ON THE AUTOMORPHISM GROUP OF CERTAIN
SIMPLE C*-ALGEBRAS

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Abstract

We show that the information contained in \( KL(A, B) \) is determined by other invariants when \( A \) and \( B \) are certain simple unital projectionless \( C^* \)-algebras. This allows us to compute the group of automorphisms modulo the group of approximately inner automorphisms in terms of the Elliott invariant.

1. Introduction

For a unital \( C^* \)-algebra \( A \) the Elliott invariant \( E_A \) consists of the ordered group \( K_0(A) \) with order unit, the group \( K_1(A) \), the compact convex set \( T(A) \) of tracial states, and the restriction map \( r_A : T(A) \to SK_0(A) \), where \( SK_0(A) \) denotes the state space of \( K_0(A) \). In [3] it was proved that the Elliott invariant is a classifying invariant for the class of unital simple infinite dimensional inductive limits of sequences of finite direct sums of building blocks. A building block is a \( C^* \)-algebra of the form

\[
A(n, d_1, d_2, \ldots, d_N) = \{ f \in C(T) \otimes M_n : f(x_i) \in M_{d_i}, \ i = 1, 2, \ldots, N \},
\]

where \( x_1, x_2, \ldots, x_N \) are (different) points in \( T \), and \( d_1, d_2, \ldots, d_N \) are integers dividing \( n \). The points \( x_1, x_2, \ldots, x_N \) will be called the exceptional points of \( A \). By allowing \( d_i = n \) we may always assume that \( N \geq 2 \).

The following calculation of the group of automorphisms modulo the group of approximately inner automorphisms is the main result of this paper.

**Theorem 1.1.** Let \( A \) be a simple unital inductive limit of a sequence of finite direct sums of building blocks.

(i) If \( K_0(A) \) is non-cyclic then \( \text{Aut}(A)/\text{Inn}(A) \) is isomorphic to the semidirect product

\[
\{ \text{Hom}(K_1(A), \text{Aff} T(A)/\rho_A(K_0(A))) \times \text{ext}(K_1(A), K_0(A)) \} \rtimes \text{Aut}(E_A),
\]

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where the action of \((\varphi_0, \varphi_1, \varphi_T) \in \text{Aut}(\mathcal{E}_A)\) is given by
\[(\eta, e) \mapsto (\tilde{\varphi}_T^{-1} \circ \eta \circ \varphi_1^{-1}, \varphi_0 \circ \varphi_1^{-1}(e))\]
for \(\eta \in \text{Hom}(K_1(A), \text{Aff}(T(A)/\rho_A(K_0(A)))), e \in \text{ext}(K_1(A), K_0(A)).\)

(ii) If \(K_0(A) \cong \mathbb{Z}\) then \(\text{Aut}(A)/\text{Inn}(A)\) is isomorphic to the semi-direct product
\[
\text{Hom}(K_1(A), \text{Aff}(T(A)/\rho_A(K_0(A)))) \rtimes \text{Aut}(\mathcal{E}_A),
\]
where the action of \((\varphi_0, \varphi_1, \varphi_T) \in \text{Aut}(\mathcal{E}_A)\) is given by
\[
\eta \mapsto \tilde{\varphi}_T^{-1} \circ \eta \circ \varphi_1^{-1}, \quad \eta \in \text{Hom}(K_1(A), \text{Aff}(T(A)/\rho_A(K_0(A)))),
\]

Here \(\text{Aut}(\mathcal{E}_A)\) denotes the group of automorphisms of \(\mathcal{E}_A\), i.e. the group of triples \((\varphi_0, \varphi_1, \varphi_T)\) where \(\varphi_0\) is an automorphism of the ordered group \(K_0(A)\) with order unit, \(\varphi_1\) is an automorphism of \(K_1(A)\), and \(\varphi_T\) is an affine homeomorphism of \(T(A)\) such that
\[
r_A \circ \varphi_T^{-1}(\omega) = r_A(\omega) \circ \varphi_0 \quad \text{on} \quad K_0(A)
\]
for every \(\omega \in T(A)\). \(\text{Aff}(T(A)/\rho_A(K_0(A)))\) denotes the continuous real-valued affine functions on \(T(A)\) and \(\rho_A : K_0(A) \to \text{Aff}(T(A))\) is the group homomorphism
\[
\rho_A(x)(\omega) = r_A(\omega)(x), \quad x \in K_0(A), \omega \in T(A).
\]

It follows from [3, Theorem 12.2] that the algebras considered under (i) are exactly the class considered by Thomsen in [9]. Therefore part (i) of the above theorem follows from Thomsen’s calculation, [9, Theorem 8.4]. Note that the term \(\text{ext}(K_1(A), K_0(A))\) is not present in case (ii). This is not because it is zero. As we shall demonstrate, it is the existence of a natural map
\[
KL(A, B)_\kappa \to \text{Hom}(\text{Tor}(U(A)/DU(A)), \text{Tor}(U(B)/DU(B))),
\]
when \(A\) and \(B\) are simple unital inductive limits of finite direct sums of building blocks with \(K_0(A) \cong K_0(B) \cong \mathbb{Z}\), which is responsible for this. \(KL(A, B)_\kappa\) denotes the subset of elements \(\kappa\) in the group \(KL(A, B)\) defined by Rørdam in [5] for which the induced map \(\kappa_\kappa : K_0(A) \to K_0(B)\) preserves the order unit, and \(U(A)/DU(A)\) is the group of unitaries \(U(A)\) in \(A\) modulo the closure of the commutator subgroup \(DU(A)\). The map above was also crucial in the proof of the classification result in [3].

It is an interesting question whether a similar map exists in greater generality – including the case where \(A\) and \(B\) are arbitrary inductive limits of subhomogeneous \(C^*\)-algebras. Such a map would probably be useful in all efforts
of classifying larger classes of simple $C^*$-algebras, and our main result suggests that it could influence the structure of the automorphism group as well.

It should be noted that the class of $C^*$-algebras considered under (ii) is quite large. It consists of matrix algebras over simple unital projectionless $C^*$-algebras, see [3, Corollary 12.5].

2. Preliminaries

The purpose of this section is to introduce the notation used in this paper and to list some results on building blocks from [3] that we will need.

Let $A$ be a unital $C^*$-algebra. Let $s(A)$ be the smallest positive integer $n$ for which there exists a unital *-homomorphism $A \to M_n$ (we set $s(A) = \infty$ if $A$ has no non-trivial finite dimensional representations). Note that if there exists a unital *-homomorphism $A \to B$ then $s(A) \leq s(B)$.

Let $A = A(n, d_1, d_2, \ldots, d_N)$ be a building block and let $x_1, x_2, \ldots, x_N$ be the exceptional points. Evaluation at $x_i$ gives rise to a unital *-homomorphism $\nu_{\Lambda_i} : A \to M_{d_i}$. This map will sometimes be denoted $\nu_{\Lambda_i \psi_i}$. For every integer $k \geq 0$ we let $\nu_{\Lambda_i \psi_k}$ be the direct sum of $k$ copies of the representation $\nu_{\Lambda_i \psi_i}$.

Let $A = A(n, d_1, d_2, \ldots, d_N)$ and $B = A(m, e_1, e_2, \ldots, e_M)$ be building blocks. Let $\varphi : A \to B$ be a unital *-homomorphism. As in [9, Chapter 1] we define $s^\varphi(j, i)$ to be the multiplicity of the representation $\nu_{\Lambda_i \psi_j}$ in the representation $\nu_{\Lambda_i \psi_j} \circ \varphi$ for $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, M$.

Let us start with the $K$-theory of a building block.

**Proposition 2.1.** Let $A = A(n, d_1, d_2, \ldots, d_N)$ be a building block and let $d = \gcd(d_1, d_2, \ldots, d_N)$. We have an isomorphism of ordered groups with order units

\[
(K_0(A), K_0(A)^+, [1]) \cong (\mathbb{Z}, \mathbb{Z}^+, d).
\]

**Proof.** This is [3, Corollary 3.6].

Let $A = A(n, d_1, d_2, \ldots, d_N)$ be a building block with exceptional points $e^{2\pi int_k}, k = 1, 2, \ldots, N$, where $0 < t_1 < t_2 < \ldots < t_N < 1$. Set $t_{N+1} = t_1 + 1$ and $t_0 = t_N$. Define continuous functions $\omega_k : \mathbb{T} \to \mathbb{T}$ for $k = 1, 2, \ldots, N$, by

\[
\omega_k(e^{2\pi i t}) = \begin{cases}
\exp \left(2\pi i \frac{t - t_k}{t_{k+1} - t_k} \right) & t_k \leq t \leq t_{k+1}, \\
1 & t_{k+1} \leq t \leq t_k + 1.
\end{cases}
\]

Let $U_k^A$ be the unitary in $A$ defined by

\[
U_k^A(z) = \text{diag}(\omega_k(z), 1, 1, \ldots, 1), \quad z \in \mathbb{T}.
\]
Set $U_0^A = U_N^A$.

