THE DISCREPANCY OF SOME REAL SEQUENCES

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Abstract

Let \((λ_n)_{n≥0}\) be a sequence of real numbers such that there exists \(δ > 0\) such that \(|λ_{n+1} - λ_n| ≥ δ\), \(n = 0, 1, \ldots\). For a real number \(y\) let \([y]\) denote its fractional part. Also, for the real number \(x\) let \(D(N, x)\) denote the discrepancy of the numbers \([λ_0 x], \ldots, [λ_{N-1} x]\). We show that given \(ε > 0\),

\[D(N, x) = o\left(N^{-\frac{3}{2}} (\log N)^{\frac{3}{2} + ε}\right)\]

almost everywhere with respect to Lebesgue measure.

1. Introduction

Recall that a sequence of real numbers \((x_n)_{n=0}^{∞}\) is uniformly distributed modulo one if for each interval \(I\) contained in \([0, 1)\), that is closed on the left and open on the right,

\[\lim_{N→∞} \frac{1}{N} \sum_{n=0}^{N-1} \chi_I([x_n]) = |I|\]

Here \(χ_I\) denotes the characteristic function of the interval \(I\) and \(|I|\) its length. Also \([y]\) is the fractional part of a real number \(y\). For a finite set of real numbers \(x_0, \ldots, x_{N-1}\), their discrepancy is

\[D(x_0, \ldots, x_{N-1}) = \sup_{I⊆[0,1)} \left| \frac{1}{N} \sum_{n=0}^{N-1} χ_I([x_n]) - |I| \right| \quad (N = 1, 2, \ldots)\]

where the supremum is taken over all intervals \(I\), closed on the left and open on the right. The discrepancy of the numbers \(x_0, \ldots, x_{N-1}\) tends to zero as \(N\) tends to infinity if and only if the sequence \((x_n)_{n=0}^{∞}\) is uniformly distributed modulo one. This means that for a uniformly distributed sequence of real numbers \((x_n)_{n=0}^{∞}\), as \(N\) tends to infinity, the rate for decay of \(D(x_0, \ldots, x_{N-1})\) provides a measure of the degree of uniformity of distribution. As usual if a property holds except for a set of Lebesgue measure zero, it is said to hold almost everywhere, abbreviated ‘a.e.’. Throughout the paper \(C\) will denote a
positive constant which will not necessarily be the same at each occurrence.
Let \((\lambda_n)_{n=0}^\infty\) denote a sequence of real numbers such that there exists \(\delta > 0\) such that
\[|\lambda_{n+1} - \lambda_n| \geq \delta > 0 \quad (n = 0, 1, \ldots).\]

In this paper we prove two theorems.

**Theorem 1.1.** Let
\[D(N, x) = D(\{\lambda_0 x\}, \ldots, \{\lambda_{N-1} x\}) \quad (N = 0, 1, \ldots).\]
Then given \(\varepsilon > 0\),
\[D(N, x) = o\left(N^{-\frac{1}{2}} (\log N)^{\frac{1}{2} + \varepsilon}\right) \quad \text{a.e.}\]

One might wish to consider sets other than intervals. However a famous example of J. M. Marstrand [5] states that there exist subsets \(B\) and \(A(B)\) of \([0, 1)\), both of positive Lebesgue measure, such that if \(x\) is in \(A(B)\), then the averages
\[\frac{1}{N} \sum_{j=0}^{N-1} \chi_B(\{jx\}) \quad (N = 1, 2, \ldots),\]
do not converge to the Lebesgue measure of \(B\). With suitable restrictions on \(B\) however positive results are however possible.

**Theorem 1.2.** Let \((R_k)_{k=1}^\infty\) be a collection of disjoint subintervals of \([0, 1)\) such that
\[(1.1) \quad |R_k| = O(a^{-k}),\]
for some \(a > 1\), and let
\[B = \bigcup_{k=1}^\infty R_k.\]
Then given \(\varepsilon > 0\), there exists \(N_0 = N_0(x, \varepsilon)\) such that if \(N > N_0\)
\[\left|\frac{1}{N} \sum_{n=0}^{N-1} \chi_B(\{\lambda_n x\}) - |B|\right| < N^{-\frac{1}{2}} (\log N)^{\frac{5}{2} + \varepsilon} \quad \text{a.e.}\]

Theorem 1.1, but with \(\frac{5}{2}\) instead \(\frac{3}{2}\) in the power of \(\log N\) appears in [2] and in [1] but with the restriction that \((\lambda_n)_{n=0}^\infty\) are integers. The extension of these results to Theorem 1.1 is made possible by Lemma 2.2 below, which is a consequence of the maximal inequality for the Carleson-Hunt inequality and the
properties of Vaaler polynomials [6]. The methods used to prove Theorem 1.1 are the basis of the proof of Theorem 1.2.


