ON THE TOPOLOGY OF SASAKIAN MANIFOLDS

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Abstract

The notion of $q$-bisectional curvature of a Sasakian manifold $M$ is defined. It is proved that if $M$ has lower bounded $q$-bisectional curvature and contains a compact invariant submanifold tangent to the structure vector field then $M$ is compact. Myers and Frankel type theorems for Sasakian manifolds with lower bounded and positive $q$-bisectional curvature, respectively, are also given.

1. Introduction

Let $M$ be a complete connected Riemannian manifold. An important theorem, proved by Myers [11], asserts that if $M$ has sectional curvature $\geq k_0 > 0$ (or more generally, if all the eigenvalues of the Ricci tensor are $\geq (\dim M - 1)k_0 > 0$) then $M$ is compact, its diameter is $\leq \pi/\sqrt{k_0}$ and has finite fundamental group.

Another remarkable theorem, due to Frankel [2], asserts that if $M$ has positive sectional curvature, then any two compact totally geodesic submanifolds $N$, $P$ of $M$ and such that $\dim N + \dim P \geq \dim M$, must intersect. He also proved that in the case of a complete connected Kähler manifold with positive sectional curvature the same conclusion holds if we replace the hypothesis totally geodesic by analytic. Such results were proved by Goldberg and Kobayashi [5] for Kähler manifolds with positive bisectional curvature. Frankel’s theorems were extended by Gray [6] to nearly Kähler manifolds, by Marchiafava [10] to quaternionic Kähler manifolds and by Ornea [12] to locally conformal Kähler manifolds and to Sasakian manifolds in the case when the submanifolds $N$ and $P$ are invariant and tangent to the structure vector field of $M$.

Recently, Kenmotsu and Xia [7], [8] proved Frankel type theorems for Kähler manifolds in the more general case when $M$ has either partially positive sectional curvature or partially positive bisectional curvature.

Our purpose is to give Myers and Frankel type theorems for a Sasakian manifold $M$ under weaker conditions on the curvature of the manifold. The second section of this paper is devoted to the notion of $q$-bisectional curvature for such a manifold. We remark that if $q = 1$ then it is exactly the $F$-bisectional curvature.
curvature of the manifold [13], [14] and so the class of Sasakian manifolds with positive (or lower bounded) $q$-bisectional curvature is richer than that of Sasakian manifolds with positive (or lower bounded) $F$-bisectional curvature. We prove that if $M$ has lower bounded $q$-bisectional curvature and contains a compact invariant submanifold tangent to the structure vector field of $M$ then $M$ must be compact (Theorem 2.3 and Corollary 2.4). In section 3 we prove a Myers type theorem for Sasakian manifolds with lower bounded $q$-bisectional curvature (Theorem 3.2). The last section is devoted to the proof of a Frankel type theorem for Sasakian manifolds with positive $q$-bisectional curvature and we obtain an extension of Ornea’s Theorem 2, [12].

2. $q$-bisectional curvature

Let $M$ be a Sasakian manifold and denote by $F$, $\xi$, $\eta$, $g$ its fundamental tensor fields. For any vector fields $X, Y \in \mathcal{X}(M)$, orthogonal to $\xi$, the $F$-bisectional curvature $\mathcal{H}$ of $M$ is defined by

$$\mathcal{H}(X, Y) = \frac{\mathcal{R}(X, FX, Y, FY)}{\|X\|^2 \|Y\|^2},$$

where $\mathcal{R}$ is the Riemann-Christoffel curvature tensor of $M$. Then for any $X', Y' \in \mathcal{X}(M)$ such that

$$\text{span}_R\{X', FX'\} = \text{span}_R\{X, FX\}, \quad \text{span}_R\{Y', FY'\} = \text{span}_R\{Y, FY\},$$

we obtain

$$\mathcal{H}(X', Y') = \mathcal{H}(X, Y).$$

Moreover

**Proposition 2.1.** For any $X, Y \in \mathcal{H}(M)$, orthogonal to $\xi$, we have

$$\mathcal{H}(X, Y) = \frac{1}{\|X\|^2 \|Y\|^2} \left[ \mathcal{R}(X, Y, X, Y) + \mathcal{R}(X, FY, X, FY) \\
+ 2 \left[ g^2(X, Y) - \|X\|^2 \|Y\|^2 + g^2(X, FY) \right] \right].$$