**Theorem 2.2.** Let $A = A(n, d_1, d_2, \ldots, d_N)$ be a building block. Set for $k = 1, 2, \ldots, N - 1$,

$$s_k = \text{lcm} \left( \frac{n}{d_1}, \frac{n}{d_2}, \ldots, \frac{n}{d_k} \right),$$

and

$$r_k = \text{gcd} \left( s_k, \frac{n}{d_{k+1}} \right) = \text{gcd} \left( \text{lcm} \left( \frac{n}{d_1}, \frac{n}{d_2}, \ldots, \frac{n}{d_k} \right), \frac{n}{d_{k+1}} \right).$$

Choose integers $\alpha_k$ and $\beta_k$ such that

$$r_k = \alpha_k s_k + \beta_k \frac{n}{d_{k+1}}, \quad k = 1, 2, \ldots, N - 1.$$

Then

$$K_1(A) \cong \mathbb{Z} \oplus \mathbb{Z}_{r_1} \oplus \mathbb{Z}_{r_2} \oplus \cdots \oplus \mathbb{Z}_{r_{N-1}}.$$

This isomorphism can be chosen such that for $k = 1, 2, \ldots, N - 1$, a generator of the direct summand $\mathbb{Z}_{r_k}$ is mapped to

$$[U_k^A] - \frac{\beta_k n}{r_k d_{k+1}} [U_{k+1}^A] - \frac{\alpha_k s_k}{r_k} [U_N^A],$$

and such that a generator of the direct summand $\mathbb{Z}$ is mapped to $[U_N^A]$.

**Proof.** See [3, Theorem 3.2].

Let $A$ and $B$ be unital $C^*$-algebras. A unital $*$-homomorphism $\varphi : A \to B$ induces morphisms $\varphi_\ast : K_0(A) \to K_0(B)$, $\varphi_+ : K_1(A) \to K_1(B)$, $\varphi^* : \text{T}(B) \to \text{T}(A)$, $\tilde{\varphi} : \text{Aff} T(A) \to \text{Aff} T(B)$, $\tilde{\varphi^*} : \text{Aff} T(A)/\rho_A(K_0(A)) \to \text{Aff} T(B)/\rho_B(K_0(B))$, and $\varphi^w : U(A)/DU(A) \to U(B)/DU(B)$. Let $\lambda'_A : U(A) \to U(A)/DU(A)$ and $\varphi_A : \text{Aff} T(A) \to \text{Aff} T(A)/\rho_A(K_0(A))$ be the canonical maps.

**Proposition 2.3.** Let $A$ be a unital inductive limit of a sequence of finite direct sums of building blocks. There exists a group homomorphism

$$\lambda_A : \text{Aff} T(A)/\rho_A(K_0(A)) \to U(A)/DU(A),$$

$$\lambda_A(q_A(\bar{a})) = q'_A(e^{2\pi i a}), \quad a \in A_{sa}.$$ 

Let $\pi_A : U(A)/DU(A) \to K_1(A)$ be the map induced by the canonical map $U(A) \to K_1(A)$. We have a short exact sequence of abelian groups

$$0 \to \text{Aff} T(A)/\rho_A(K_0(A)) \to \text{Aff} T(A)/DU(A) \to U(A)/DU(A) \to K_1(A) \to 0.$$
This sequence is natural in $A$ and splits unnaturally.

**Proof.** See [3, Proposition 5.2].

**Proposition 2.4.** Let $A = A(n, d_1, d_2, \ldots, d_N)$ be a building block. Let $u \in A$ be a unitary. Assume that

$$\text{Det}(u(z)) = 1, \quad z \in T,$$

$$\text{Det}(\Lambda_i(u)) = 1, \quad i = 1, 2, \ldots, N.$$

Then $u \in DU(A)$.

**Proof.** See [3, Proposition 5.3].

**Lemma 2.5.** Let $A = A(n, d_1, d_2, \ldots, d_N)$ and adopt the notation of Theorem 2.2. For $k = 1, 2, \ldots, N - 1$, there exists a unitary $v_k^A \in A$ such that

$$\text{Det}(v_k^A(z)) = 1, \quad z \in T,$$

and

$$\text{Det}(\Lambda_j(v_k^A)) = \begin{cases} 
\exp\left(2\pi i \frac{\alpha_j s_k}{r_k} \frac{d_j}{n}\right) & j = 1, 2, \ldots, k, \\
\exp\left(-2\pi i \frac{\beta_k}{r_k}\right) & j = k + 1, \\
1 & j = k + 2, k + 3, \ldots, N.
\end{cases}$$

Furthermore, $[v_k^A]$ has order $r_k$ in $K_1(A)$, and $[v_1^A], [v_2^A], \ldots, [v_{N-1}^A]$ generate the torsion subgroup of $K_1(A)$. There is a group homomorphism $\sigma_A : \text{Tor}(K_1(A)) \to \text{Tor}((U(A)/DU(A))$ given by $\sigma_A([v_k^A]) = q_A'(v_k^A)$, $k = 1, 2, \ldots, N - 1$.

**Proof.** The existence of $v_k^A$ follows from [3, Lemma 5.4].

Fix $k = 1, 2, \ldots, N$. By [3, Lemma 5.4] there is a unitary $u \in A$ with $\text{Det}(u(z)) = 1, z \in T$, and

$$\text{Det}(\Lambda_j(u)) = \begin{cases} 
1 & j \neq k, \\
\exp\left(2\pi i \frac{d_k}{n}\right) & j = k.
\end{cases}$$

Set $w = u U_{k-1}^A U_k^{A*}$. By Proposition 2.4 we have that $w$ modulo $DU(A)$
equals the unitary \( z \mapsto e^{2\pi i \lambda(z)} \), where \( \lambda : T \to \mathbb{R} \) is the continuous function

\[
\lambda(e^{2\pi i t}) = \begin{cases} 
\frac{1}{n} \frac{t - t_{k-1}}{t_k - t_{k-1}} & t_{k-1} \leq t \leq t_k, \\
\frac{1}{n} \frac{t_{k+1} - t}{t_{k+1} - t_k} & t_k \leq t \leq t_{k+1}, \\
0 & \text{otherwise.}
\end{cases}
\]

In particular, \( w \) is trivial in \( K_1(A) \), i.e., \([u] = [U_k^A] - [U_{k-1}^A] \) in \( K_1(A) \).

As a consequence of this observation,

\[
[v_k^A] = \sum_{j=1}^{k} \frac{\alpha_k s_k}{r_k} ([U_j^A] - [U_{j-1}^A]) - \frac{\beta_k n}{r_k d_{k+1}} ([U_{k+1}^A] - [U_k^A])
\]

in \( K_1(A) \). Hence by Theorem 2.2 we see that \([v_k^A] \) has order \( r_k \) in \( K_1(A) \) and that the elements \([v_1^A], [v_2^A], \ldots, [v_{N-1}^A] \) generate \( \text{Tor}(K_1(A)) \). Since \( r_k q'_A(v_k^A) = 0 \) and \( \pi_A(q'_A(v_k^A)) = [v_k^A] \) it follows that \( q'_A(v_k^A) \) has order \( r_k \) in \( U(A)/DU(A) \). The existence of \( \sigma_A \) follows.

We remark that the map \( \sigma_A \) is neither natural nor unique, and that \( \pi_A \circ \sigma_A \) is the identity map on \( \text{Tor}(K_1(A)) \).

In [5] Rørdam defined the bifunctor \( KL \) to be a certain quotient of \( KK \).

Recall from [5] that the Kasparov product yields a product \( KL(A, B) \times KL(A, C) \to KL(A, C) \) which we will denote by \( \cdot \). Furthermore, if \( K_*(A) \) is finitely generated then \( KK(A, \cdot) \cong KL(A, \cdot) \). If \( \varphi \) is a unital \(*\)-homomorphism, we let \([\varphi]\) denote the induced element in \( KL(A, B) \). For unital \( C^*\) algebras \( A \) and \( B \) we let \( KL(A, B)_e \) denote the elements of \( KL(A, B) \) for which the induced map \( K_0(A) \to K_0(B) \) preserves the order unit.

