STRONG PERFORATION IN INFINITELY GENERATED 
$K_0$-GROUPS OF SIMPLE $C^*$-ALGEBRAS

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Abstract

Let $(G, G^+)$ be an ordered abelian group. We say that $G$ has strong perforation if there exists $x \in G, x \notin G^+$, such that $nx \in G^+, nx \neq 0$ for some natural number $n$. Otherwise, the group is said to be weakly unperforated. Examples of simple $C^*$-algebras whose ordered $K_0$-groups have this property and for which the entire order structure on $K_0$ is known, until now, been restricted to the case where $K_0$ is group isomorphic to the integers. We construct simple, separable, unital $C^*$-algebras with strongly perforated $K_0$-groups group isomorphic to an arbitrary infinitely generated subgroup of the rationals, and determine the order structure on $K_0$ in each case.

1. Introduction

Elliott’s classification of AF $C^*$-algebras via the $K_0$-group ([2]) began a widespread effort to classify nuclear $C^*$-algebras. The $K_0$-group, which is an ordered group for stably finite $C^*$-algebras ([1]), has figured prominently in almost all work on this problem. (For an overview of the classification problem for nuclear $C^*$-algebras, see [3].) So far, every result on the classification of $C^*$-algebras has required the assumption that the ordered $K_0$-group be weakly unperforated whenever it is not zero. This assumption was shown to be non-trivial by Villadsen ([8]); the ordered abelian group $\mathbb{Z}_n := (\mathbb{Z}, \{0, n, n+1, \ldots\})$ may arise as a saturated sub-ordered group of the $K_0$-group of a simple nuclear $C^*$-algebra. In [4], Elliott and Villadsen refined the results of [8] to obtain, for each natural number $n$, a simple nuclear $C^*$-algebra $A_n$ whose ordered $K_0$-group is order isomorphic to $\mathbb{Z}_n$. This result was further generalised by the author in [7], where it was shown that a certain class of order structures on the integers (which might possibly comprise all such order structures giving a simple ordered group) could arise as the ordered $K_0$-group of a simple nuclear $C^*$-algebra.

The classification of a category by an invariant is not complete until one knows the range of the invariant, and any classification of simple nuclear stably finite $C^*$-algebras will necessarily capture the ordered $K_0$-group. Thus, the range of the $K_0$ functor bears investigation. This range is known when $K_0$

Received August 8, 2003.
is a weakly unperforated ordered group, whence our interest in instances of the ordered $K_0$-group which exhibit strong perforation.

2. Essential Results

In this section we review results from [4] that will be used in the sequel.

Let $C, D$ be $C^*$-algebras, and let $\phi_0, \phi_1$ be $*$-homomorphisms from $C$ to $D$. The generalised mapping torus of $C$ and $D$ with respect to $\phi_0$ and $\phi_1$ is

$$A := \{(c, d) \mid d \in C([0, 1]; D), \ c \in C, \ d(0) = \phi_0(c), \ d(1) = \phi_1(c)\}$$

We will write $A(C, D, \phi_0, \phi_1)$ for $A$ when clarity demands it. We now list without proof some theorems, specialised to our needs, which will be used in the sequel.

**Theorem 2.1** (Elliott and Villadsen ([4]), Sec. 2, Thm. 2). The index map $b_* : K_* C \rightarrow K_1 D$ in the six term periodic sequence for the extension

$$0 \rightarrow SD \rightarrow A \rightarrow C \rightarrow 0$$

is the difference $K_* \phi_1 - K_* \phi_0 : K_* C \rightarrow K_* D$.

Thus, the six-term exact sequence may be written as the short exact sequence

$$0 \rightarrow \text{Coker } b_{1-*} \rightarrow K_* A \rightarrow \text{Ker } b_* \rightarrow 0.$$ 

In particular, if $b_{1-*}$ is surjective, then $K_* A$ is isomorphic to its image, $\text{Ker } b_*$, in $K_* C$.

Suppose that cancellation holds for each pair of projections in $D \otimes \mathcal{H}$ obtained as the images under the maps $\phi_0$ and $\phi_1$ of a single projection in $C \otimes \mathcal{H}$. Then, if $b_1$ is surjective,

$$(K_0 A)^+ \cong (K_0 C)^+ \cap K_0(e_{\infty})(K_0 A),$$

where $e_{\infty}$ denotes the evaluation of $A$ at the fibre at infinity.

**Theorem 2.2** (Elliott and Villadsen ([4]), Sec. 3, Thm. 3). Let $A_1$ and $A_2$ be building block algebras as described above,

$$A_i = A(C, D, \phi_i^0, \phi_i^1), \ i = 1, 2.$$ 

Let there be given three maps between the fibres,

$$\gamma : C_1 \rightarrow C_2,$$
$$\delta, \delta' : D_1 \rightarrow D_2,$$
such that \( \delta \) and \( \delta' \) have mutually orthogonal images, and

\[
\delta \phi_0^1 + \delta' \phi_1^1 = \phi_0^2 \gamma, \\
\delta \phi_1^1 + \delta' \phi_0^1 = \phi_1^2 \gamma.
\]