**Proof.** By the Lemma, pg. 93, [1], on a contact manifold we have

$$\mathcal{R}(FY, Y, X, FX) = \mathcal{R}(Y, X, X, Y)$$

$$+ \mathcal{R}(FY, X, X, FY) - 2 \mathcal{P}(X, Y, X, FY),$$

where

$$\mathcal{P}(X, Y, Z, U) = d\eta(X, Z) g(Y, U) - d\eta(X, U) g(Y, Z)$$

$$- d\eta(Y, Z) g(X, U) + d\eta(Y, U) g(X, Z).$$
But $M$ is Sasakian, hence $d\eta = \Omega$, with $\Omega(X, Y) = g(X, FY)$, and then

$$\mathcal{P}(X, Y, X, FY) = g^2(X, Y) + g^2(X, FY) - \|X\|^2\|Y\|^2.$$  

Now, from (1) and taking into account (2), (3), we obtain the announced formula.

Let $T_xM$ be the tangent space to the Sasakian manifold $M$ at the point $x$ and we denote by $\mathcal{S} = \{X_1, \ldots, X_q\} \subset T_xM$ an orthonormal system of vectors orthogonal to $\xi$. Then the vectors of the system $F\mathcal{S} = \{FX_1, \ldots, FX_q\}$ are orthogonal to $\xi$. $\mathcal{S}$ is called an $F$-orthonormal $q$-system of tangent vectors at $x$ if $\mathcal{S} \cup F\mathcal{S}$ is orthonormal. We remark that $q \leq \left\lfloor \frac{1}{2}\dim M \right\rfloor$ and for such a system $\mathcal{S}$ and for any tangent vector $X \in T_xM$, orthogonal to $\xi$, we can consider the scalar

$$\mathcal{H}_q(X, \mathcal{S}) = \sum_{i=1}^{q} \mathcal{H}(X, X_i).$$

Now, taking into account Proposition 2.1, we obtain

**Proposition 2.2.** Let $X$ be a unit tangent vector at $x \in M$ and $\mathcal{S}$ be an $F$-orthonormal $q$-system at $x$. If $\mathcal{S}' \subset T_xM$ is an orthonormal system such that $\text{span}_R \mathcal{S}' = \text{span}_R \mathcal{S}$ then:

a) $\mathcal{S}'$ is an $F$-orthonormal $q$-system

b) $\mathcal{H}_q(X, \mathcal{S}') = \mathcal{H}_q(X, \mathcal{S})$.

From Proposition 2.2 it follows that $\mathcal{H}_q(X, \mathcal{S})$ is depending only on the subspace of $T_xM$ spanned by $\mathcal{S}$, but not on the $F$-orthonormal $q$-system $\mathcal{S}$. We call $\mathcal{H}_q(X, \mathcal{S})$ the $q$-bisectional curvature of $M$ at the point $x$ and we remark that for $q = 1$ it is exactly the $F$-bisectional curvature of $M$.

In the following of this section we shall construct $F$-orthonormal systems and these will be used in order to give information about the topology of the manifold.

Let $N$ be a $2r$-dimensional ($r \geq 1$) submanifold of the complete connected Sasakian manifold $M$ and we assume $N$ to be invariant (i.e. $FT_xN \subseteq T_xN$ for any $x \in N$) and it is tangent to $\xi$. If $\{e_1, \ldots, e_r, Fe_1, \ldots, Fe_r, \xi\}$ is an adapted basis of $T_xN$ then $\mathcal{B} = \{e_1, \ldots, e_r\}$ is, obviously, an $F$-orthonormal $r$-system. Moreover, if $\gamma : [0, \infty) \rightarrow M$ is the geodesic starting from $x$ and orthogonal to $N$ at $x$ then the system $\mathcal{B}$, obtained from $\mathcal{B}$ by parallel translation along $\gamma$ is an $F$-orthogonal $r$-system, too. Indeed, if $E_i$ is obtained by parallel translation of $e_i$ along $\gamma$ then we have

$$\nabla_{\gamma'} E_i = 0, \quad E_i(\gamma(t)) = E_i(t) = e_i.$$
hence $E_i$ is normal to $\gamma$. Similar equalities hold for the vector fields $\tilde{E}_i$, obtained from $Fe_i$ by parallel translation along $\gamma$, and therefore $\tilde{E}_i$ are normal to $\gamma$, too. But by using the well-known equality, true on a Sasakian manifold, ([1], Theorem, pg. 73)
\[(\nabla_X F) Y = g(X, Y)\xi - \eta(Y)X,\]
we have
\[\nabla_{\gamma'} (FE_i) = 0,
\]
and because $FE_i(\gamma(t)) = Fe_i$, it follows $\tilde{E}_i = FE_i$, which proves that $\tilde{B}$ is an $F$-orthonormal $r$-system.