Let \( A \) and \( B \) be building blocks. \( KL(A, B) \) is conveniently described in terms of the \( K \)-homology groups \( K^0(A) = KK(A, \mathbb{C}) \cong KL(A, C) \) and \( K^0(B) \). Recall that the Kasparov product gives rise to a group homomorphism \( \kappa^* : K^0(B) \to K^0(A) \) for every \( \kappa \in KL(A, B) \).

**Theorem 2.6.** Let \( A = A(n, d_1, d_2, \ldots, d_N) \) and \( B = A(m, e_1, e_2, \ldots, e_M) \) be building blocks such that \( s(B) \geq Nn \).

(i) If \( v \in KL(A, B)_e \) then there exists a unital \(*\)-homomorphism \( \varphi : A \to B \) such that \([\varphi] = v \) in \( KL(A, B) \).
(ii) Let \( \phi : A \to B \) be a unital \(*\)-homomorphism and let \( \kappa \in KL(A, B) \).
If \( \phi^* = \kappa^* \) on \( K^0(B) \) and \( \phi_*([U^A_N]) = \kappa_*([U^B_N]) \) in \( K_1(B) \) then \( [\phi] = [\kappa] \) in \( KL(A, B) \).

**Proof.** This follows from [3, Theorem 4.7].

The next result says that \( K^0(\cdot) \) and the torsion subgroup of \( U(\cdot)/DU(\cdot) \) are related for building blocks.

**Proposition 2.7.** Let \( A = A(n, d_1, d_2, \ldots, d_N) \) be a building block. There exists a finite set \( F \subseteq Tor(U(A)/DU(A)) \) such that if \( B \) is a building block and \( \phi, \psi : A \to B \) are unital \(*\)-homomorphisms with \( \phi^*(x) = \psi^*(x) \), \( x \in F \), then \( \phi^* = \psi^* \) on \( K^0(B) \).

**Proof.** This is part of [3, Theorem 5.5].

We also need a description of the structure of the group \( K^0(\cdot) \) for a building block.

**Proposition 2.8.** Let \( A = A(n, d_1, d_2, \ldots, d_N) \) be a building block. Then \( K^0(A) \) is generated by \( [\Lambda^A_1], [\Lambda^A_2], \ldots, [\Lambda^A_N] \). Furthermore, for \( a_1, a_2, \ldots, a_N \in \mathbb{Z} \) we have that
\[
a_1[\Lambda^A_1] + a_2[\Lambda^A_2] + \cdots + a_N[\Lambda^A_N] = 0
\]
if and only if there exist \( b_1, b_2, \ldots, b_N \in \mathbb{Z} \) such that \( \sum_{i=1}^N b_i = 0 \) and
\[
a_i = b_i \frac{n}{d_i}, \quad i = 1, 2, \ldots, N.
\]

**Proof.** This is [3, Proposition 4.2].

We conclude with a technical proposition which is needed in the next section.

**Proposition 2.9.** Let \( A = A(n, d_1, d_2, \ldots, d_N) \) and \( B = A(m, e_1, e_2, \ldots, e_M) \) be building blocks with \( s(B) \geq Nn \). Let \( \chi \in K_1(B) \) and let \( h : K^0(B) \to K^0(A) \) be a homomorphism of the form
\[
\begin{pmatrix}
  h([\Lambda^B_1]) \\
  h([\Lambda^B_2]) \\
  \vdots \\
  h([\Lambda^B_M])
\end{pmatrix} = \begin{pmatrix}
  h_{11} & h_{12} & \cdots & h_{1N} \\
  h_{21} & h_{22} & \cdots & h_{2N} \\
  \vdots & \vdots & \ddots & \vdots \\
  h_{M1} & h_{M2} & \cdots & h_{MN}
\end{pmatrix} \begin{pmatrix}
  [\Lambda^A_1] \\
  [\Lambda^A_2] \\
  \vdots \\
  [\Lambda^A_M]
\end{pmatrix}
\]
with \( \sum_{j=1}^M h_{ji}d_i = e_j \) for \( j = 1, 2, \ldots, M \). There exists a unital \(*\)-homomorphism \( \phi : A \to B \) such that \( \phi^* = h \) on \( K^0(B) \) and \( \phi_*([U^A_N]) = \chi \) in \( K_1(B) \).
Proof. By Proposition 2.8 we have that $\frac{n}{d}[\Lambda^L_i] = \frac{n}{d}[\Lambda^L_N]$ in $K^0(A)$. Hence we may assume that $0 \leq h_{ji} < \frac{n}{d}$ for $i \neq N$ and still have that $\sum_{i=1}^{N} h_{ji}d_i = e_j$ for $j = 1, 2, \ldots, M$. Note that for $j = 1, 2, \ldots, M$, 

$$Nn \leq \sum_{i=1}^{N} h_{ji}d_i < (N - 1)n + h_{jN}d_N$$

and hence $h_{jN} > \frac{n}{d}$. The conclusion follows from [3, Proposition 4.4].

3. KL and other invariants

Let $A$ and $B$ be unital C*-algebras and let $\varphi_0 : K_0(A) \to K_0(B)$ be an order unit preserving group homomorphism. Assume that $K_0(A) \cong \mathbb{Z}$. Then $\rho_A(K_0(A))$ is closed in $\text{Aff } T(A)$, and we have a well-defined map

$$\tilde{\varphi}_0 : \text{Tor}(\text{Aff } T(A)/\rho_A(K_0(A))) \to \text{Tor}(\text{Aff } T(B)/\rho_B(K_0(B)))$$

given by $\tilde{\varphi}_0(q_A(\frac{1}{k}\rho_A(x))) = q_B(\frac{1}{k}\rho_B(\varphi_0(x)))$ for $x \in K_0(A)$ and every positive integer $k$.

Theorem 3.1. Let $A = A(n, d_1, d_2, \ldots, d_N)$ and $B = A(m, e_1, e_2, \ldots, e_M)$ be building blocks with $s(B) \geq Nn$. If $\varphi_0 : K_0(A) \to K_0(B)$ is an order unit preserving group homomorphism, if $\varphi : \text{Tor}(U(A)/DU(A)) \to \text{Tor}(U(B)/DU(B))$ is a group homomorphism such that the diagram

$$\begin{array}{ccc}
\text{Tor}(\text{Aff } T(A)/\rho_A(K_0(A))) & \xrightarrow{\lambda_A} & \text{Tor}(U(A)/DU(A)) \\
\downarrow{\varphi_0} & & \downarrow{\varphi} \\
\text{Tor}(\text{Aff } T(B)/\rho_B(K_0(B))) & \xrightarrow{\lambda_B} & \text{Tor}(U(B)/DU(B))
\end{array}$$

commutes, and if $\chi \in K_1(B)$, then there is a unital *-homomorphism $\psi : A \to B$ such that $\psi^* = \varphi$ on $\text{Tor}(U(A)/DU(A))$ and $\psi[A^L_N] = \chi$ in $K_1(B)$.

Proof. We adopt the notation of Theorem 2.2. Set $\alpha_0 = 1$. If $i = 1$ and $k$ are integers, $1 \leq i \leq k \leq N$, we define an integer $c_i^k$ by

$$c_i^k = \alpha_{i-1}\beta_i\beta_{i+1}\ldots\beta_{k-1}.$$ 

We claim that

$$\frac{1}{s_k} = \sum_{i=1}^{k} \frac{c_i^k d_i}{n}, \quad k = 1, 2, \ldots, N.$$
As \( c^1_i = 1 \), this is clear for \( k = 1 \). Assume it is correct for \( k, 1 \leq k \leq N - 1 \).

Then

\[
\sum_{i=1}^{k+1} c_i^{k+1} \frac{d_i}{n} = \frac{c_i^{k+1} d_{k+1}}{n} + \sum_{i=1}^{k} c_i^{k+1} \frac{d_i}{n} = \frac{c_i^{k+1} d_{k+1}}{n} + \beta_k \sum_{i=1}^{k} c_i^k \frac{d_i}{n}
\]

\[
= \alpha_k \frac{d_{k+1}}{n} + \frac{\beta_k \frac{1}{s_k}}{n} = \frac{d_{k+1}}{n} \left( \alpha_k \frac{1}{s_k} + \beta_k \frac{n}{d_{k+1}} \right)
\]

\[
= \frac{r_k}{s_k} \frac{n}{d_{k+1}} = \frac{1}{s_{k+1}},
\]

proving (1).