Then there exists a unique map

\[
\theta : A_1 \to A_2,
\]

respecting the canonical ideals, giving rise to the map \( \gamma : C_1 \to C_2 \) between the quotients (or fibres at infinity), and such that for any \( 0 < s < 1 \), if \( e_s \) denotes evaluation at \( s \),

\[
e_s \theta = \delta e_s + \delta' e_{1-s}.
\]

Let \( A_1 \) and \( A_2 \) be building block algebras as in Theorem 2.1 with \( \theta : A_1 \to A_2 \) as in Theorem 2.2. Let there be given a map \( \beta : D_1 \to C_2 \) such that the composed map \( \beta \phi_1^1 \) is a direct summand of the map \( \gamma \), and such that the composed maps \( \phi_0^1 \beta \) and \( \phi_1^1 \beta \) are direct summands of the maps \( \delta' \) and \( \delta \), respectively. Suppose that the decomposition of \( \gamma \) as the orthogonal sum of \( \beta \phi_1^1 \) and another map is such that the image of the second map is orthogonal to the image of \( \beta \). (Note that this requirement is automatically satisfied if \( C_1 \), \( D_1 \), and the map \( \beta \phi_1^1 \) are unital.)

Let

\[
A_1 \xrightarrow{\theta_1} A_2 \xrightarrow{\theta_2} \cdots
\]

be a sequence of separable building block \( C^* \)-algebras,

\[
A_i = A(C_i, D_i, \phi_0^i, \phi_1^i), \quad i = 1, 2, \ldots
\]

with each map \( \theta_i : A_i \to A_{i+1} \) obtained by the construction of Theorem 2.2. For each \( i = 1, 2, \ldots \), let \( \beta_i : D_i \to C_{i+1} \) be a map verifying the hypotheses of the preceding paragraph.

Suppose that for every \( i = 1, 2, \ldots \), the intersection of the kernels of the boundary maps \( \phi_0^i \) and \( \phi_1^i \) from \( C_i \) to \( D_i \) is zero.

Suppose that, for each \( i \), the image of each of \( \phi_0^{i+1} \) and \( \phi_1^{i+1} \) generates \( D_{i+1} \) as a closed two-sided ideal, and that this is in fact true for the restriction of \( \phi_0^{i+1} \) and \( \phi_1^{i+1} \) to the smallest direct summand of \( C_{i+1} \) containing the image of \( \beta_i \). Suppose that the closed two-sided ideal of \( C_{i+1} \) generated by the image of \( \beta_i \) is a direct summand.

Suppose that, for each \( i \), the maps \( \delta_i' - \phi_0^i \beta_i \) and \( \delta_i - \phi_1^i \beta_i \) from \( D_i \) to \( D_{i+1} \) are injective.
Suppose that, for each $i$, the map $\gamma_i - \beta_i \phi_i^1$ takes each non-zero direct summand of $C_i$ into a subalgebra of $C_{i+1}$ not contained in any proper closed two-sided ideal.

Suppose that, for each $i$, the map $\beta_i : D_i \to C_{i+1}$ can be deformed – inside the hereditary sub-$C^*$-algebra generated by its image – to a map $\alpha_i : D_i \to C_{i+1}$ with the following property: There is a direct summand of $\alpha_i$, say $\bar{\alpha}_i$, such that $\bar{\alpha}_i$ is non-zero on an arbitrary given element $x_i$ of $D_i$, and has image a simple sub-$C^*$-algebra of $C_{i+1}$, the closed two-sided ideal generated by which contains the image of $\beta_i$.

**Theorem 2.3** (Elliott and Villadsen ([4]), Sec. 5, Thm. 5). If the hypotheses above are satisfied, there is a map $\theta_i'$ homotopic inside $A_i$ to $\theta_i$ for each $i$ such that the inductive limit of the sequence

$$A_1 \xrightarrow{\theta_1'} A_2 \xrightarrow{\theta_2'} \cdots$$

is simple.

### 3. Infinitely Generated Subgroups of the Rational Numbers

A generalised integer is a symbol $n = a_1^{n_1}a_2^{n_2}a_3^{n_3}\ldots$, where the $a_i$'s are pairwise distinct prime numbers and each $n_i$ is either a non-negative integer or $\infty$. The subgroup $G_n$ of the rational numbers associated to the generalised integer $n$ is the group of all rationals whose denominators (when in lowest terms) are products of powers of the $a_i$'s not exceeding $a_i^{n_i}$. If $n_i = \infty$, then an arbitrarily large power of $a_i$ may appear in the denominator.

**Theorem 3.1.** For each pair $(n, k)$ consisting of a generalised integer $n$ and a positive rational $k < 1$, there exists a simple, separable, unital, nuclear $C^*$-algebra $A(n, k)$ such that

$$(K_0(A(n, k)), K_0(A(n, k))^+, [1_{A(n, k)}]) = (G_n, G_n \cap (k, \infty), 1).$$

**Proof.** Given a 2-tuple $(n, k)$ we will construct a sequence

$$A_1 \xrightarrow{\theta_1} A_2 \xrightarrow{\theta_2} \cdots$$

where $A_j = A(C_j, D_j, \phi_j^0, \phi_j^1)$, and the $\theta_j$ constructed as in Theorem 2.2 from maps

$$\gamma_j : C_j \to C_{j+1}, \quad \delta_j, \delta_j' : D_j \to D_{j+1}.$$

In order to obtain a simple inductive limit, we will require a map

$$\beta_j : D_j \to C_{j+1}$$
having the properties listed in Section 2.