**Theorem 2.3.** Let $M$ be a complete connected Sasakian manifold of dimension $2n + 1 \geq 5$. If for some $r \geq 1$ there exists a $2r + 1$-dimensional compact invariant submanifold $N$, tangent to $\xi$ and such that
\[(4) \quad \lim_{t \to \infty} \inf \int_0^t \mathcal{H}_r(\gamma'(s), \tilde{B}) \, ds > 0,
\]
for any $x \in N$ and for any $F$-orthonormal $r$-system $B$ of $T_xM$ then the manifold $M$ is compact.

For the proof of this theorem we drew one's inspiration from [8].

**Proof.** If $M$ is not compact then, by Theorem 1 of [3], there exists $x \in N$ and a geodesic $\gamma : [0, \infty) \to M$, orthogonal to $N$ at $x$, and such that
distance$(\gamma(t), N) = \text{length} \gamma_{[0, t]}$,

hence $\gamma$ has no conjugate points. By putting $\mathcal{H}(t) = \frac{1}{2r} \mathcal{H}_r(\gamma'(t), \tilde{B})$ and taking into account (4), from [15] (see also [8]) it follows that the scalar Jacobi equation
\[f'' + \mathcal{H}(t) f = 0\]
has a solution $\Phi : [0, \infty) \to \mathbb{R}$, satisfying the conditions $\Phi(0) = 1$, $\Phi'(0) = 0$ and $\Phi(t_0) = 0$ for some $t_0 > 0$.

In the following we shall use the well-known expressions of the index form $I$ of the geodesic $\gamma$ along $\gamma_{[0, t]}$ (see [9], t. II, Theorems 5.4 and 5.5, pg. 81)
\[I^b_a(X, Y) = \int_a^b [g(X', Y') - \mathcal{R}(X, \gamma', Y, \gamma')] \, dt
\]
\[= g(X', Y)(b) - g(X', Y)(a)
\]
\[- \int_a^b [g(X'', Y) + \mathcal{R}(X, \gamma', Y, \gamma')] \, dt
\]
for all vector fields $X$ and $Y$ along $\gamma$. Indeed, we consider it for the vector fields $X_i, Y_i$, defined along $\gamma|_{[0, t_0]}$, by

\begin{align}
(6) \quad & X_i(t) = \Phi(t)E_i(t), \quad Y_i(t) = \Phi(t)\tilde{E}_i(t).
\end{align}

They are tangent to $N$ at $\gamma(0)$ and $X_i(t_0) = Y_i(t_0) = 0$. Moreover we have

\begin{align}
(7) \quad & X'_i(t) = \Phi'E_i, \quad Y'_i(t) = \Phi'\tilde{E}_i,
\end{align}

hence $X'_i(0) = Y'_i(0) = 0$ and $X''_i = \Phi''E_i, Y''_i = \Phi''\tilde{E}_i$. Then we have

\begin{align}
(8) \quad & g(X''_i, X_i) + R(X_i, \gamma', X_i, \gamma') = \Phi''\Phi + \Phi^2R(E_i, \gamma', E_i, \gamma'),
\end{align}

\begin{align}
(9) \quad & g(Y''_i, Y_i) + R(Y_i, \gamma', Y_i, \gamma') = \Phi''\Phi + \Phi^2R(\tilde{E}_i, \gamma', \tilde{E}_i, \gamma').
\end{align}