Choose a unitary \( u_k \in B \) such that \( \Phi(q_k^j(u_k^A)) = q_k^j(u_k^B), k = 1, 2, \ldots, N - 1 \). Let \( q_k^j \in \mathbb{R} \) be numbers such that

\[
\det(\Lambda_j(u_k)) = e^{2\pi i q_k^j}, \quad k = 1, 2, \ldots, N - 1, \quad j = 1, 2, \ldots, M.
\]

Set \( q_0^j = 0 \) and set \( d = \gcd(d_1, d_2, \ldots, d_N) \). By Proposition 2.1 \( d \) divides \( e_j \), \( j = 1, 2, \ldots, M \). Define for \( i = 1, 2, \ldots, N \), \( j = 1, 2, \ldots, M \),

\[
h_{ji} = c_i^N \frac{e_j}{d} - \frac{n}{d_i} q_{i-1}^j + \sum_{l=1}^{N-i} c_i^{N-l} s_{N-l}^i q_{N-l}^j.
\]

Since \( r_k q_k^j(u_k^A) = 0 \) in \( U(A)/DU(A) \) by Lemma 2.5, we see that \( u_k^A \in DU(B) \) and hence \( r_k q_k^j \in \mathbb{Z} \), \( k = 1, 2, \ldots, N - 1 \). It follows that \( h_{ji} \in \mathbb{Z} \) for every \( i \) and \( j \). Since \( q_k^j(u_k) \) has finite order, \( \det(u_k(\cdot)) \) is constantly equal to \( e^{2\pi i a_k} \) for some \( a_k \in \mathbb{R} \). Note that

\[
e^{2\pi i a_k} = e^{2\pi i q_k^j \frac{n}{d_i}}, \quad j = 1, 2, \ldots, M, \quad k = 1, 2, \ldots, N - 1.
\]

Thus if we set \( a_0 = 0 \) we find that

\[
\frac{m}{e_j} h_{ji} = c_i^N \frac{m}{d} - \frac{n}{d_i} q_{i-1}^j \frac{m}{e_j} + \sum_{l=1}^{N-i} c_i^{N-l} s_{N-l}^i q_{N-l}^j \frac{m}{e_j}
\]

\[
\equiv c_i^N \frac{m}{d} - \frac{n}{d_i} a_{i-1} + \sum_{l=1}^{N-i} c_i^{N-l} s_{N-l}^i a_{N-l} \mod \frac{n}{d_i}
\]

for \( i = 1, 2, \ldots, N \), \( j = 1, 2, \ldots, M \). Hence

(2) \[ \frac{m}{e_j} h_{ji} \equiv \frac{m}{e_M} h_{M_i} \mod \frac{n}{d_i}. \]
For $k = 1, 2, \ldots, N$,

$$
\sum_{i=1}^{k} c_{i}^{N-l} s_{N-l}^{-1} q_{N-l}^{i} \frac{d_{i}}{n} = \sum_{l=1}^{N-1 \min(k, N-l)} c_{i}^{N-l} s_{N-l}^{-1} q_{N-l}^{i} \frac{d_{i}}{n}
$$

$$
= \sum_{l=1}^{N-1} c_{i}^{N-l} s_{N-l}^{-1} q_{N-l}^{i} \frac{d_{i}}{n} + \sum_{l=N-k+1}^{N-1} c_{i}^{N-l} s_{N-l}^{-1} q_{N-l}^{i} \frac{d_{i}}{n}
$$

$$
= \sum_{l=1}^{N-k} \beta_{k}^{\beta_{k+1}} \cdots \beta_{N-1}^{\beta_{l}} \sum_{i=1}^{k} c_{i}^{l} s_{N-l}^{-1} q_{N-l}^{i} + \sum_{l=N-k+1}^{N-1} q_{N-l}^{i}
$$

Hence

$$
\sum_{i=1}^{k} h_{ji} \frac{d_{i}}{n} = \sum_{i=1}^{k} c_{i}^{N-l} \frac{d_{i}}{n} - \sum_{i=1}^{k} q_{N-l}^{i} + \sum_{l=1}^{N-1} c_{i}^{N-l} s_{N-l}^{-1} q_{N-l}^{i} \frac{d_{i}}{n}
$$

$$
= \frac{e_{j}}{d} \beta_{k}^{\beta_{k+1}} \cdots \beta_{N-1}^{\beta_{l}} \sum_{i=1}^{k} c_{i}^{l} \frac{d_{i}}{n} + \sum_{l=1}^{N-k} \beta_{k}^{\beta_{k+1}} \cdots \beta_{N-1}^{\beta_{l}} \frac{1}{s_{k}} s_{N-l}^{-1} q_{N-l}^{i}
$$

$$
= \frac{e_{j}}{d} \beta_{k}^{\beta_{k+1}} \cdots \beta_{N-1}^{\beta_{l}} \frac{1}{s_{k}} + \sum_{l=1}^{N-k} \beta_{k}^{\beta_{k+1}} \cdots \beta_{N-1}^{\beta_{l}} \frac{1}{s_{k}} s_{N-l}^{-1} q_{N-l}^{i}
$$

By setting $k = N$ we see that

$$
\sum_{i=1}^{N} h_{ji} d_{i} = \frac{e_{j}}{d} n \frac{1}{s_{N}} = e_{j}.
$$

Combining this equation with (2) and Proposition 2.8 it is an elementary exercise to prove that we can define a homomorphism $h : K^{0}(B) \to K^{0}(A)$ by

$$
\begin{pmatrix}
 h((\Lambda_{1}^{B})) \\
 h((\Lambda_{2}^{B})) \\
 \vdots \\
 h((\Lambda_{M}^{B}))
\end{pmatrix} =
\begin{pmatrix}
 h_{11} & h_{12} & \cdots & h_{1N} \\
 h_{21} & h_{22} & \cdots & h_{2N} \\
 \vdots & \vdots & \ddots & \vdots \\
 h_{M1} & h_{M2} & \cdots & h_{MN}
\end{pmatrix}
\begin{pmatrix}
 [\Lambda_{1}^{A}] \\
 [\Lambda_{2}^{A}] \\
 \vdots \\
 [\Lambda_{M}^{A}]
\end{pmatrix}.
$$
By Proposition 2.9 there exists a unital *-homomorphism \( \psi : A \to B \) such that \( \psi^* = h \) on \( K^0(B) \) and \( \psi_*([U_N^A]) = \chi \) on \( K_1(B) \). Fix \( j = 1, 2, \ldots, M \). Let \( t_i = s^\psi(j, i) \). By [3, Lemma 2.1] there exist a unitary \( w \in M_e \) and \( z_1, z_2, \ldots, z_L \in T \) such that

\[
\Lambda^B_j \circ \psi(f) = w \text{diag}(\Lambda^A_1(f), \Lambda^A_2(f), \ldots, \Lambda^A_N(f), f(z_1), f(z_2), \ldots, f(z_L))w^* \]

for \( f \in A \). Since point-evaluations are homotopic *-homomorphisms \( A \to M_n \), we see that

\[
\psi^*[\Lambda^B_j] = [\Lambda^B_j \circ \psi] = \sum_{i=1}^N t_i [\Lambda^A_i] + L \frac{n}{d_N} [\Lambda^A_N].
\]

in \( K^0(A) \). On the other hand, \( \psi^*[\Lambda^B_j] = \sum_{i=1}^N h_{ji} [\Lambda^A_i] \). It follows from Proposition 2.8 that

\[
s^\psi(j, i) \equiv h_{ji} \mod \frac{n}{d_i}, \quad i = 1, 2, \ldots, N, \quad j = 1, 2, \ldots, M.
\]

Note that for \( k = 1, 2, \ldots, N-1, j = 1, 2, \ldots, M \),

\[
\frac{\alpha_k s_k}{r_k} \sum_{i=1}^k h_{ji} \frac{d_i}{n} = \frac{e_j}{d} \frac{\alpha_k}{r_k} \beta_k \beta_{k+1} \ldots \beta_{N-1} + \sum_{i=1}^{N-k} \frac{\alpha_k}{r_k} \beta_k \beta_{k+1} \ldots \beta_{N-i-1} s_{N-i-l} q_{N-i}^j
\]

\[
= \frac{e_j}{d} \frac{\beta_k}{r_k} c_{k+1}^N + \sum_{i=1}^{N-k-1} \frac{\beta_k}{r_k} c_{k+1}^{N-i} q_{N-i}^j + \frac{\alpha_k}{r_k} s_k q_k^j
\]

\[
= \frac{\beta_k}{r_k} (h_{j(k+1)} + \frac{n}{d_k} q_k^j) + \frac{\alpha_k}{r_k} s_k q_k^j
\]

\[
= \frac{\beta_k}{r_k} h_{j(k+1)} + q_k^j.
\]