2. Proof of Theorem 1.1

To prove Theorem 1.1, we need the following lemmas.

**Lemma 2.1** ([3]). Given real numbers \( x_0, \ldots, x_{N-1} \), there exists \( C > 0 \) such that for all natural numbers \( L \)

\[
ND(x_0, \ldots, x_{N-1}) \leq C \left( \frac{N}{L} + \sum_{h=1}^{L} \frac{1}{h} \left| \sum_{n=0}^{N-1} e^{2\pi i h x_n} \right| \right).
\]

**Lemma 2.2** ([6]). Suppose we are given \( \delta > 0 \), real numbers \( (\lambda_n)_{n=0}^{N-1} \) such that \( \lambda_{n+1} - \lambda_n \geq \delta > 0 \), real numbers \( T \) and \( T_0 \) with \( T > 0 \) and complex numbers \( (a_n)_{n=1}^{N} \). Then there exists \( C > 0 \) such that

\[
\int_{T_0}^{T_0+T} \max_{0 \leq v \leq N-1} \left| \sum_{n=0}^{v} a_n e^{i\lambda_n x} \right|^2 dt \leq C (T + 2\pi \delta^{-1}) \sum_{n=0}^{N-1} |a_n|^2.
\]

Note that Lemma 2.2 in the special case where the \( (\lambda_n)_{n=1}^{\infty} \) are all integers reduces to the maximal inequality of Carleson-Hunt [4]. Plainly in proving Theorem 1.1, we may without loss of generality assume \( x \) belongs a finite interval \( [T_0, T_0 + T] \). Let \( f : [T_0, T_0 + T] \to \mathbb{R} \) be square integrable and let

\[
\|f\| = \left( \frac{1}{T} \int_{T_0}^{T_0+T} |f|^2 \, dx \right)^{\frac{1}{2}}.
\]

Then by applying Minkowski’s inequality to Lemma 2.1,

\[
\max_{1 \leq v \leq N} vD(v, x) \leq C \left( \frac{N}{L} + \sum_{h=1}^{L} \frac{1}{h} \max_{1 \leq v \leq N} \left| \sum_{n=1}^{v} e^{2\pi i \lambda_n x} \right| \right).
\]

whence by Lemma 2.2

\[
\max_{1 \leq v \leq N} vD(v, x) \leq C \left( \frac{N}{L} + \sum_{h=1}^{L} \frac{1}{h} N^2 \right).
\]
Choosing $L$ optimally this gives

\begin{equation}
\left\| \max_{1 \leq v \leq N} vD(v, x) \right\| \leq CN^{\frac{1}{2}}(\log N). \tag{2.1}
\end{equation}

To deduce Theorem 1.1, we argue as in [2] and let

$$E(\varepsilon) = \left\{ x \in [T_0, T_0 + T] : \limsup_{l \to \infty} \frac{ID(l, x)}{f(l, \varepsilon)} > 0 \right\},$$

where

$$f(l, \varepsilon) = l^\frac{1}{2}(\log N)^{\frac{1}{2}+\varepsilon}.$$ \vspace{1em}

We need to show that the Lebesgue measure $|E(\varepsilon)|$ of $E(\varepsilon)$ is zero for all $\varepsilon > 0$. Set

$$A_s(\varepsilon) = \left\{ x \in [T_0, T_0 + T] : \max_{1 \leq l \leq 4^s} ID(l, x) > \frac{1}{4} f\left(4^s, \frac{\varepsilon}{2}\right) \right\}.$$ \vspace{1em}

If $x \in E(\varepsilon)$ then there exist $c(\varepsilon, x) > 0$ and arbitrarily large positive integers $s$ such that for some integer $l$ in $[4^{s-1}, 4^s)$

$$lD(l, x) \geq c(\varepsilon, x) f\left(4^{s-1}, \varepsilon\right) \geq \frac{1}{4} f\left(4^s, \frac{\varepsilon}{2}\right).$$

The last inequality here being evident from the identity

$$f\left(4^{s-1}, \frac{\varepsilon}{2}\right) = \frac{1}{2} \left(\log s - \frac{1}{\log s}\right)^{\frac{1}{2}+\frac{\varepsilon}{2}} f\left(4^s, \frac{\varepsilon}{2}\right)$$

and the fact that for large enough $s$ we have $\frac{1}{2} \left(\log s - \frac{1}{\log s}\right)^{\frac{1}{2}+\frac{\varepsilon}{2}} > \frac{1}{4}$, In particular, we know that for infinitely many $s$,

$$\max_{1 \leq l \leq 4^s} |ID(l, x)| > \frac{1}{4} f\left(4^{s-1}, \frac{\varepsilon}{2}\right).$$