By using Gauss formula we have

\begin{align}
g(X', Y) = g(h(X, Y), \gamma')
\end{align}

for all $X, Y$ tangent to $N$ and normal to $\gamma$. But $N$ is invariant and tangent to $\xi$ and then we have

\begin{align}
(10) \quad & h(Fe_i, Fe_i) = -h(e_i, e_i).
\end{align}

Now, from (5), (8), (9) and (10) we give

\begin{align}
\sum_{i=1}^{r}[I^0_0(X_i, X_i) + I^0_0(Y_i, Y_i)] &= -2r \int_0^{t_0} \Phi'\Phi'' + H(t)\Phi dt \\
&= -2r \int_0^{t_0} \|\gamma''\|^2 dt \\
&= -2r \int_0^{t_0} \|\gamma''\|^2 dt < 0,
\end{align}

hence $I^0_0(X_i, X_i) < 0$ or $I^0_0(Y_i, Y_i) < 0$ for some $i \in \{1, 2, \ldots, r\}$ and therefore $\gamma|_{[0, t_0]}$ has a conjugate point. But this contradicts the hypothesis that $\gamma$ has no conjugate points.

We say that $M$ has \textit{lower bounded $q$-bisectional curvature} at the point $x \in M$ if there exists $k_0 \in \mathbb{R}$ such that $\mathcal{H}_q(X, \mathcal{S}) \geq k_0$ for any unit tangent vector $X \in T_xM$ and for any $F$-orthonormal $q$-system $\mathcal{S}$. If $k_0 = 0$ then we say that $M$ has \textit{nonnegative $q$-bisectional curvature} and taking into account (1) we remark that if $M$ has nonnegative $F$-bisectional curvature then its $q$-bisectional curvature is also nonnegative for any $q \leq \left[\frac{1}{2} \dim M\right]$. Hence the family of Sasakian manifolds with nonnegative $q$-bisectional curvature is richer
than the one containing all Sasakian manifolds with nonnegative $F$-bisectional curvature.

By using the above notions, from Theorem 2.3 we deduce

**Corollary 2.4.** Let $M$ be a complete connected Sasakian manifold with positive lower bounded $r$-bisectional curvature, $r \leq \left[ \frac{1}{2} \dim M \right]$. If $M$ contains a $2r + 1$-dimensional compact invariant submanifold, tangent to $\xi$, then $M$ is compact.

### 3. A Myers type theorem

Let $M$ be a $2n + 1$-dimensional Sasakian manifold and we denote by $M^*$ its universal covering space. It is well-known (see for instance [9], t. I, pg. 162) that on $M^*$ there is a Riemannian metric $g^*$ such that the projection $\pi : M^* \to M$ is an isometric immersion and we define the 1-form $\eta^*$ on $M^*$ by $\eta^*_p = \pi^* \pi_! \eta_{\pi(p)}$, where $\pi^*_p$ is the codifferential of $\pi$ at the point $p \in M^*$. If $\xi^{**}$ is its dual vector field with respect to $g^*$, i.e. the only vector field satisfying

$$\eta^*(X^*) = g^*(X^*, \xi^{**}),$$

for any $X^* \in \mathfrak{X}(M^*)$, then $\xi^{**}$ is nowhere zero, hence we can consider its associated unit vector field $\xi^*$ and we have $\pi_{s,p} \xi^*_p = \xi_{\pi(p)}$, where $\pi_{s,p}$ is the differential of $\pi$.

Let $\nabla^*$ be the Levi-Civita connection on $M^*$, associated with the metric $g^*$. Then we can define the morphism $F^* : \mathfrak{X}(M^*) \to \mathfrak{X}(M^*)$ by

$$F^* X^* = -\nabla^*_X^* \xi^*$$

for any $X^* \in \mathfrak{X}(M^*)$ and a straightforward computation shows that the tensor fields $F^*$, $\xi^*$, $\eta^*$, $g^*$ define a Sasakian structure on $M^*$ (see for instance [1], Theorem, pg. 73).

If $\mathfrak{J} = \{e_1, \ldots, e_q\}$ is an $F$-orthonormal $q$-system of local vector fields in $M$ then we consider the 1-forms $\omega_1, \ldots, \omega_{2q}$, defined by

$$\omega_i(X) = g(e_i, X), \quad \omega_{q+i}(X) = g(F e_i, X)$$

for any $X \in \mathfrak{X}(M)$ and $i \in \{1, \ldots, q\}$. We obtain $2q$ local 1-forms $\omega_1^*, \ldots, \omega_{2q}^*$, defined by

$$\omega_{j,p}^* = \pi^*_p \omega_{j, \pi(p)}$$

for $j \in \{1, \ldots, 2q\}$. Their dual local vector fields $e_1^*, \ldots, e_{2q}^*$, given by formulæ similar to (11), satisfy

$$g(\pi_* e_j^*, \pi_* Y^*) = (\pi^* \omega_j)(Y^*) = \omega_j(\pi_* Y^*) = g(e_j, \pi_* Y^*).$$
and taking into account \( \pi_* \) is injective, we deduce \( e_j = \pi_* e_j^* \). Hence \( \mathcal{S}^* = \{ e_1^*, \ldots , e_q^* \} \) is an \( F \)-orthonormal \( q \)-system of local vector fields in \( M^* \) and by a straightforward computation we deduce