Since \( \text{Det}(v_k^A(z)) = 1, z \in T \), we see that

\[
\text{Det}(\Lambda_j(v_k^A)) = \prod_{i=1}^N \text{Det}(\Lambda_i(v_k^A))^{s^\psi(j, i)} = \prod_{i=1}^N \text{Det}(\Lambda_i(v_k^A))^{h(j, i)}
\]

\[
= \exp \left( 2\pi i \left( \sum_{i=1}^k \frac{\alpha_k s_k}{r_k} h_{ji} \frac{d_i}{n} - \frac{\beta_k}{r_k} h_{j(k+1)} \right) \right)
\]

\[
= \exp(2\pi i q_k^j) = \text{Det}(\Lambda_j(u_k)).
\]
Thus \( \text{Det}(\psi(v^A_k)) \) and \( \text{Det}(u_k(\cdot)) \) agree at the exceptional points of \( B \), and hence they agree everywhere. It follows from Proposition 2.4 that

\[
q'_B(\psi(v^A_k)) = q'_B(u_k) = \Phi(q'_A(v^A_k)), \quad k = 1, 2, \ldots, N - 1.
\]

As \( \tilde{\psi} = \tilde{\varphi}_0 \) on \( \text{Tor}\left(\text{Aff} T(A)/\rho_A(K_0(A))\right) \), we conclude from Lemma 2.5 and Proposition 2.3 that \( \psi^\# \) and \( \Phi \) agree on all of \( \text{Tor}(U(A)/DU(A)) \).

Our next result says that the information contained in \( KL(A, B) \) can be detected by other invariants when \( A \) and \( B \) are building blocks.

**Proposition 3.2.** Let \( A = A(n, d_1, d_2, \ldots, d_N) \) and \( B \) be building blocks with \( s(B) \geq Nn \). Let \( \varphi_0 : K_0(A) \to K_0(B) \) be an order unit preserving group homomorphism, and let \( \Phi : \text{Tor}(U(A)/DU(A)) \to \text{Tor}(U(B)/DU(B)) \) and \( \varphi_1 : K_1(A) \to K_1(B) \) be group homomorphisms such that the diagram

\[
\begin{array}{ccc}
\text{Tor}(\text{Aff} T(A)/\rho_A(K_0(A))) & \xrightarrow{\lambda_A} & \text{Tor}(U(A)/DU(A)) \\
\downarrow \tilde{\varphi}_0 & & \downarrow \Phi \\
\text{Tor}(\text{Aff} T(B)/\rho_B(K_0(B))) & \xrightarrow{\lambda_B} & \text{Tor}(U(B)/DU(B))
\end{array}
\]

commutes.

(i) There exists a unital *-homomorphism \( \varphi : A \to B \) such that \( \varphi^\# = \varphi_0 \) on \( K_0(A) \), \( \varphi^* = \varphi_1 \) on \( K_1(A) \) and \( \varphi^\# = \Phi \) on \( \text{Tor}(U(A)/DU(A)) \).

(ii) If \( \psi : A \to B \) is another unital *-homomorphism such that \( \psi^\# = \varphi_0 \) on \( K_0(A) \), \( \psi^* = \varphi_1 \) on \( K_1(A) \) and \( \psi^\# = \Phi \) on \( \text{Tor}(U(A)/DU(A)) \), then \( [\varphi] = [\psi] \) in \( KL(A, B) \).

**Proof.** Choose by Theorem 3.1 a unital *-homomorphism \( \varphi : A \to B \) such that \( \varphi^\# = \Phi \) on \( \text{Tor}(U(A)/DU(A)) \) and \( \varphi^* = \varphi_1 \) on \( K_1(A) \). Then \( \varphi^\# = \varphi_1 \) on \( K_1(A) \), and thus \( \varphi^\# = \varphi_1 \) on all of \( K_1(A) \) by Theorem 2.2. Obviously \( \varphi^* = \varphi_0 \) since \( \varphi \) is unital. This proves (i).

To prove (ii), note that since \( \varphi^\# = \Phi \) on \( \text{Tor}(U(A)/DU(A)) \) we have that \( \varphi^* = \psi^* \) on \( K^0(B) \) by Proposition 2.7. Hence \( [\varphi] = [\psi] \) by Theorem 2.6.

Let \( A \) and \( B \) be simple unital infinite dimensional inductive limits of sequences of finite direct sums of building blocks. In [3, Chapter 10] a group homomorphism

\[
s_\kappa : \text{Tor}(U(A)/DU(A)) \to \text{Tor}(U(B)/DU(B)),
\]

was constructed for every \( \kappa \in KL(A, B)_T \) (the map was constructed for slightly different \( B \) but can be applied in our case by [3, Lemma 10.3], [3, Lemma 9.6] and [3, Theorem 9.9]). Recall from [3] that \( KL(A, B)_T \) is the
set of elements $\kappa \in KL(A,B)$, for which there exists an affine continuous map $\varphi_T : T(B) \to T(A)$ such that $r_B(\omega)(\kappa_\omega(x)) = r_A(\varphi_T(\omega))(x)$ for $x \in K_0(A), \omega \in T(B)$. Note that if $K_0(A) \cong \mathbb{Z}$ and $K_0(B) \cong \mathbb{Z}$ then $KL(A,B)_T = KL(A,B)_e$.

Recall furthermore from [3, Chapter 10] that $s_{[\mu]} = \mu^#$ on $\text{Tor}(U(A)/DU(A))$ for every unital $\ast$-homomorphism $\mu : A \to B$, and that if $C$ is a finite direct sum of building blocks, and $\varphi : C \to A$, $\psi : C \to B$ are unital $\ast$-homomorphisms such that $[\psi] = \kappa \cdot [\varphi]$ in $KL(C, B)$, then $\psi^# = s_{\kappa} \circ \varphi^#$ on $\text{Tor}(U(C)/DU(C))$.

We can now generalize Theorem 3.2 to simple inductive limits for which $K_0(A)$ and $K_0(B)$ are cyclic:

**Theorem 3.3.** Let $A$ and $B$ be unital simple infinite dimensional inductive limits of sequences of finite direct sums of building blocks. Assume that $K_0(A)$ and $K_0(B)$ are cyclic groups. Let $\varphi_0 : K_0(A) \to K_0(B)$ be an order unit preserving group homomorphism, and let $\varphi_1 : K_1(A) \to K_1(B)$ and $\Phi : U(A)/DU(A) \to U(B)/DU(B)$ be group homomorphisms such that the diagram

\[
\begin{array}{ccc}
\text{Tor}(\text{Aff} T(A)/\rho_A(K_0(A))) & \xrightarrow{\lambda_A} & \text{Tor}(U(A)/DU(A)) \\
\downarrow \varphi_0 & & \downarrow \Phi \\
\text{Tor}(\text{Aff} T(B)/\rho_B(K_0(B))) & \xrightarrow{\lambda_B} & \text{Tor}(U(B)/DU(B)) \\
& \downarrow \Phi & \downarrow \varphi_1 \\
& \text{Tor}(K_1(A)) & \text{Tor}(K_1(B))
\end{array}
\]

commutes. There exists a unique element $\kappa \in KL(A,B)$ such that $\kappa_\omega = \varphi_0$ on $K_0(A)$, $\kappa = \varphi_1$ on $K_1(A)$ and $s_{\kappa} = \Phi$ on $\text{Tor}(U(A)/DU(A))$.

**Proof.** We may by [3, Theorem 9.9] assume that $A$ is the inductive limit of a sequence

\[
A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \ldots
\]

of finite direct sums of building blocks with unital and injective connecting maps. Similarly we may assume that $B$ is the inductive limit of a sequence

\[
B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3 \xrightarrow{\beta_3} \ldots
\]

of finite direct sums of building blocks with unital and injective connecting maps. Since $K_0(A) \cong \mathbb{Z}$ it is easy to see that we may furthermore assume that each $A_k$ is a building block, rather than a finite direct sum of building blocks. Similarly we may assume that each $B_k$ is a building block. Let $\alpha_{k,\infty} : A_k \to A$ and $\beta_{k,\infty} : B_k \to B$ denote the canonical $\ast$-homomorphisms.