For each \( j \) let

\[ C_j = p_j(C(X_j) \otimes \mathcal{H})p_j \]

where \( p_j \) is a projection in \( C(X_j) \otimes \mathcal{H} \) and \( \mathcal{H} \) denotes the compact operators.

Express \( k \) in lowest terms, say \( \frac{a}{b} \), and set \( X_1 = S^{2 \times (a+1)} \). Let \( X_{j+1} = X_j \times n_j \), where \( n_j \) is a natural number to be specified.

Let \( D_j = C_j \otimes M_{\dim(p_j)k_j} \), where \( k_j \) is a natural number to be specified. Let \( \mu_j \) and \( \nu_j \) be maps from \( C_j \) to \( C_j \otimes M_{\dim(p_j)} \) given by

\[ \mu_j(a) = p_j \otimes a(x_j) \cdot 1_{\dim(p_j)} \]

(where \( x_j \) is a point to be specified in \( X_j \) and \( 1_{\dim(p_j)} \) is the unit of \( M_{\dim(p_j)} \)) and

\[ \nu_j(a) = a \otimes 1_{\dim(p_j)}. \]

For \( t \in \{0, 1\} \), let \( \phi^t_j : C_j \to D_j \) be the direct sum of \( l^j_1 \) and \( l^j_0 - l^j_1 \) copies of \( \mu_j \) and \( \nu_j \), respectively, where the \( l^j_t \) are non-negative integers such that \( l^j_0 \neq l^j_1 \) for all \( j \geq 1 \).

Note that both \( C_j \) and \( D_j \) are unital, as are the maps \( \phi^t_j \). The \( \phi^t_j \) are also injective and as such satisfy the hypotheses of Section 2 concerning them alone.

By Theorem 2.1, for each \( e \in K_0(C_j) \),

\[ b_0(e) = (l^j_1 - l^j_0)(K_0(\mu_j) - K_0(\nu_j))(e) \]

\[ = (l^j_1 - l^j_0)(\dim(p_j) \cdot K_0(p_j) - \dim(p_j) \cdot e). \]

Since \( l^j_1 - l^j_0 \) is non-zero for every \( j \) and \( K_0(X_j) \) is torsion free, \( b_0(e) = 0 \) implies that \( e \) belongs to the maximal free cyclic subgroup of \( K_0(C_j) \) containing \( K_0(p_j) \). As \( K_1(C_j) = 0, b_1 \) is surjective. \( K_0(A_j) \) is thus group isomorphic (by Theorem 2.1) to its image, in \( K_0(C_j) \) – which is isomorphic as a group to \( \mathbb{Z} \).

In order for \( K_0(A_j) \) to be isomorphic as an ordered group to its image in \( K_0(C_j) \), with the relative order, it is sufficient (by Theorem 2.1) that for any projection \( q \) in \( C_j \otimes \mathcal{H} \) such that the images of \( q \) under \( \phi^0_j \otimes \text{id} \) and \( \phi^1_j \otimes \text{id} \) have the same \( K_0 \) class, these images be in fact equivalent. For any such \( q \), the image of \( K_0(q) \) under \( b_0 = K_0(\phi^1_j) - K_0(\phi^0_j) \) is zero, so that \( K_0(q) \) belongs to \( \text{Ker} b_0 \). It will be clear from the construction below that the dimension of both \( \phi^1_j(q) \) and \( \phi^0_j(q) \) is at least half the dimension of \( X_j \). Thus, by Theorem 8.1.5 of [5], \( \phi^1_j(q) \) and \( \phi^0_j(q) \) are equivalent, as they have the same \( K_0 \) class.

Let us now specify the projection \( p_1 \). Let \( \xi \) be the Hopf line bundle over \( S^2 \).

Set \( g_1 = [\xi \times (a+1)] - [\theta_a] \in K^0(X_1) \), where \([ \cdot ]\) denotes the stable isomorphism class of a vector bundle and \( \theta_l \) denotes the trivial vector bundle of fiber dimension \( l \). By Theorem 8.1.5 of [5], we have that \( (a + 1) \cdot g_1 \) and hence \( b \cdot g_1 \).
are positive. Let \( p_1 \) be a projection in \( C(X_1) \otimes \mathcal{K} \) corresponding to the \( K^0 \) class \( b \cdot g_1 \). By \([8]\) we know that the ordered, saturated, free cyclic subgroup of \( K_0(C_1) \) generated by \( g_1 \) is equal to
\[
(\mathbb{Z}, \{0, a + 1, a + 2, \ldots\}),
\]
where the class of the unit is the integer \( b \geq a + 1 \).

Decompose \( b \) into powers of primes, \( b = a^1\eta_1 a^2\eta_2 \cdots a^n\eta_n \). Set \( \mathbf{n}' = \frac{n}{\eta} \), with the convention that \( \infty - l = \infty \) for all natural numbers \( l \). Let \( L_j \) be an enumeration of the primes appearing in \( \mathbf{n}' \) for \( j \geq 2, j \in \mathbb{N} \), and set \( L_1 = b \).