**Proposition 3.1.** Let \( M \) be a Sasakian manifold and \( M^* \) its universal covering. If \( \mathcal{K}_q, \mathcal{K}_q^* \) are the \( q \)-bisectional curvatures of \( M \) and \( M^* \) respectively, then

\[
\mathcal{K}_q^*(X^*, \mathcal{S}^*) = \mathcal{K}_q(\pi_* X^*, \mathcal{S}),
\]

for any unit vector field \( X^* \in \mathfrak{X}(M^*) \) and for any \( F \)-orthonormal \( q \)-system \( \mathcal{S} \) of \( M \).

Now we shall prove a Myers type theorem for Sasakian manifolds, namely

**Theorem 3.2.** Let \( M \) be a complete connected Sasakian manifold with lower bounded \( q \)-bisectional curvature \( \mathcal{K}_q \geq k_0 > 0 \). Then:

a) \( M \) is compact;

b) the diameter of \( M \) is at most equal to \( \pi \sqrt{\frac{2q}{k_0 + 2q}} \);

c) \( M \) has finite fundamental group.

**Proof.** For two arbitrary points \( x \) and \( y \) of \( M \) denote by \( \gamma : [a,b] \to M \) the minimizing geodesic joining \( x \) to \( y \). We assume that \( \gamma \) is parametrized by its arc length \( s \) and \( b \) is such that \( \gamma(b) \) is the first conjugate point of \( \gamma \). If \( \mathcal{S} = \{ e_1, \ldots , e_q \} \subset T_x M \) is an \( F \)-orthonormal \( q \)-system normal to \( \gamma \) at \( \gamma(a) \) then, as above, by parallel translation of \( \mathcal{S} \) along \( \gamma \), we obtain another \( F \)-orthonormal \( q \)-system \( \tilde{\mathcal{S}} = \{ E_1, \ldots , E_q \} \) of vector fields defined along \( \gamma \) and normal to \( \gamma \).

Now, if \( \Phi : [a,b] \to \mathbb{R} \) is a nonzero differentiable function such that \( \Phi(a) = \Phi(c) = 0 \) for some \( c \in (a,b) \), then definitions similar to (6) give the vector fields \( X_i, Y_i \) along \( \gamma \) and by using (7) we obtain

\[
\sum_{i=1}^{q} \left[ I_a^c (X_i, X_i) + I_a^c (Y_i, Y_i) \right]
= \int_a^c \left[ 2q \Phi^2 - \Phi^2 \sum_{i=1}^{q} \left[ R(E_i, \gamma', E_i, \gamma') + R(F E_i, \gamma', F E_i, \gamma') \right] \right] ds.
\]

But \( \gamma_{[a,c]} \) has no conjugate points and taking into account Proposition 2.1, from (12) we deduce

\[
0 < \int_a^c \left[ 2q \Phi^2 - \Phi^2 [\mathcal{K}_q(\gamma', \tilde{\mathcal{S}}) + 2q] \right] ds \leq \int_a^c \left[ 2q \Phi^2 - (k_0 + 2q) \Phi^2 \right] ds
\]
For $\Phi(s) = \sin \frac{s - a}{c - a}$, from the above inequalities we obtain

$$(c - a)^2 < \frac{2q}{2q + k_0} \pi^2.$$ 

But $c$ is arbitrary in $(a, b)$, hence

$$b - a \leq \pi \sqrt{2q \over 2q + k_0}.$$ 

and because $\gamma$ is parametrized by its arc length, it follows

$$\text{distance}(x, y) = \text{distance}(\gamma(a), \gamma(b)) \leq \pi \sqrt{2q \over 2q + k_0}. \tag{13}$$

Thus b) is proved. From (13) we also deduce that $M$ is bounded and because it is complete, a) is proved too (see [9], t. I, Theorem 4.1, pg. 172). Now, because $M$ is compact, it is well-known that $M^*$ is complete and connected ([9], t. I, pg. 76) and taking into account Proposition 3.1, it follows that $M^*$ satisfies the hypothesis of Theorem 3.2. Hence, by a) it follows that $M^*$ is compact and then the fundamental group of $M$ is finite.