By passing to subsequences we may assume that for every positive integer $k$ there exist an order unit preserving group homomorphism $\mu_k : K_0(A_k) \to$
$K_0(B_k)$ and a group homomorphism $\eta_k : K_1(A_k) \to K_1(B_k)$ such that

$$\beta_{k,\infty} \circ \mu_k = \varphi_0 \circ \alpha_{k,\infty} \quad \text{on} \quad K_0(A_k),$$

$$\beta_{k,\infty} \circ \eta_k = \varphi_1 \circ \alpha_{k,\infty} \quad \text{on} \quad K_1(A_k).$$

By passing to a subsequence again, we may assume that

$$\mu_{k+1} \circ \alpha_{k,*} = \beta_{k,*} \circ \mu_k \quad \text{on} \quad K_0(A_k),$$

$$\eta_{k+1} \circ \alpha_{k,*} = \beta_{k,*} \circ \eta_k \quad \text{on} \quad K_1(A_k).$$

Let $A_k = A(n_k, d_1^k, d_2^k, \ldots, d_{N_k}^k)$. By Proposition 2.3, Lemma 2.5, and [3, Lemma 10.8], we may also assume that for every positive integer $k$, there exists a group homomorphism $\Phi_k : \text{Tor}(U(A_k)/DU(A_k)) \to \text{Tor}(U(B_k)/DU(B_k))$ such that

$$\lambda_{B_k} \circ \widetilde{\mu}_k = \Phi_k \circ \lambda_{A_k}$$

on $\text{Tor}(\text{Aff} T(A_k)/\rho_{A_k}(K_0(A_k)))$ and

$$\beta_{k,\infty} \circ \Phi_k(q'_{A_k}(v_{A_k}^j)) = \Phi \circ \alpha_{k,\infty}(q'_{A_k}(v_{A_k}^j))$$

for $j = 1, 2, \ldots, N_k - 1$. Since for every positive integer $k$,

$$\beta_{k,\infty} \circ \Phi_k \circ \lambda_{A_k} = \beta_{k,\infty} \circ \lambda_{B_k} \circ \widetilde{\mu}_k = \lambda_B \circ \beta_{k,\infty} \circ \widetilde{\mu}_k$$

$$= \lambda_B \circ \widetilde{\psi}_0 \circ \alpha_{k,\infty} = \Phi \circ \lambda_{A_k} \circ \alpha_{k,\infty} = \Phi \circ \alpha_{k,\infty} \circ \lambda_{A_k},$$

on $\text{Tor}(\text{Aff} T(A_k)/\rho_{A_k}(K_0(A_k)))$, we conclude from Proposition 2.3 and Lemma 2.5 that

$$\beta_{k,\infty} \circ \Phi_k = \Phi \circ \alpha_{k,\infty}$$

on $\text{Tor}(U(A_k)/DU(A_k))$.

It follows from the above equation and [3, Lemma 10.4] that by passing to subsequences we may assume that for every positive integer $k$,

$$\beta_{k}^\# \circ \Phi_k(q'_{A_k}(v_{A_k}^j)) = \Phi_{k+1} \circ \alpha_{k}^\#(q'_{A_k}(v_{A_k}^j))$$

for $j = 1, 2, \ldots, N_k - 1$. Since for every positive integer $k$,

$$\beta_{k}^\# \circ \Phi_k \circ \lambda_{A_k} = \beta_{k}^\# \circ \lambda_{B_k} \circ \widetilde{\mu}_k = \lambda_{B_{k+1}} \circ \widetilde{\beta}_k \circ \widetilde{\mu}_k$$

$$= \lambda_{B_{k+1}} \circ \mu_{k+1} \circ \alpha_{k+1} = \Phi_{k+1} \circ \lambda_{A_{k+1}} \circ \alpha_{k+1} = \Phi_{k+1} \circ \alpha_{k}^\# \circ \lambda_{A_k},$$

on $\text{Tor}(\text{Aff} T(A_k)/\rho_{A_k}(K_0(A_k)))$, we see that

$$\beta_{k}^\# \circ \Phi_k = \Phi_{k+1} \circ \alpha_{k}^\#$$
on Tor\left(U(A_k)/DU(A_k)\right).

Note that for every positive integer \(k\),
\[
\beta_{k,\infty} \circ \eta_k \circ \pi_{Ak} = \psi_1 \circ \alpha_{k,\infty} \circ \pi_{Ak} = \varphi_1 \circ \pi_A \circ \alpha_{k,\infty}^*
\]
\[
= \pi_B \circ \Phi \circ \alpha_{k,\infty}^* = \pi_B \circ \beta_{k,\infty}^* \circ \Phi_k = \beta_{k,\infty} \circ \pi_{B_k} \circ \Phi_k
\]
on Tor\left(U(A_k)/DU(A_k)\right). By passing to subsequences again we may assume that
\[
\eta_k \circ \pi_{Ak} \left(\gamma_{Ak}(v_{Ak})\right) = \pi_{B_k} \circ \Phi_k \left(\gamma_{Ak}(v_{Ak})\right)
\]
in \(K_1(B)\) for \(j = 1, 2, \ldots, N_k - 1\). Since \(\pi_{B_k} \circ \Phi_k \circ \lambda_{Ak} = 0\) on the torsion subgroup of \(\text{Aff}(T(A_k)/\rho_{Ak}(K_0(A_k)))\), it follows that \(\eta_k \circ \pi_{Ak} = \pi_{B_k} \circ \Phi_k\) on Tor\left(U(A_k)/DU(A_k)\right). Thus the diagram
\[
\begin{array}{ccc}
\text{Tor(\text{Aff}(T(A_k)/\rho_{Ak}(K_0(A_k))))} & \longrightarrow & \text{Tor(U(A_k)/DU(A_k))} \\
\mu \downarrow & & \phi \downarrow \\
\text{Tor(\text{Aff}(T(B_k)/\rho_{B_k}(K_0(B_k))))} & \longrightarrow & \text{Tor(U(B_k)/DU(B_k))}
\end{array}
\]
commutes. Finally we may by [3, Lemma 9.6] assume that \(s(B_k) \geq N_k n_k\).

It follows from Proposition 3.2 that for every positive integer \(k\), there exists a unital *-homomorphism \(\psi_k : A_k \to B_k\) such that \(\psi_{k,n} = \mu_k\) on \(K_0(A_k)\), \(\psi_{k,n} = \eta_k\) on \(K_1(A_k)\), and \(\psi_k^* = \Phi_k\) on Tor\left(U(A_k)/DU(A_k)\right). By the uniqueness part of the same proposition, \([\beta_k] \cdot [\psi_k] = [\psi_{k+1}] \cdot [\alpha_k]\) in \(KL(A_k, B_{k+1})\). By [6, Theorem 1.12] and [7, Theorem 7.1] there exists an element \(\kappa \in KL(A, B)\) such that \(\kappa \cdot [\alpha_{k,\infty}] = [\beta_{k,\infty}] \cdot [\psi_k]\) in \(KL(A_k, B_k)\) for every positive integer \(k\). Then \(\kappa_n = \varphi_0\) on \(K_0(A)\), \(\kappa_n = \varphi_1\) on \(K_1(A)\), and
\[
s_n \circ \alpha_{k,\infty}^* = (\beta_{k,\infty} \circ \psi_k)^* = \beta_{k,\infty}^* \circ \Phi_k = \Phi \circ \alpha_{k,\infty}^*
\]
on Tor\left(U(A_k)/DU(A_k)\right).

By [3, Lemma 10.8] we see that \(s_n = \Phi\) on Tor\left(U(A)/DU(A)\right).

To prove uniqueness, let \(v \in KL(A, B)\) be another element such that \(\nu_n = \varphi_0\) on \(K_0(A)\), \(\nu_n = \varphi_1\) on \(K_1(A)\) and \(\nu = \Phi\) on Tor\left(U(A)/DU(A)\right).