We now define a family of continuous maps from \( S^2 \) to \( S^2 \), indexed by the integers, to be used in the construction of the maps \( \gamma_j \) from \( C_j \) to \( C_{j+1} \).

Consider \( S^2 \) as being embedded in \( \mathbb{R}^3 = \mathbb{C} \times \mathbb{R} \) as the unit sphere with center the origin, with the identification \((x, y, z) = (x + yi, z)\). For each \( \eta \in \mathbb{N} \), let \( \omega'_{\eta} : \mathbb{C} \times \mathbb{R} \rightarrow \mathbb{C} \times \mathbb{R} \) be defined by \( \omega'_{\eta}(w, z) = (\frac{w^\eta}{|w^\eta - 1|}, z) \) when \( w \neq 0 \) and otherwise by \( \omega'_{\eta}(0, z) = (0, z) \). This defines a map from \( S^2 \) to itself by restriction. Let \( \omega_{\eta} \) be the composition of \( \omega'_{\eta} \) with the antipodal map. Note that \( \omega'_{\eta} \) is the suspension of the \( \eta \)-th power map on \( S^1 \), and thus has the same degree, namely \( -\eta \), as this map \([6]\). As the antipodal map has degree \( -1 \), the composed map \( \omega_{\eta} \) has degree \( \eta \). In the language of vector bundles, \( K^0(\omega_{\eta})([\xi]) = [\xi^{\otimes \eta}] \).

Define a map \( \gamma'_{\eta} \) from \( C(X_j) \) to \( M_{\eta_j} \otimes C(X_{j+1}) = M_{\eta_j}(C(X_j^{\otimes \eta})) \) as follows:
\[
\gamma'_{\eta}(f(x)) = (f(\omega_{L_{j+1}}(x))\otimes 1 \otimes \cdots \otimes 1) \oplus (1 \otimes f(\omega_{L_{j+1}}(x))\otimes \cdots \otimes 1) \oplus \cdots \oplus (1 \otimes 1 \otimes \cdots \otimes f(\omega_{L_{j+1}}(x))).
\]

Let
\[
\beta'_{\eta} = 1 \cdot e_{x_j}
\]
be a map from \( C(X_j) \) to \( C(X_{j+1}) \), where \( e_{x_j} \) denotes the evaluation of an element of \( C(X_j) \) at a point \( x_j \in X_j \) and \( 1 \) is the unit of \( C(X_{j+1}) \). Fix \( x_1 \in S^2 \) and define \( x_{j+1} := (\omega_{L_{j+1}}(x_1), \ldots, \omega_{L_{j+1}}(x_j)) \in X_j^{\otimes \eta_j} = X_{j+1} \).

Let us define \( \gamma_{\eta} : C(X_j) \rightarrow M_{\eta_j}(C(X_{j+1})) \otimes M_2(\mathcal{K}) \) inductively as the direct sum of two maps. For the first map, take the restriction to \( C_j \subseteq C(X_j) \otimes \mathcal{K} \) of the tensor product of \( \gamma'_{\eta} \) with the identity map from \( \mathcal{K} \) to \( \mathcal{K} \). The second map is obtained as follows: compose the map \( \phi'_{\eta} \) with the direct sum of \( q_j \) copies of the tensor product of \( \beta'_{\eta} \) with the identity map from \( \mathcal{K} \) to \( \mathcal{K} \) (restricted to \( D_j \subseteq C(X_j) \otimes \mathcal{K} \)), where \( q_j \) is to be specified. The induction consists of first considering the case \( j = 1 \) (since \( p_1 \) has already been chosen), then setting \( p_2 = \gamma_{\eta}(p_1) \), so that \( C_2 \) is specified as the cut-down of \( C(X_j) \otimes M_2(\mathcal{K}) \), and continuing in this way.
With $\beta_j$ taken to be the restriction to $D_j \subseteq C(X_j) \otimes \mathcal{K}$ of $\beta'_j \otimes \text{id}$ we have, by construction, that $\beta_j \phi_j^1$ is a direct summand of $\gamma_j$ and, furthermore, the second direct summand and $\beta_j$ map into orthogonal blocks (and hence orthogonal subalgebras) as desired.

We will now need to verify that $p_j$ has the following property: the set of all rational multiples of $K_0(p_j)$ in the ordered group $K_0(C_j) = K^0(X_j)$ is isomorphic (as a sub ordered group) to

$$(\mathbb{Z}, \{0, l_j + 1, l_j + 2, \ldots\}),$$

where

$$l_j := L_j l_{j-1}, \quad l_1 := a$$

and the class of the unit (i.e., of $p_j$) is $\prod_{k=1}^{j} L_k$.

Our verification will proceed by induction. The case $j = 1$ has been established by construction. Suppose that the assertion of the preceding paragraph holds for all $p_k$, $k \leq j$. Suppose further that the group of rational multiples of $K_0(p_k)$ (being isomorphic as a group to $\mathbb{Z}$) is generated by a $K_0$ class of the form $[\xi^{\times n}] - [\theta_m]$, where $m < n$ and (this is again true by construction for $k = 1$). We will show that $K_0(p_j)$ has both the property of the preceding paragraph and the property just mentioned.

