k_0 so large that the measure of $E = \cup_{k \ge k_0} E_k$ is arbitrarily small. But clearly

$$\limsup_{\substack{t_1,t_2\to 0\\ |\eta_j|<h_j(t_j),\ j=1,2}} \mathscr{P}_0\chi_{E\times\mathsf{T}}\left((1-t_1)e^{i(\theta_1-\eta_1)}, (1-t_2)e^{i(\theta_2-\eta_2)}\right) \ge C\varepsilon$$

for each $(\theta_1, \theta_2) \in \mathsf{T}^2$. We have shown that a.e. convergence to $\chi_{E\times\mathsf{T}}$ along the region defined by h_1 and h_2 fails. This completes the proof.

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