By passing to a subsequence, we may assume that there is an element \(v_k\) in \(KL(A_k, B_k)\) such that \([\beta_{k,\infty}] \cdot v_k = v \cdot [\alpha_{k,\infty}]\). By passing to a subsequence again we may assume that \(\psi_{k,n} = v_{k,n}\) on \(K_0(A)\) as well as on \(K_1(A)\). By Theorem 2.6 there exists a unital *-homomorphism \(\xi_k : A_k \to B_k\) such that \([\xi_k] = v_k\) in \(KL(A_k, B_k)\). Then
\[
\beta_{k,\infty}^* \circ \xi_k^* = s_n \circ \alpha_{k,\infty}^* = s_n \circ \alpha_{k,\infty}^* = \beta_{k,\infty}^* \circ \psi_k^*
\]
on Tor\left(U(A_k)/DU(A_k)\right). By passing to subsequences again, we may by [3, Lemma 10.4] assume that \(\xi_k^* = \psi_k^*\) on any given finite subset of Tor\left(U(A_k)/
Hence, we can arrange that $\xi_k^* = \psi_k^*$ on $K^0(B_k)$ by Proposition 2.7. It follows from Theorem 2.6 that $[\xi_k] = [\psi_k]$ in $KL(A_k, B_k)$. Thus, $\kappa \cdot [\alpha_k, \infty] = \nu \cdot [\alpha_k, \infty]$ for all $k$. It follows that $\kappa = \nu$ by [5, Lemma 5.8].

4. Main results

In [3] the existence result [3, Theorem 11.2] was subsequently simplified in the case where $K_0(A)$ is non-cyclic. The following theorem shows that a similar simplification is possible when $K_0(A)$ and $K_0(B)$ are cyclic, but this time without $KL$.

**Theorem 4.1.** Let $A$ and $B$ be unital simple inductive limits of sequences of finite direct sums of building blocks and assume that $K_0(A) \cong \mathbb{Z}$, $K_0(B) \cong \mathbb{Z}$ and that $B$ is infinite dimensional. Let $\varphi_T : T(B) \to T(A)$ be an affine continuous map, let $\varphi_0 : K_0(A) \to K_0(B)$ be an order unit preserving group homomorphism, let $\varphi_1 : K_1(A) \to K_1(B)$ be a group homomorphism, and let $\Phi : U(A)/DU(A) \to U(B)/DU(B)$ be a homomorphism such that the diagram

$$
\begin{array}{ccc}
\text{Aff} T(A)/\rho_A(K_0(A)) & \xrightarrow{\lambda_A} & U(A)/DU(A) \\
\varphi_T \downarrow & & \downarrow \Phi \\
\text{Aff} T(B)/\rho_B(K_0(B)) & \xrightarrow{\lambda_B} & U(B)/DU(B)
\end{array}
$$

commutes. There exists a unital $^*$-homomorphism $\psi : A \to B$ such that $\psi^* = \varphi_T$ on $T(B)$, $\psi^\# = \Phi$ on $U(A)/DU(A)$, and $\psi^\# = \varphi_0$ on $K_0(A)$.

**Proof.** We may assume that $A$ is infinite dimensional. As in the proof of Theorem 3.3 there exists an element $\kappa \in KL(A, B)$ such that $\kappa_s = \varphi_0$ on $K_0(A)$, $\kappa_s = \varphi_1$ on $K_1(A)$, and $s_s = \Phi$ on $\text{Tor}(U(A)/DU(A))$. By [3, Theorem 11.2] there exists a unital $^*$-homomorphism $\psi : A \to B$ such that $[\psi] = \kappa$ in $KL(A, B)$, $\psi^* = \varphi_T$ on $T(B)$, and $\psi^\# = \Phi$ on $U(A)/DU(A)$.

The next result says that $KL$ can also be removed from the uniqueness theorem, [3, Theorem 11.5], when $K_0(B)$ is cyclic.

**Theorem 4.2.** Let $A$ and $B$ be simple unital inductive limit of sequences of finite direct sums of building blocks such that $K_0(A) \cong \mathbb{Z}$ and $K_0(B) \cong \mathbb{Z}$. Let $\varphi, \psi : A \to B$ be unital $^*$-homomorphisms with $\varphi^\# = \psi^\#$ on $U(A)/DU(A)$. Then $\varphi$ and $\psi$ are approximately unitarily equivalent.

**Proof.** We may assume that $A$ is infinite dimensional. As in the proof of Theorem 3.3 we see that $A$ is the inductive limit of a sequence

$$
A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \ldots
$$
of building blocks with unital and injective connecting maps. Similarly $B$ is the inductive limit of a sequence

$$B_1 \overset{\beta_1}{\to} B_2 \overset{\beta_1}{\to} B_3 \overset{\beta_3}{\to} \ldots$$

of building blocks with unital and injective connecting maps. By [3, Lemma 8.5] we have that $s(B_k) \to \infty$. Obviously $\varphi_\ast = \psi_\ast$ on $K_0(A)$ and $\varphi_\ast = \psi_\ast$ on $K_1(A)$, such that $[\varphi] = [\psi]$ in $KL(A, B)$ by Theorem 3.3. Finally note that $\varphi^\# = \psi^\#$ implies $\tilde{\varphi} = \tilde{\psi}$. Thus the linear map $\tilde{\varphi} - \tilde{\psi}$ takes values in $\rho_B(K_0(B))$, and hence it must be 0. Therefore $\varphi^* = \psi^*$ on $T(B)$. It follows from [3, Theorem 11.5] that $\varphi$ and $\psi$ are approximately unitarily equivalent.

We need the following isomorphism version of Theorem 4.1.

**Theorem 4.3.** Let $A$ and $B$ be simple unital infinite dimensional inductive limits of sequences of finite direct sums of building blocks with $K_0(A) \cong \mathbb{Z}$. Let $\varphi_0 : K_0(A) \to K_0(B)$ be an isomorphism of ordered groups with order units, let $\varphi_1 : K_1(A) \to K_1(B)$ be an isomorphism of groups, let $\varphi_T : T(B) \to T(A)$ be an affine homeomorphism, and let $\Phi : U(A)/DU(A) \to U(B)/DU(B)$ be an isomorphism of groups, such that the diagram

$$\begin{array}{ccc}
\text{Aff } T(A)/\rho_A(K_0(A)) & \overset{\lambda_A}{\to} & U(A)/DU(A) \overset{\pi_A}{\to} K_1(A) \\
\tilde{\varphi} \downarrow & & \downarrow \varphi \\
\text{Aff } T(B)/\rho_B(K_0(B)) & \overset{\lambda_B}{\to} & U(B)/DU(B) \overset{\pi_B}{\to} K_1(B)
\end{array}$$

commutes. Then there exists an isomorphism $\psi : A \to B$ such that $\psi_\ast = \varphi_1$ on $K_1(A)$, $\varphi^* = \varphi_T$ on $T(B)$, and $\psi^* = \Phi$ on $U(A)/DU(A)$.

**Proof.** By Theorem 4.1 there exists a unital $^*$-homomorphism $\mu : A \to B$ such that $\mu^* = \Phi$ on $U(A)/DU(A)$, $\mu^* = \varphi_T$ on $T(B)$, and $\mu_\ast = \varphi_1$ on $K_1(A)$. Similarly, there exists a unital $^*$-homomorphism $\xi : B \to A$ such that $\xi^* = \Phi^{-1}$ on $U(B)/DU(B)$, $\xi^* = \varphi_T^{-1}$ on $T(A)$, and $\xi_\ast = \varphi_1^{-1}$ on $K_1(B)$. By Theorem 4.2 we see that $\mu \circ \xi$ and $\xi \circ \mu$ are approximately inner. Hence by [4, Proposition A] $\mu$ is approximately unitarily equivalent to an isomorphism $\psi : A \to B$.

We are now in a position to prove part (ii) of Theorem 1.1.

**Theorem 4.4.** Let $A$ be a simple unital inductive limit of a sequence of finite direct sums of building blocks with $K_0(A) \cong \mathbb{Z}$. Then

$$\text{Aut}(A)/\text{Inn}(A) \cong \text{Hom}\left(K_1(A), \text{Aff } T(A)/\rho_A(K_0(A))\right) \rtimes \text{Aut}(\overline{K}_A),$$
where the action of \((\varphi_0, \varphi_1, \varphi_T) \in \mathrm{Aut}(\mathcal{E}_A)\) is given by
\[
\eta \mapsto \widetilde{\varphi_T}^{-1} \circ \eta \circ \varphi_1^{-1}, \quad \eta \in \hom(K_1(A), \aff T(A)/\rho_A(K_0(A))).
\]

**Proof.** We may assume that \(A\) is infinite dimensional. By Proposition 2.3 we may identify \(U(A)/DU(A)\) with \(G_1 \oplus G_2\), where \(G_1 = \aff T(A)/\rho_A(K_0(A))\) and \(G_2 = K_1(A)\). Thus an endomorphism \(\psi\) of the group \(U(A)/DU(A)\) can be identified with a \(2 \times 2\) matrix
\[
\begin{pmatrix}
\psi_{11} & \psi_{12} \\
\psi_{21} & \psi_{22}
\end{pmatrix}
\]
where \(\psi_{ij} : G_j \to G_i\) is a homomorphism, \(i, j = 1, 2\). Note that if \(\psi\) is induced by an automorphism of \(A\) then \(\psi_{21} = 0\) since the short exact sequence of Proposition 2.3 is natural.