Let $g_k \in K^0(X_k)$ be the generator of the group of rational multiples of $p_k$. Note that, as is the case for all maps on $K^0(S^2)$ induced by a continuous map from $S^2$ to itself, $K_0(\omega)([\theta_1]) = [\theta_1]$. Write $g_k = [\xi^{\times d_k}] - [\theta_{m_k}]$. Then

$$K_0(\gamma_j)(g_j) = ([\xi^{\times L_{j+1}}]^{\times d_j n_j} - [\theta_{m_{j+1}}])$$

for some integers $d_j > 0$ and $m_{j+1}$. We may assume that the multiplicity of the map $K_0(\gamma_j)$ is divisible by $L_{j+1}$, as we have yet to specify $n_j$. We recall that for any integer $l$, the $K_0$ class $[\xi^{\times l}]$ corresponds to the element $(1, l)$ in $K^0(S^2) = ([\theta_1]) \oplus \langle e(\xi) \rangle$, which is also the difference of $K_0$ classes $l[\xi] - [\theta_{l-1}]$. Thus we have

$$K_0(\gamma_j)(g_j) = L_{j+1}([\xi^{\times (a+1)n_{j+2} \cdots n_j}] - [\theta_{m_{j+1}}]).$$

for some integer $m_{j+1}$. Setting $g_j := [\xi^{\times (a+1)n_{j+2} \cdots n_j}] - [\theta_{m_{j+1}}]$, we have established that $K_0(\gamma_j)(g_j) = L_{j+1}g_{j+1}$ for all natural numbers $j$.

We now show that $n_j$ may be chosen so as to ensure that the maximal, free, cyclic subgroup of $K_0C_{j+1}$ generated by $g_{j+1}$ is indeed isomorphic as an ordered group to the integers with positive cone $\{0, l_{j+1} + 1, l_{j+1} + 2, \ldots\}$. That $\prod_{k=1}^{j} L_k$ is the class of the unit follows directly from the fact that $L_1 = b$ (the class of the unit in $K_0C_1$) and that $K_0(\gamma_j)(g_j) = L_{j+1}g_{j+1}$. 


As the Euler class of the Hopf line bundle on $S^2$ is non-zero we have, by [8], that for $q, m, h \in \mathbb{N}$ such that $0 < h(q - m) < q$,

$$h((\xi^q) - [\theta_m]) \notin (K^0S^2)^+.$$

To apply this we note that

$$g_{j+1} = [\xi^{(a+1)n_1n_2...n_j}] - [\theta_{n_j}].$$

With $q = (a + 1)n_1n_2...n_j$ and $m = m_j$ we wish to have

$$0 < l_j(q - m) < q$$

as then $0 < h(q - m) < q$ for all $0 < h < l_j + 1$.

Since

$$q - m = \dim g_{j+1} = \frac{n_j + k_jq_j \dim p_j}{L_j+1} \dim g_j$$

we want

$$\dim g_{j+1} < \frac{(a + 1)n_1n_2...n_j}{l_j+1}.$$  

Assume inductively that $n_1, n_2, ..., n_{j-1}$ have been chosen so that

$$\dim g_j < \frac{(a + 1)n_1n_2...n_{j-1}}{l_j}.$$  

Choose $n_j$ large enough so that

$$\frac{n_j + k_jq_j \dim p_j}{n_j} \dim g_j < \frac{(a + 1)n_1n_2...n_{j-1}}{l_j}.$$  

Then we have that

$$\frac{n_j + k_jq_j \dim p_j}{L_j+1} \dim g_j < \frac{(a + 1)n_1n_2...n_j}{L_j+1l_j}.$$  

Recalling that $l_{j+1} = L_j+1l_j$ we conclude that

$$\dim g_{j+1} = \frac{n_j + k_jq_j \dim p_j}{L_j+1} \dim g_j < \frac{(a + 1)n_1n_2...n_j}{l_{j+1}}.$$  

as desired.

Note that $\gamma_j - \beta_j\phi_j^j$ is non-zero and so, as required in the hypotheses of Theorem 2.4, takes $C_j$ into a subalgebra of $C_{j+1}$ not contained in any proper closed two-sided ideal.
It remains to construct maps $\delta_j$ and $\delta'_j$ from $D_j$ to $D_{j+1}$ with orthogonal images such that

$$\delta_j \phi_j^0 + \delta'_j \phi_j^1 = \phi_{j+1} Y_j,$$

$$\delta_j \phi_j^1 + \delta'_j \phi_j^0 = \phi_{j+1}^1 Y_j,$$

and $\phi_{j+1}^0 \beta_j$ and $\phi_{j+1}^1 \beta_j$ are direct summands of $\delta'_j$ and $\delta_j$ respectively. To do this we shall have to modify $\phi_{j+1}^0$ and $\phi_{j+1}^1$ by inner automorphisms; this is permissible since it has no effect on $K_0$-groups. The definition of $\delta_j$ and $\delta'_j$ along with the proof that they satisfy the hypotheses of section 2 is taken from [4].