4. A Frankel type theorem

THEOREM 4.1. Let $M$ be a complete connected Sasakian manifold with positive $q$-bisectional curvature. If $N$ and $P$ are two compact invariant submanifolds of $M$, tangent to $\xi$ and such that $\dim N + \dim P \geq \dim M + 2q - 1$, then $N \cap P \neq \emptyset$.

PROOF. If $N \cap P = \emptyset$ then there is a geodesic $\gamma : [0, l] \to M$, parametrized by the arc length, joining two points $x_0 \in N$, $y_0 \in P$ and realizing the minimum of the distance between $N$ and $P$. We denote by $V_{y_0}$ the subspace of $T_{y_0}M$, obtained by parallel translation of $T_{x_0}N$ along $\gamma$ at the point $y_0$. From the hypothesis concerning the dimensions and because $N$ and $P$ are tangent to $\xi$, it follows that $\dim \left( V_{y_0} \cap T_{y_0}P \right) \geq 2q$. But $N$ and $P$ are invariant, hence $V_{y_0} \cap T_{y_0}P$ is invariant under $F$ and because $\xi_{y_0} \in V_{y_0} \cap T_{y_0}P$, it follows that $\dim \left( V_{y_0} \cap T_{y_0}P \right) \geq 2q + 1$. Moreover, we can find an $F$-orthonormal $q$-system $\mathcal{J}_{y_0} = \{e_1, \ldots, e_q\} \subset V_{y_0} \cap T_{y_0}P$. By parallel translation of $\mathcal{J}_{y_0}$ along $\gamma$, we obtain a system of $q$ unit vector fields $\mathcal{F} = \{E_1, \ldots, E_q\}$, defined along $\gamma$ and such that $a$ vectors of $\mathcal{F} \cup F\mathcal{F}$ are from $\mathcal{F}$ and $b (a + b = 2q)$ of them are from $F\mathcal{F}$.

If $a \geq q$ then we can assume that these vector fields are $E_1, \ldots, E_q$ and because $P$ is invariant, it follows that $FE_1, \ldots, FE_q$ are tangent to $P$ at $y_0$.
too. The same argument used if $b \geq q$ shows that always we can find an $F$-orthonormal system $\tilde{\mathcal{F}}$ of vector fields along $\gamma$, tangent to $N$ at $x_0$ and tangent to $P$ at $y_0$. Then, by a computation similar to the one used in [8] (the proof of Theorem 3.2.) and taking into account (10), we obtain

$$
\sum_{i=1}^{q} \left[ I_0^i(E_i, E_i) + I_0^i(FE_i, FE_i) \right] = -\int_0^l \sum_{i=1}^{q} \left[ \mathcal{R}(E_i, \gamma', E_i, \gamma') - \mathcal{R}(FE_i, \gamma', FE_i, \gamma') \right] ds.
$$

Now, by using Proposition 2.1 and taking into account

$$
\sum_{i=1}^{q} \left[ g^2(\gamma', E_i) + g^2(\gamma', FE_i) \right] \leq \|\gamma'\|^2 = 1,
$$

from (14) it follows

$$
\sum_{i=1}^{q} \left[ I_0^i(E_i, E_i) + I_0^i(FE_i, FE_i) \right] = -ql + \int_0^l \sum_{i=1}^{q} \left[ g^2(\gamma', E_i) + g^2(\gamma', FE_i) \right] ds - \int_0^l H_q(\gamma', \tilde{\mathcal{F}}) ds \leq l(1 - q) - \int_0^l H_q(\gamma', \tilde{\mathcal{F}}) ds < 0
$$

We deduce that $I_0^i (E_i, E_i) < 0$ or $I_0^i (FE_i, FE_i) < 0$ for some $i \in \{1, \ldots, q\}$, contradicting the hypothesis that $\gamma$ has minimal length. Hence $N$ and $P$ must have nonempty intersection.

REFERENCES