This tells us that $x$ is in infinitely many of the sets $A_s(\varepsilon)$. Hence we can conclude that

$$E(\varepsilon) \subseteq \bigcap_{s=1}^{\infty} \bigcup_{r=1}^{\infty} A_s(\varepsilon).$$

From (2.1) there exists $C > 0$ such that

$$|A_s(\varepsilon)| \left(f\left(4^s, \frac{\varepsilon}{2}\right)\right)^2 \leq C 4^s (\log 4^s)^2.$$
Hence

\[ |A_s(\varepsilon)| \leq \frac{C}{s^{1+\varepsilon}} \]

and

\[ \sum_{s=1}^{\infty} |A_s(\varepsilon)| < \infty, \]

so by the Borel-Cantelli Lemma, Theorem 1.1 is proved.

3. Proof of Theorem 1.2

For

\[ z(N) = \log_a N \quad (N = 1, 2, \ldots) \]

let

\[ t(N) = \bigcup_{1 \leq k \leq z(N)} R_k \quad (N = 1, 2, \ldots) \]

and

\[ s(N) = \bigcup_{k > z(N)} R_k \quad (N = 1, 2, \ldots). \]

Note that if, for any \( S \subseteq [0, 1) \) we set

\[ K(S, l, x) = \frac{1}{l} \sum_{n=0}^{l-1} \chi_S(\{\lambda_n x\}) - |S| \quad (l = 1, 2, \ldots), \]

then

\[ K(B, l, x) = K(t(N), l, x) + K(s(N), l, x). \]

Hence

\[ \left\| \max_{0 \leq l \leq N-1} K(B, l, x) \right\| \leq \left\| \max_{0 \leq l \leq N-1} |K(t(N), l, x)| \right\| + \left\| \max_{0 \leq l \leq N-1} |K(s(N), l, x)| \right\|. \]

Note that

\[ K(t(N), l, x) = \sum_{1 \leq k \leq z(N)} \left( \sum_{j=0}^{l-1} \chi_{R_k}(\{\lambda_j x\}) - l|R_k| \right) \]

so,

\[ \left\| \max_{0 \leq l \leq N-1} K(t(N), l, x) \right\| \leq z(N) \left\| \max_{0 \leq l \leq N-1} lD(l, x) \right\|. \]
This by (2.1) gives
\[ \left\| \max_{0 \leq l \leq N-1} \mathbf{K}(t(N), l, x) \right\| \leq Cz(N)N^{1/2}(\log N). \]

By definition
\[ \left\| \max_{0 \leq l \leq N-1} |\mathbf{K}(s(N), l, x)| \right\| = \left\| \max_{0 \leq l \leq N-1} \left| \sum_{j=0}^{l-1} \chi_{s(N)}(\{\lambda_j x\}) - l|s(N)| \right| \right\|. \]

Also evidently
\[ \left(3.1\right) \quad \left\| \max_{0 \leq l \leq N-1} \left| \sum_{j=0}^{l-1} \chi_{s(N)}(\{\lambda_j x\}) - l|s(N)| \right| \right\| \leq \sum_{j=0}^{N-1} \left\| \chi_{s(N)}(\{\lambda_j x\}) \right\| + N|s(N)|. \]

As \( \chi_{R_k}^2 = \chi_{R_k} \), if
\[ E_{k,j} = \left\{ x \in [T_0, T_0 + T] : \{\lambda_j x\} \in R_k \right\}, \]

we see that
\[ \sum_{j=0}^{N-1} \left\| \chi_{s(N)}(\{\lambda_j x\}) \right\| \leq C \sum_{j=0}^{N-1} \left( \sum_{k > z(N)} |E_{k,j}| \right)^{1/2}. \]

It is very easy to check that there exists \( C = C(T_0, T) > 0 \) such that
\[ |E_{k,j}| \leq C |R_k|. \]

This means that
\[ \left\| \max_{0 \leq l \leq N-1} |\mathbf{K}(s(N), l, x)| \right\| \leq CN(|s(N)|^{1/2} + |s(N)|). \]

Also
\[ |s(N)| = \sum_{k > z(N)} |R_k| \leq C \sum_{k > z(N)} a^{-k} \leq Ca^{-z(N)}. \]

So we have shown that
\[ \left\| \max_{0 \leq l \leq N-1} |\mathbf{K}(B, l, x)| \right\| \leq CN^{1/2}(\log N)^2. \]

The argument of the previous section now gives Theorem 1.2.
REFERENCES