In order to carry out this step we define $x_{j+1} := \omega_{L_j+1}(x_j)$, so that

$$e_{x_{j+1}} Y_j = \text{mult}(\gamma_j) e_{x_j},$$

where $\text{mult}(\gamma_j)$ denotes the factor by which $\gamma_j$ multiplies dimension. It follows that

$$\mu_{j+1} Y_j = p_{j+1} \otimes e_{x_{j+1}} Y_j$$

$$= \gamma_j(p_j) \otimes \text{mult}(\gamma_j) e_{x_j}$$

$$= \text{mult}(\gamma_j) \gamma_j(p_j \otimes e_{x_j})$$

$$= \text{mult}(\gamma_j) \gamma_j \mu_j,$$

and

$$\nu_{j+1} Y_j = \gamma_j \otimes 1_{\dim(p_{j+1})}$$

$$= \text{mult}(\gamma_j) \gamma_j \otimes 1_{\dim(p_j)}$$

$$= \text{mult}(\gamma_j) \gamma_j \nu_j.$$ 

Take $\delta_j$ and $\delta'_j$ to be the direct sum of $r_j$ and $s_j$ copies of $\gamma_j$, where $r_j$ and $s_j$ are to be specified. The condition, for $t = 0, 1$, that

$$\delta_j \phi_j^t + \delta'_j \phi_j^{1-t} = \phi_{j+1}^t Y_j,$$

understood up to unitary equivalence, then becomes the condition

$$r_j \gamma_j(l_j^t \mu_j + (k_j - l_j^t) \nu_j) + s_j \gamma_j(l_j^{1-t} \mu_j + (k_j - l_j^{1-t}) \nu_j)$$

$$= (l_j^{t+1} \mu_{j+1} + (k_{j+1} - l_j^{t+1}) \nu_{j+1}) \gamma_j,$$

also up to unitary equivalence. As $K_0(\mu_j)$ and $K_0(\nu_j)$ are independent this is equivalent to the two equations

$$r_j l_j^t + s_j l_j^{1-t} = \text{mult}(\gamma_j) l_j^{t+1},$$

$$(r_j + s_j) k_j = \text{mult}(\gamma_j) k_{j+1}.$$
Choose \( r_j = 2 \text{ mult}(\gamma_j) \) and \( s_j = \text{ mult}(\gamma_j) \), so that
\[
k_{j+1} = 3k_j
\]
and
\[
l_{j+1}^t = 2l_j^t + l_j^{1-t}
\]

Taking \( k_1 = 1, l_0^t = 0, \) and \( l_1^t = 1 \) we have \( k_j = 3^{j-1} \) and \( l_j^1 - l_j^0 = 1 \) for all \( j \) and, in particular, these quantities are non-zero, as required above.

Next let us show that, up to unitary equivalence preserving the equations
\[
\delta_j \phi_t^j + \delta_j^t \phi_j^{1-t} = \delta_j^{j+1} \gamma_j,
\]
\( \phi_j^t \beta_j \) is a direct summand of \( \delta_j^t = \text{ mult}(\gamma_j) \gamma_j \), and \( \phi_j^1 \beta_j \) is a direct summand of \( \gamma_j \).

Note that \( \delta_j^t = \text{ mult}(\gamma_j) \gamma_j \) contains a copy of \( \phi_j^t \beta_j \) for \( t = 0, 1 \). In particular \( \delta_j^t \) contains a copy of \( \phi_j^t \beta_j \) for \( t = 0, 1 \). In particular \( \delta_j^t \) contains a copy of \( \phi_j^t \beta_j \) for \( t = 0, 1 \). In particular \( \delta_j^t \) contains a copy of \( \phi_j^t \beta_j \) for \( t = 0, 1 \).

With this choice of \( q_j \), let us show that for each \( t = 0, 1 \) there exists a unitary \( u_t \in D_{j+1} \) commuting with the image of \( \phi_j^t \gamma_j \), such that \((\text{Ad} u_0) \phi_j^0 \beta_j \) is a direct summand of \( \delta_j^0 \) and \( \text{Ad} u_1 \phi_j^1 \beta_j \) is a direct summand of \( \delta_j^1 \). In other words, for each \( t = 0, 1 \) we must show that the partial isometry constructed in the preceding paragraph, producing a copy of \( \phi_j^t \beta_j \) inside either \( \delta_j^t \) or \( \delta_j^1 \), may be chosen in such a way that it extends to a unitary element of \( D_{j+1} \) — which in addition commutes with the image of \( \phi_j^t \gamma_j \).
We will consider the case \( t = 0 \). The case \( t = 1 \) is similar. Let us first show that the partial isometry in \( D_{j+1} \), transforming \( \phi^0_{j+1} \beta_j \) into a direct summand of \( \delta_j^0 \), may be chosen to lie in the commutant of the image of \( \phi^0_{j+1} \gamma_j \). Note first that the unit of the image of \( \phi^0_{j+1} \beta_j \) – the initial projection of the partial isometry – lies in the commutant of the image of \( \phi^0_{j+1} \gamma_j \). Indeed, this projection is the image by \( \phi^0_{j+1} \beta_j \) of the unit of \( C_j \). The property that \( \beta_j \phi^0_{j+1} \) is a direct summand of \( \gamma_j \) implies in particular that the image by \( \beta_j \phi^0_{j+1} \) of the unit of \( C_j \) commutes with the image of \( \gamma_j \). The image by \( \phi^0_{j+1} \beta_j \phi^0_{j+1} \) of the unit of \( C_j \) (i.e. the unit of the image of \( \phi^0_{j+1} \beta_j \)) therefore commutes with the image of \( \phi^0_{j+1} \gamma_j \), as asserted.