Let \(H = \hom(K_1(A), \aff T(A)/\rho_A(K_0(A)))\). Let \(\eta \in H\). Choose by Theorem 4.3 an element \(\psi \in \mathrm{Aut}(A)\) such that \(\psi^* = \mathrm{id}\) on \(T(A)\), \(\psi_\ast = \mathrm{id}\) on \(K_1(A)\), and
\[
\psi^\# = \begin{pmatrix}
\mathrm{id} & \eta \\
0 & \mathrm{id}
\end{pmatrix}
\]
on \(U(A)/DU(A)\). By Theorem 4.2 we obtain a well-defined group homomorphism
\[
\iota : H \to \mathrm{Aut}(A)/\mathrm{Inn}(A)
\]
by setting \(\iota(\eta) = p(\psi)\), where \(p : \mathrm{Aut}(A) \to \mathrm{Aut}(A)/\mathrm{Inn}(A)\) denotes the canonical map. Let \(\pi : \mathrm{Aut}(A)/\mathrm{Inn}(A) \to \mathrm{Aut}(\mathcal{E}_A)\) be the homomorphism
\[
\pi(p(\psi)) = (\psi_\ast, \psi_\ast, (\psi^\ast)^{-1}).
\]
We have a short exact sequence
\[
0 \to H \to \mathrm{Aut}(A)/\mathrm{Inn}(A) \to \pi \to \mathrm{Aut}(\mathcal{E}_A) \to 0
\]
of groups. Let \((\varphi_0, \varphi_1, \varphi_T) \in \mathrm{Aut}(\mathcal{E}_A)\). Choose by Theorem 4.1 an element \(\psi\) in \(\mathrm{Aut}(A)\) such that \(\psi_\ast = \varphi_1\), \(\psi^* = \varphi_T^{-1}\), and
\[
\psi^\# = \begin{pmatrix}
\widetilde{\varphi_T}^{-1} & 0 \\
0 & \varphi_1
\end{pmatrix}.
\]
By Theorem 4.2 we obtain a well-defined map \(\sigma : \mathrm{Aut}(\mathcal{E}_A) \to \mathrm{Aut}(A)/\mathrm{Inn}(A)\) by setting \(\sigma(\varphi_0, \varphi_1, \varphi_T) = p(\psi)\). Note that \(\sigma\) splits the sequence above. Hence \(\mathrm{Aut}(A)/\mathrm{Inn}(A)\) is isomorphic to a semi-direct product \(H \rtimes \mathrm{Aut}(\mathcal{E}_A)\). Since
\[
\iota(\widetilde{\varphi_T}^{-1} \eta \varphi_1^{-1}) = \sigma(\varphi_0, \varphi_1, \varphi_T) \iota(\eta) \sigma(\varphi_0, \varphi_1, \varphi_T)^{-1},
\]
it follows that the action of Aut(\(E_A\)) on \(H\) is the desired one.

Let us finally show that our main result can be simplified when \(K_1(A)\) is a torsion group. Recall that ext(\(G, H\)) is defined as Ext(\(G, H\))/Pext(\(G, H\)) for abelian groups \(G\) and \(H\), where Pext(\(G, H\)) is the subgroup of pure (i.e. locally trivial) extensions in Ext(\(G, H\)), see [5, Chapter 5].

**Corollary 4.5.** Let \(A\) be a simple unital inductive limit of a sequence of finite direct sums of building blocks such that \(K_1(A)\) is a torsion group. Then

\[
\text{Aut}(A)/\text{Inn}(A) \cong \text{ext}(K_1(A), K_0(A)) \rtimes \text{Aut}(E_A),
\]

where the action of \((\varphi_0, \varphi_1, \varphi_T) \in \text{Aut}(E_A)\) is given by \(e \mapsto \varphi_0 \circ \varphi_1^{-1}(e)\) for \(e \in \text{ext}(K_1(A), K_0(A))\).

**Proof.** If \(K_0(A)\) is non-cyclic, then Aff(\(T(A)/\rho_A(K_0(A))\)) is torsion-free by [3, Lemma 10.3], and hence the result follows in this case from (i) in Theorem 1.1. Therefore we may assume that \(K_0(A) \cong \mathbb{Z}\). Then \(\rho_A\) is injective and has closed range. Hence we have a short exact sequence

\[
0 \longrightarrow K_0(A) \xrightarrow{\rho_A} \text{Aff}(T(A)) \xrightarrow{q_A} \text{Aff}(T(A)/\rho_A(K_0(A)) \longrightarrow 0.
\]

Let \(E\) denote the corresponding class in Ext(\(\text{Aff}(T(A)/\rho_A(K_0(A))), K_0(A))\). Note that Aff(\(T(A)\)) is divisible, and therefore Ext(\(K_1(A), \text{Aff}(T(A))\)) = 0. Hence by applying [2, Theorem III.3.4] we get an isomorphism

\[
E_* : \text{Hom}(K_1(A), \text{Aff}(T(A)/\rho_A(K_0(A)))) \rightarrow \text{Ext}(K_1(A), K_0(A)),
\]

where \(E_*(\eta) = \eta^*(E)\). By a result of C. U. Jensen, see e.g. [8, Theorem 6.1], we have that Pext(\(K_1(A), K_0(A))\)=0. Thus Ext(\(K_1(A), K_0(A)) = \text{ext}(K_1(A), K_0(A))\). To see that the two actions of Aut(\(E_A\)) can be identified as well, note that the diagram

\[
0 \longrightarrow K_0(A) \xrightarrow{\rho_A} \text{Aff}(T(A)) \xrightarrow{q_A} \text{Aff}(T(A)/\rho_A(K_0(A)) \longrightarrow 0
\]

\[
\varphi_0 \downarrow \quad \varphi^{-1}_T \downarrow
\]

\[
0 \longrightarrow K_0(A) \xrightarrow{\rho_A} \text{Aff}(T(A)) \xrightarrow{q_A} \text{Aff}(T(A)/\rho_A(K_0(A)) \longrightarrow 0
\]

commutes, such that \(\varphi_T^{-1} = \varphi_0(E)\) by [2, Proposition III.1.8]. The corollary follows.

We mention without proof that the \(C^*\)-algebras considered in the above corollary are exactly the simple unital inductive limits of sequences of finite direct sums of building blocks of the form

\[
\{ f \in C[0, 1] \otimes M_n : f(x_i) \in M_{d_i}, i = 1, 2, \ldots, N\}.
\]
Corollary 4.5 suggests that only $KL$, and not $U(\cdot)/DU(\cdot)$, is needed in an approximate intertwining argument to show that the Elliott invariant is a classifying invariant for these $C^*$-algebras. This was demonstrated by Jiang and Su [1] for a large subclass of this class of $C^*$-algebras.

Let us finally emphasize the following surprising consequence of the corollary above. Let $A$ be a simple unital inductive limit of a sequence of finite direct sums of building blocks. If $K_0(A)$ is non-cyclic then $\operatorname{Aut}(A)/\operatorname{Inn}(A)$ is isomorphic to a semi-direct product

$$\left(\operatorname{Hom}(K_1(A), \operatorname{Aff} T(A)/\rho_A(K_0(A))) \times \operatorname{ext}(K_1(A), K_0(A))\right) \rtimes \operatorname{Aut}(E_A).$$

The term $\operatorname{Hom}(K_1(A), \operatorname{Aff} T(A)/\rho_A(K_0(A)))$ vanishes e.g. if $A$ has real rank zero, whereas the term $\operatorname{ext}(K_1(A), K_0(A))$ vanishes e.g. if $A$ is an inductive limit of a sequence of finite direct sums of circle algebras. When $K_0(A) \cong \mathbb{Z}$ and $K_1(A)$ is a torsion group, however, these two terms agree, but only one of them appear in the expression for $\operatorname{Aut}(A)/\operatorname{Inn}(A)$.

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