Note also that the final projection of the partial isometry commutes with the image of \( \phi^0_{j+1} \gamma_j \). Indeed, it is the unit of the image of a direct summand of \( \delta_j^0 \), and since \( D_j \) is unital it is the image of the unit of \( D_j \) by this direct summand; since \( C_j \) is unital and \( \phi^0_{j+1} : C_j \rightarrow D_j \) is unital, the projection in question is the image of the unit of \( C_j \) by a direct summand of \( \delta_j^0 \). But \( \delta_j^0 \phi^0_{j+1} \) is itself a direct summand of \( \phi^0_{j+1} \gamma_j \) (as \( \delta_j^0 \phi^0_{j+1} = \delta_j^0 \phi^0_{j+1} = \delta_j^0 \phi^0_{j+1} \)), and so the projection in question is the image of the unit of \( C_j \) by a direct summand of \( \phi^0_{j+1} \gamma_j \), and in particular commutes with the image of \( \phi^0_{j+1} \gamma_j \).

Note that both direct summands of \( \phi^0_{j+1} \gamma_j \) under consideration \( (\phi^0_{j+1} \beta_j \phi^0_{j+1} \) and a copy of it) factor through the evaluation of \( C_j \) at the point \( x_j \), and so are contained in the largest such direct summand of \( \phi^0_{j+1} \gamma_j \); this largest direct summand, say \( \pi_j \), is seen to exist by inspection of the construction of \( \phi^0_{j+1} \gamma_j \). Since both projections under consideration (the images of the unit of \( C_j \) by the two copies of \( \phi^0_{j+1} \beta_j \phi^0_{j+1} \)) are less than \( \pi_j \), to show that they are unitarily equivalent in the commutant of the image of \( \phi^0_{j+1} \gamma_j \) (in \( D_{j+1} \)) it is sufficient to show that they are unitarily equivalent in the commutant of the image of \( \pi_j \) in \( \pi_j(1)D_{j+1} \). Note that this image is isomorphic to \( M_{\dim \beta_j} (C_j) \). By construction, the two projections in question are Murray-von Neumann equivalent – in \( D_{j+1} \) and therefore in \( \pi_j(1)D_{j+1} \) – but all we shall use from this is that they have the same class in \( K^0 X_{j+1} \). Note that the dimension of these projections is \( (k_{j+1} \dim(p_{j+1}))(k_j \dim(p_j)) \), and that the dimension of \( \pi_j \) is \( k_j \dim(p_{j+1}) + l_{j+1} \dim(p_{j+1}) \). Since the two projections under consideration commute with \( \pi_j(C_j) \), and this is isomorphic to \( M_{\dim(p_j)} (C_j) \), to prove unitary equivalence in the commutant of \( \pi_j(C_j) \) in \( \pi_j(1)D_{j+1} \) it is sufficient to prove unitary equivalence of the product of these projections with a fixed minimal projection of \( \pi_j(C_j) \), say \( e \). Since \( K^0 X_{j+1} \) is torsion free, the products of the two projections under consideration with \( e \) still have the same class in \( K^0 X_{j+1} \). To prove that they are unitarily equivalent in \( e D_{j+1} e \) , it is sufficient (and necessary) to prove that both they and their complements inside \( e \) are Murray von-Neumann equi-
valent. Since both the cut-down projections and their complements inside $e$ have the same class in $K^0X_{j+1}$, to prove that the two pairs are equivalent it is sufficient, by Theorem 8.1.5 of [Hu], to show that all four projections have dimension at least $\frac{1}{2} \dim X_{j+1}$. Dividing the dimensions above by $\dim(p_j)$ (the order of the matrix algebra), we see that the dimension of the first pair of projections is $k_j+1k_j \dim(p_{j+1}) = k_j+1k_j \mult(\gamma_j) \dim(p_j)$. The dimension of $e$ is $k_j+1 \mult(\gamma_j) + l_j+1 \mult(\gamma_j) \dim(p_{j+1})$, so that the dimension of the second pair of projections is $\mult(\gamma_j)(k_j+1+l_j+1 \dim(p_{j+1}) - k_j+1k_j \dim(p_j))$. Since $\dim(p_j) \geq \frac{1}{2} \dim X_j$, $\dim(p_{j+1}) = \mult(\gamma_j) \dim(p_j)$, $\dim X_{j+1} = n_j \dim X_j$, and $\mult(\gamma_j) \geq n_j$ (for all $j$), we have $\dim(p_{j+1}) \geq \frac{1}{2} \dim X_{j+1}$ (for all $j$). Since $k_j+1k_j$ is non-zero for all $j$, the first inequality holds. Since $l_j+1$ is non-zero for all $j$, the second inequality holds if $\mult(\gamma_j)$ is strictly greater than $k_j+1k_j$. (One then has, using $\dim(p_{j+1}) = \mult(\gamma_j) \dim(p_j)$ twice, that the dimension of the second pair of projections is at least $\dim(p_{j+1})$.) Since $k_j+1k_j = 3k_j^2$, and $k_j$ was specified before $n_j$, we may modify the choice of $n_j$ so that $\mult(\gamma_j) - \gamma_j$ is greater than $n_j - \gamma_j$ is sufficiently large.

This shows that the two projections in $D_{j+1}$ under consideration are unitarily equivalent by a unitary in the commutant of the image of $\phi_{j+1}^0 \gamma_j$. Replacing $\phi_{j+1}^0$ by its composition with the corresponding inner automorphism, we may suppose that the two projections in question are equal. In other words $\phi_{j+1}^0 \beta_j$ is unitarily equivalent to the cut-down of $\delta_j'$ by the projection $\phi_{j+1}^0 \beta_j(1)$.

Now consider the compositions of these two maps with $\phi_j^1$, namely $\phi_{j+1}^0 \beta_j \phi_j^1$ and the cut-down of $\delta_j' \phi_j^1$ by the projection $\phi_{j+1}^0 \beta_j(1)$. Since both of these maps can be viewed as the cut-down of $\phi_{j+1} \gamma_j$ by the same projection, they are in fact the same map. Thus any unitary inside the cut-down of $D_{j+1}$ by $\phi_{j+1}^0 \beta_j(1)$ taking $\phi_{j+1}^0 \beta_j$ into the cut-down of $\delta_j'$ by this projection (such a unitary is known to exist) must commute with the image of $\phi_{j+1}^0 \beta_j \phi_j^1$ and hence with the image of $\phi_{j+1}^0 \gamma_j$, since this commutes with the projection $\phi_{j+1}^0 \beta_j(1) = \phi_{j+1}^0 (\beta_j \phi_j^1(1))$. The extension of such a partial unitary to a unitary $u_0$ in $D_{j+1}$ equal to one inside the complement of this projection then belongs to the commutant of the image of $\phi_{j+1}^0 \gamma_j$, and transforms $\phi_{j+1}^0 \beta_j$ into the cut-down of $\delta_j'$ by this projection, as desired.

As stated above, the proof for the case $t = 1$ is similar.

Inspection of the construction of the maps $\delta_j' - \phi_j^0 \beta_j$ and $\delta_j - \phi_j^1 \beta_j$ shows that they are injective, as required by the hypotheses of section 2.

Replacing $\phi_{j+1}^0$ with $(\Ad u_t) \phi_{j+1}^0$, we have an inductive sequence

$$A_1 \overset{\theta_1}{\rightarrow} A_2 \overset{\theta_2}{\rightarrow} \cdots$$
satisfying the hypotheses of section 2. (The existence of $\alpha_j$ homotopic to $\beta_j$ and non-zero on a given element of $D_j$, defined by another point evaluation, is clear.)

By Theorem 2.3 there exists a sequence

$$A_1 \xrightarrow{\theta'_1} A_2 \xrightarrow{\theta'_2} \cdots,$$

with $\theta'_j$ homotopic to $\theta_j$ (and so agreeing with $\theta_j$ on $K_0$), the inductive limit of which is simple.

Since the map $K_0(\theta'_j)$ (considered as a map between single copies of the integers) takes the canonical generator $1 \in \mathbb{Z}$ to $L_j + 1$, we may conclude that the simple inductive limit in question has the desired $K_0$-group. That the positive elements are all those greater than $k$ follows from the fact that at each stage, $l_j + 1$ is the smallest positive element in $K_0A_j = \mathbb{Z}$ and

$$\lim \frac{l_j + 1}{\prod_{k=1}^j L_k} = \lim \frac{a \prod_{k=2}^j L_j + 1}{b \prod_{k=2}^j L_j} = k + \lim \frac{1}{\prod_{k=1}^j L_k} = k.$$

Theorem 3.1 follows.

Finally, one might reasonably ask whether $K_0(A_{(n,k)})^+$ can be made to contain $k$. There is no reason a priori why this should not be possible, but the construction above does not seem amenable to modifications which would achieve this result. Roughly speaking, the $K_0$-group in Theorem 3.1 can be thought of as an inductive limit of sub-ordered groups of ordered $K_0$-groups of homogeneous $C^*$-algebras. In order that the inductive limit of Theorem 3.1 be simple, one must introduce point evaluations via the maps $\beta_j$. In the absence of these point evaluations, one could have maps $\Psi : \mathbb{Z}_{mk} \to \mathbb{Z}_{mnk}$ with $\Psi(nk) = m nk$ at the level of $K_0$ between the building blocks $A_i$ and $A_{i+1}$. With these point evaluations, however, one is forced into a situation where $\Psi(nk)$ is necessarily strictly less than $mnk$.

Acknowledgement. This work was supported by an NSERC Postdoctoral Fellowship.

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